Optical Lattices in Atom Interferometry: An Analytical Treatment and Proposal of Applications

Tim Kovachy, Department of Physics, Harvard University

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I. Introduction

Atom interferometry has opened new frontiers in precision metrology. Highly sensitive gravimeters, gravity gradiometers, and gyroscopes have been constructed, and promising work has been done to integrate these sensors into a robust apparatus that can operate outside the laboratory with applications in inertial navigation and geodesy [1]. Moreover, atom interferometers have been used to make competitive measurements of the fine structure constant [2, 3]. By directly measuring the electron magnetic moment and subsequently calculating the corresponding value of the fine structure constant using Quantum Electrodynamics (QED), Hanneke *et al.* have determined the fine structure constant to within an uncertainty of 0.37 ppb, providing the tightest bound on the value of the fine structure constant to date [4]. The uncertainty they achieve is smaller than the uncertainty in the most accurate atom interferometric measurements of the fine structure constant by approximately an order of magnitude. Because their determination of the fine structure constant assumes the validity of QED, comparison with an independent measurement that does not depend on QED could provide extremely precise tests of QED. Atom interferometry is one of the most promising candidates to provide such an independent measurement. It is therefore a matter of fundamental importance to decrease the uncertainty in determinations of the fine structure constant based on atom interferometry to a level comparable to that reached by Hanneke *et al.* [5]. In addition, an experiment to test Einstein's Equivalence Principle with unprecedented precision is underway [6], and atom interferometric gravitational wave detectors offer the possibility to study gravitational radiation in frequency ranges inaccessible to LIGO or LISA [7]. Atom interferometers have traditionally relied on matter gratings or light pulses to act as beam splitters and mirrors, with atomic wave packets traveling freely between these interaction zones. Light-pulse schemes using either Raman pulses (where the internal state of the atom is changed) or Bragg pulses (where the internal state of the atom remains unchanged) have been implemented, such as those described in [8, 9, 10, 11].

In many applications, such as measurements of gravity and rotation, the sensitivity of

a light-pulse atom interferometer is proportional to the separation in momentum that can be attained between its two arms [1]. In measurements of the fine structure constant, the sensitivity scales as the square of this separation [2]. Therefore, significant efforts have been devoted to the development of Large Momentum Transfer (LMT) beam splitters. LMT beam splitters achieving momentum splittings of $24\hbar k$ using multi-photon Bragg pulses have recently been demonstrated [9]. However, the required laser intensities to implement multi-photon Bragg transitions of order $\sim$1000 may prove to be prohibitive [12]. In contrast, LMT beam splitters that use several two-photon Bragg pulses or a multi-photon Bragg pulse of relatively small order to separate the two arms of the interferometer in momentum space, followed by the acceleration of one of the arms with an optical lattice, could potentially provide $1000\hbar k$ momentum separations with relatively modest laser intensity requirements. An atom interferometer that uses this method to achieve $12\hbar k$ LMT beam splitters has been successfully operated in a proof of principle experiment [13]. In a separate experiment, an atom interferometer with $10\hbar k$ LMT beam splitters has been realized using a similar technique [14]. Alternatively, both arms of the interferometer could be simultaneously accelerated in opposite directions by two different optical lattices after the initial splitting. Using this second scheme, Mueller *et al.* have demonstrated an interferometer with $24\hbar k$ LMT beam splitters that achieves 15 percent contrast and an individual beam splitter that provides an $88\hbar k$ momentum separation [15].

The utility of atom interferometry hinges upon our ability to precisely calculate the phase accumulated along the different arms of an interferometer. Indeed, the phase obtained by an atom during a Raman or Bragg pulse is well-understood [1, 12]. Analagously, in order to take full advantage of the potential of lattice beam splitters, we must have a detailed understanding of the phase evolution of an atom in an optical lattice. In this thesis, we provide a complete, analytical, and rigorous treatment of this problem. To our knowledge, such a treatment has not been previously presented in the literature, where we note that a sufficient analysis must account for corrections to the adiabatic approximation (that is, we

cannot assume that all population remains in the ground state of the lattice) and must be sufficiently general to allow for a broad class of external potentials.

In addition, we propose a new type of atom interferometer in which optical lattices are used as waveguides for the atoms, so that the atomic trajectories are precisely controlled for the duration of the interferometer sequence. Our analysis indicates that such interferometers will offer unprecendented sensitivities for a wide variety of applications and that they will be able to operate effectively over distance scales previously considered too small to be studied by atom interferometry. We will perform phase shift calculations for these lattice interferometers using the theoretical groundwork formulated in this thesis, and we will discuss how lattice interferometers can both exceed the performance of conventional atom interferometers in many standard applications and expand the types of measurements that can effectively be carried out using atom interferometry.

II. The Hamiltonian in Different Frames

We begin our discussion of the lattice-atom interaction by finding a useful form for our Hamiltonian. As is typical for many applications of atom interferometry, to minimize decoherence we assume that we work with atomic gases dilute enough so that the effects of atom-atom interactions are negligible. We first consider the Hamiltonian in the lab frame, where for now we assume constant gravitational acceleration g so that we have a gravitational potential given by mgx. We expose the atom to a superposition of an upward propagating lattice beam with phase $\phi_{up}(t)$ and a downward propagating lattice beam with phase $\phi_{down}(t)$, which couples an internal ground state $|g\rangle$ to an internal excited state $|e\rangle$. Where we let $\Omega_{up}(t)$ denote the single photon Rabi frequency of the upward propagating beam, $\Omega_{down}(t)$ denote the single-photon Rabi frequency of the downward propagating beam, and Δ denote the detuning from the excited state, we define the two-photon Rabi frequency $\Omega(t) \equiv \frac{\Omega_{up}(t)\Omega_{down}(t)}{4\Delta}$. If we choose the frequencies of our lasers appropriately, we will obtain

the following Hamiltonian–where the periodic term in the potential arises from a spatially varying AC stark shift and where $V_0 \equiv 4\hbar\Omega(t)$ [16, 17]:

$$\hat{H}_{Lab} = \frac{\hat{p}^2}{2m} + V_0 \sin^2\left[k\hat{x} - \frac{1}{2}(\phi_{up}(t) - \phi_{down}(t))\right] + mg\hat{x} \tag{1}$$

Note that where the difference between the frequency of the upward propagating beam and the frequency of the downward propagating beam is denoted by $\Delta\omega(t)$, we will have the relation $\Delta\phi(t) \equiv \phi_{up}(t) - \phi_{down}(t) = \int_0^t \Delta\omega(t')dt' + \phi_{up}(0) - \phi_{down}(0)$. For a given $\Delta\phi(t)$, the lattice standing wave will be translated by $D_{Lab}(t) \equiv \frac{\Delta\phi(t)}{2k}$ in the x direction from the origin. Thus, the velocity of the lattice in the lab frame will be:

$$v_{Lab}(t) = \frac{d}{dt}D_{Lab}(t) = \frac{\Delta\omega(t)}{2k} \tag{2}$$

and we rewrite the lab frame Hamiltonian as:

$$\hat{H}_{Lab} = \frac{\hat{p}^2}{2m} + V_0 \sin^2\left[k\hat{x} - kD_{Lab}(t)\right] + mg\hat{x} \tag{3}$$

In order to most readily describe the dynamics of an atom in an accelerating optical lattice, it will be useful to work in momentum space. The $mg\hat{x}$ term that appears in the lab frame Hamiltonian makes such an approach difficult, especially when considering non-adiabatic corrections to the phase shift. However, we can change frames by performing a unitary transformation in order to obtain a Hamiltonian that is easier to handle analytically. In the end, we will see that approaching the problem from the point of view of dressed states provides us with a convenient Hamiltonian for our purposes.

To consider how the Hamiltonian changes under a unitary transformation, we consider two general frames related by a unitary transformation $\hat{U}(t)$. Let $|\Psi_1\rangle$ be the state vector for frame 1, with corresponding state vector $|\Psi_2\rangle$ in frame 2 such that $|\Psi_1\rangle = \hat{U}(t)\,|\Psi_2\rangle$. We let $\hat{H}_1$ be the Hamiltonian in frame 1 and $\hat{H}_2$ be the Hamiltonian in frame two. We recall that by definition the Hamiltonian $\hat{H}_A$ in any frame A is the operator such that for any

state vector $|\Psi_A\rangle$ in frame A's Hilbert space, the time evolution of $|\Psi_A\rangle$ is described by the equation $i\hbar\frac{\partial}{\partial t}|\Psi_A\rangle = \hat{H}_A|\Psi_A\rangle$. Thus:

$$i\hbar\frac{\partial}{\partial t}|\Psi_1\rangle = i\hbar\frac{\partial}{\partial t}\left(\hat{U}(t)|\Psi_2\rangle\right) = \hat{U}(t)\left(i\hbar\frac{\partial}{\partial t}|\Psi_2\rangle\right) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}(t)\right)|\Psi_2\rangle \tag{4}$$

$$= \left[\hat{U}(t)\hat{H}_2 + i\hbar\left(\frac{\partial}{\partial t}\hat{U}(t)\right)\right]|\Psi_2\rangle \tag{5}$$

Therefore, since $|\Psi_2\rangle = \hat{U}^\dagger(t)|\Psi_1\rangle$:

$$\hat{H}_1 = \hat{U}(t)\hat{H}_2\hat{U}^\dagger(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}(t)\right)\hat{U}^\dagger(t) \tag{6}$$

We will now transform from the lab frame to a frame which is falling freely with gravity. In this frame, we no longer will have a gravitational potential term–the effects of gravity will manifest purely through the dynamics of the lattice. Moreover, since we are falling with gravity, a lattice that is stationary in the lab frame will have an upwards velocity of gt in this frame, where we choose the initial velocity of the freely falling frame with respect to the lab frame to be 0. Thus, the Hamiltonian in the freely falling frame should be:

$$\hat{H}_{FF} = \frac{\hat{p}^2}{2m} + V_0\sin^2\left[k\hat{x} - k(D_{Lab}(t) + \frac{1}{2}gt^2)\right] \tag{7}$$

To verify that $\hat{H}_{FF}$ is indeed physically equivalent to $\hat{H}_{Lab}$, we must find a unitary transformation $\hat{U}_{FF}(t)$ that links the two frames. We will ansatz a transformation that consists of a boost in position space, a boost in momentum space, and a time-dependent change of phase. Such a transformation has the general form:

$$\hat{U}(t) = e^{\frac{i}{\hbar}d(t)\hat{p}}e^{-\frac{i}{\hbar}mv(t)\hat{x}}e^{\frac{i}{\hbar}\theta(t)} \tag{8}$$

The operators $\hat{x}$ and $\hat{p}$ transform under $\hat{U}(t)$ as follows:

$$\hat{U}(t)\hat{x}\hat{U}^{\dagger}(t) = \hat{x} + d(t) \tag{9}$$

$$\hat{U}(t)\hat{p}\hat{U}^{\dagger}(t) = \hat{p} + mv(t) \tag{10}$$

Since the freely falling frame is separated from the lab frame by $\frac{1}{2}gt^2$ in position space and by mgt in momentum space, a logical guess is that $\hat{U}_{FF}(t)$ is a transformation with $d(t) = \frac{1}{2}gt^2$ and $v(t) = gt$. Given this ansatz, we will find an appropriate phase factor $\theta(t)$ so that $\hat{U}_{FF}(t)$ indeed transforms between our two frames. We require that:

$$\hat{H}_{Lab} = \hat{U}_{FF}(t)\hat{H}_{FF}\hat{U}^{\dagger}_{FF}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{FF}(t)\right)\hat{U}^{\dagger}_{FF}(t) \tag{11}$$

$$= \frac{(\hat{p} + mgt)^2}{2m} + V_0 \sin^2\left[k\hat{x} - kD_{Lab}(t)\right] - gt\hat{p} + mg(\hat{x} + \frac{1}{2}gt^2) - \dot{\theta}(t) \tag{12}$$

We thus obtain that $\dot{\theta}(t) = mg^2t^2$, and we can integrate to find $\theta(t)$, where we drop the integration constant since it will only add a time-independent phase factor to the unitary transformation that has no physical significance. Our result is therefore:

$$\theta(t) = \frac{1}{3}mg^2t^3 \tag{13}$$

So we indeed see that $\hat{H}_{Lab}$ and $\hat{H}_{FF}$ describe the same physics from the point of view of two different frames, as expected. Since the only position dependence in $\hat{H}_{FF}$ comes from the lattice potential, we could now readily approach the problem of finding the dynamics of the system by working in the freely falling frame. However, it will prove to be useful from a calculational standpoint to transform to a third frame, with a Hamiltonian resembling that describing the atom-light interaction from the point of view of dressed states. Our Hamiltonian in this frame will be:

$$\hat{H}_{DS} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt)\hat{p} + V_0 \sin^2(k\hat{x}) \tag{14}$$

The unitary transformation $\hat{U}_{DS}$ such that $\hat{H}_{FF} = \hat{U}_{DS}(t)\hat{H}_{DS}\hat{U}^\dagger_{DS}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{DS}(t)\right)\hat{U}^\dagger_{DS}(t)$ corresponds to a boost in position space by $D_{Lab}(t) + \frac{1}{2}gt^2$ with no boost in momentum space and with no time-dependent phase factor:

$$\hat{U}_{DS} = e^{-\frac{i}{\hbar}(D_{Lab}(t) + \frac{1}{2}gt^2)\hat{p}} \tag{15}$$

We note that our ability to boost to a frame in which the Hamiltonian contains no position dependent terms outside of the lattice potential is contingent upon the assumption that the external potential in the lab frame (not including the lattice potential) is linear in x. However, real-world potentials–such as the potential corresponding to earth's gravitational field–will deviate somewhat from this assumption. Any such deviations would manifest as residual position dependent terms in the dressed state Hamiltonian, which we collectively refer to as V'. Under the semiclassical approximation, we neglect the effects of V' on the time evolution of our atomic wavepacket. This approximation will be valid when the energy scale of V' (which will be on the order of magnitude of the expectation value of V' in the atomic wavepacket) is much smaller than the energy scale of the lattice potential, which is the case for a wide class of experimental parameters. For example, in the case of an atomic wavepacket with a spatial spread of $\sim 100\mu$m in the gravitational field at the earth's surface (which has a gradient of $\sim 10^{-6}\ s^{-2}$), the energy scale of V' will be $\sim h \times (10^{-6}$ Hz). This energy scale is smaller than that of a lattice of typical experimental depth ($\sim 5E_r$) by a factor of $\sim 10^{10}$. We will discuss the effects of linear gradients and of more general potentials in greater depth in Appendix A.

Now, we will show how working in momentum space allows us to represent $\hat{H}_{DS}$ as an infinite dimensional, discrete matrix. This matrix will be discrete because the optical lattice potential term, $V_0 \sin^2(k\hat{x})$, only couples a momentum eigenstate $|p\rangle$ to the eigenstates

$|p + 2\hbar k\rangle$ and $|p - 2\hbar k\rangle$ [17]. For the moment, we will examine the evolution of individual Bloch states as described by $\hat{H}_{DS}$. These Bloch states reduce to single momentum eigenstates $|p\rangle$ when $V_0 = 0$. The knowledge of how each of these Bloch states evolves under $\hat{H}_{DS}$ will allow us to describe the dynamics of an entire wave packet.

Let us consider a discrete Hilbert space spanned by the momentum eigenstates $|2n\hbar k\rangle$, where n can be any integer. We can express any vector in this Hilbert space in the following form:

$$|\Psi(t)\rangle = \sum_{n=-\infty}^{\infty} c_n(t) \, |2n\hbar k\rangle \tag{16}$$

Since our Hilbert space is discrete, it is natural to adopt the normalization convention that $\langle 2m\hbar k|2n\hbar k\rangle = \delta_{mn}$. When considered as an operator acting on this discrete Hilbert space, $\hat{H}_{DS}$ can be written as [12, 18]:

$$\hat{H}_{DS}^{discrete} = \sum_{n=-\infty}^{\infty} \frac{(2n\hbar k)^2}{2m} \, |2n\hbar k\rangle \langle 2n\hbar k| - \sum_{n=-\infty}^{\infty} (v_{Lab}(t) + gt)(2n\hbar k) \, |2n\hbar k\rangle \langle 2n\hbar k| \tag{17}$$

$$+ \sum_{n=-\infty}^{\infty} \hbar\Omega(t) \left(|2n\hbar k\rangle \langle 2(n-1)\hbar k| + |2n\hbar k\rangle \langle 2(n+1)\hbar k| \right) \tag{18}$$

Note that we neglect the light shift $\frac{\Omega_{up}(t)^2}{4\Delta} + \frac{\Omega_{down}(t)^2}{4\Delta}$ since it simply amounts to adding a constant to the energy, which will have no net effect on the phase shift between the two arms of an atom interferometer provided that the effective two-photon Rabi frequencies for the two arms differ negligibly. Now, it is convenient to define $\omega_{bragg} \equiv \frac{2\hbar k^2}{m}$. We note that ω_{bragg} is the frequency such that an atom in the momentum eigenstate $|2n\hbar k\rangle$ has kinetic energy $n^2\hbar\omega_{bragg}$. We also introduce the recoil velocity $v_r \equiv \frac{\hbar k}{m}$, which is the velocity of an atom with momentum $\hbar k$. In order to make our notation as compact as possible, we will be interested in the quantity $\alpha(t) \equiv \frac{v_{Lab}(t)+gt}{v_r}$, which is the velocity of the lattice in the freely

falling frame in units of v_r. We can express the second term of $\hat{H}_{DS}^{discrete}$ in a useful way by noting that $(v_{Lab}(t) + gt)2n\hbar k = \frac{v_{Lab}(t)+gt}{v_r} v_r 2n\hbar k = n\hbar\alpha(t)\omega_{bragg}$. Furthermore, we define $\tilde{\Omega}(t) \equiv -\frac{\Omega(t)}{\omega_{bragg}}$. We can now write our discrete Hamiltonian in a simplified form:

$$\hat{H}_{DS}^{discrete} = \hbar\omega_{bragg} \sum_{n=-\infty}^{\infty} [n^2 \left|2n\hbar k\right\rangle \left\langle 2n\hbar k\right| - n\alpha(t) \left|2n\hbar k\right\rangle \left\langle 2n\hbar k\right| \tag{19}$$

$$-\tilde{\Omega}(t) \left(\left|2n\hbar k\right\rangle \left\langle 2(n-1)\hbar k\right| + \left|2n\hbar k\right\rangle \left\langle 2(n+1)\hbar k\right|\right)] \tag{20}$$

We proceed to write the Schrodinger equation for our discrete Hilbert space in matrix notation. In Dirac notation, the Schrodinger equation takes the form:

$$i\hbar \frac{\partial}{\partial t} \left|\Psi(t)\right\rangle = \hat{H}_{DS}^{discrete} \left|\Psi(t)\right\rangle \tag{21}$$

Recalling the definition of $\left|\Psi(t)\right\rangle$ from equation (16) and projecting onto the eigenstate $\left|2m\hbar k\right\rangle$ for arbitrary integer m, we obtain:

$$i\hbar \left\langle 2m\hbar k\right| \sum_{n=-\infty}^{\infty} \dot{c}_n(t) \left|2n\hbar k\right\rangle = \left\langle 2m\hbar k\right| \hat{H}_{DS}^{discrete} \sum_{n=-\infty}^{\infty} c_n(t) \left|2n\hbar k\right\rangle \tag{22}$$

The orthonormality of the momentum eigenstates in our Hilbert space dictates that:

$$\left\langle 2m\hbar k\right| \sum_{n=-\infty}^{\infty} \dot{c}_n(t) \left|2n\hbar k\right\rangle = \sum_{n=-\infty}^{\infty} \dot{c}_n(t)\delta_{mn} = \dot{c}_m(t) \tag{23}$$

Therefore, where we adopt the notation $H_{mn} \equiv \left\langle 2m\hbar k\right| \hat{H}_{DS}^{discrete} \left|2n\hbar k\right\rangle$, we can write equation (22) as:

$$i\hbar\dot{c}_m(t) = \sum_{n=-\infty}^{\infty} H_{mn} c_n(t) \tag{24}$$

which we recall holds for all integers m. We can rewrite equation (24) in terms of matrix multiplication. We let $\vec{\Psi}(t)$ be the column vector whose nth entry is $c_n(t)$, so:

$$\vec{\Psi}(t) = \begin{pmatrix} \vdots \\ c_{-2}(t) \\ c_{-1}(t) \\ c_0(t) \\ c_1(t) \\ c_2(t) \\ \vdots \end{pmatrix} \tag{25}$$

And we let H be the matrix such that the element in the mth row and nth column is given by H_{mn}. Therefore, we have:

$$i\hbar\frac{\partial}{\partial t}\vec{\Psi}(t) = H\vec{\Psi}(t) \tag{26}$$

Equation (26) is simply the Schrodinger equation rewritten in terms of matrix notation, and the matrix H acts as the Hamiltonian. Using equation (19) to evaluate the matrix elements H_{mn}, we see that:

$$H_{mn} = \hbar\omega_{bragg}\sum_{j=-\infty}^{\infty}\left[\left(j^2 - j\alpha(t)\right)\delta_{mj}\delta_{jn} - \tilde{\Omega}(t)\left(\delta_{mj}\delta_{j-1,n} + \delta_{mj}\delta_{j+1,n}\right)\right] \tag{27}$$

$$= \hbar\omega_{bragg}\left[\left(n^2 - n\alpha(t)\right)\delta_{mn} - \tilde{\Omega}(t)\left(\delta_{m,n+1} + \delta_{m,n-1}\right)\right] \tag{28}$$

H is thus a tridiagonal matrix which takes the following form:

$$H = \hbar\omega_{bragg} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 0 & -\tilde{\Omega}(t) & 4+2\alpha(t) & -\tilde{\Omega}(t) & 0 & & & \\ & & 0 & -\tilde{\Omega}(t) & 1+\alpha(t) & -\tilde{\Omega}(t) & 0 & & \\ & & & 0 & -\tilde{\Omega}(t) & 0 & -\tilde{\Omega}(t) & 0 & \\ & & & & 0 & -\tilde{\Omega}(t) & 1-\alpha(t) & -\tilde{\Omega}(t) & 0 \\ & & & & & 0 & -\tilde{\Omega}(t) & 4-2\alpha(t) & -\tilde{\Omega}(t) \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \quad (29)$$

III. Phase Evolution Under the Adiabatic Approximation

Now that we have determined our Hamiltonian matrix H, we have the appropriate machinery in place to describe the phase evolution of an atom in an optical lattice. We will focus on the problem of adiabatic rapid passage between momentum eigenstates. This process involves ramping up the lattice depth in order to load the atom into the lattice potential, accelerating the lattice to transfer momentum to the atom, and ramping down the lattice depth to deliver the atom into a target momentum eigenstate, as illustrated in Figure 1 and Figure 2. As in our previous discussion, we consider the evolution of single Bloch states, noting that we can easily generalize our results to the case of a wave packet of finite width, as we will address later.

When we are ramping up the lattice depth, we set the frequency difference between the

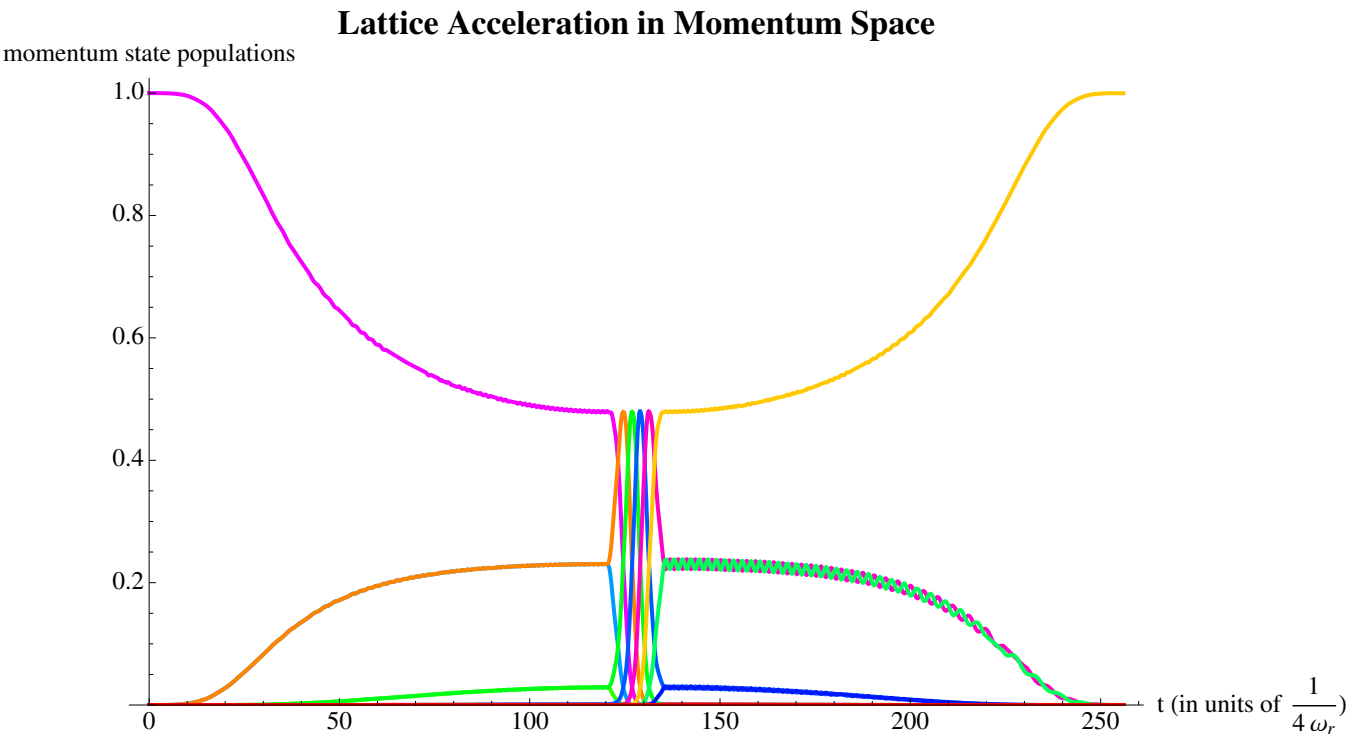

Figure 1: This plot shows a lattice acceleration in momentum space that transfers $10\hbar k$ of momentum. First, we adiabatically ramp up the lattice. As the lattice depth increases, the spread of the Bloch state in momentum space increases. Subsequently, we accelerate and ramp down the lattice, leaving the atom in a single momentum eigenstate.

lattice beams so that the lattice is resonant with the atoms. That is, we want the velocity of the lattice to be within v_r of the velocity of the atom. If this resonance condition is satisfied, the atoms will be loaded into the ground state of the lattice as long as we increase the lattice depth sufficiently slowly so that the process can be considered adiabatic. To see why this is true, we examine the eigenvalues of H when $\tilde{\Omega}(t) = 0$, so that H is a diagonal matrix. The eigenstates of our Hamiltonian are thus the momentum eigenstates $|2n\hbar k\rangle$ for integer n with corresponding eigenvalues $E_n = \hbar\omega_{bragg}\left(n^2 - n\alpha(t)\right)$. Our resonance condition states that if our initial state is $|2n_0\hbar k\rangle$, then the initial velocity of the lattice must be within v_r of $2n_0 v_r$. Thus, the initial value α_0 of the parameter α must be within 1 of $2n_0$. Now, where we write $\alpha_0 = 2n_0 + \delta$ and let $n = n_0 + \Delta n$, the eigenvalues of H with the lattice depth set to 0 are:

$$E_n = \hbar\omega_{bragg}\left[(n_0 + \Delta n)^2 - (n_0 + \Delta n)(2n_0 + \delta)\right] = \hbar\omega_{bragg}\left[-n_0^2 + (\Delta n)^2 - n_0\delta - \Delta n\delta\right] \tag{30}$$

For $|\delta| < 1$, E_n will be most negative when $\Delta n = 0$, so $|2n_0\hbar k\rangle$ will be the ground state of our Hamiltonian when $\tilde{\Omega}(t) = 0$. Thus, our atom is initially in the ground state of the Hamiltonian. And as long as we ramp up the lattice slowly enough and keep the lattice velocity constant while doing so, the adiabatic theorem tells us that the atom will remain in the lattice ground state throughout the ramp up process provided that the ground state never passes through a point of degeneracy, which is indeed the case for fixed α and $|\delta| < 1$.

Note that the maximum rate at which we can increase the lattice depth while still maintaining the validity of the adiabatic approximation depends on $|\delta|$. To see why this is true, we observe that:

$$E_{n_0+\Delta n} - E_{n_0} = \hbar\omega_{bragg}\Delta n(\Delta n - \delta) \tag{31}$$

As $|\delta|$ increases toward 1, the gap between E_n and $E_{n+sign(\delta)}$ decreases (note that $sign(\delta)$ is equal to 1 if δ is positive and -1 if δ is negative). Therefore, for the adiabatic approximation

to hold, the maximum rate at which we can ramp up the lattice decreases as we increase $|\delta|$.

After ramping up the lattice, we accelerate it until its velocity is within v_r of $2n_f v_r$ where $|2n_f\hbar k\rangle$ is our target momentum eigenstate. The ground state of the Hamiltonian when the lattice has finished ramping down will therefore be $|2n_f\hbar k\rangle$, and as long as our entire process is adiabatic so that the atom remains in the ground state, the atom will indeed end up in our target state. We must make sure that there are no points of degeneracy for the ground state during the acceleration, which can be achieved with a sufficiently deep lattice. Analagously to our previous discussion, the maximum rate at which we can ramp down the lattice increases as the difference between the final lattice velocity and $2n_f v_r$ goes to 0.

Since we assume that the atom always stays in the ground state of the Hamiltonian and that the ground state avoids any energy level crossings, the phase $\phi(t)$ of the atom evolves as follows:

$$\phi(t) - \phi_0 = -\frac{1}{\hbar}\int_{t_0}^{t} \varepsilon_0(t')dt' \tag{32}$$

where $\varepsilon_0(t)$ is the ground state eigenvalue of the Hamiltonian and ϕ_0 is the initial phase of the atom. In principle, we could also have a Berry's phase term, but we empirically find that this term is 0 for the case of an external potential that is linear in position. We therefore will have no contribution from the Berry's phase under the semiclassical approximation, where we note that any such contribution would arise from the residual terms V' in the dressed state Hamiltonian. We will now proceed to determine the form of $\varepsilon_0(t)$ using our expression for the matrix H. In doing so, we will prove some useful facts about the spectrum of H. First, we note that we can express H as the sum of a diagonal matrix which depends on α but not on $\tilde{\Omega}$ and a matrix with all diagonal elements equal to 0 which depends on $\tilde{\Omega}$ but not on α:

$$H(\alpha, \tilde{\Omega}) = H_\alpha(\alpha) + H_{\tilde{\Omega}}(\tilde{\Omega}) \tag{33}$$

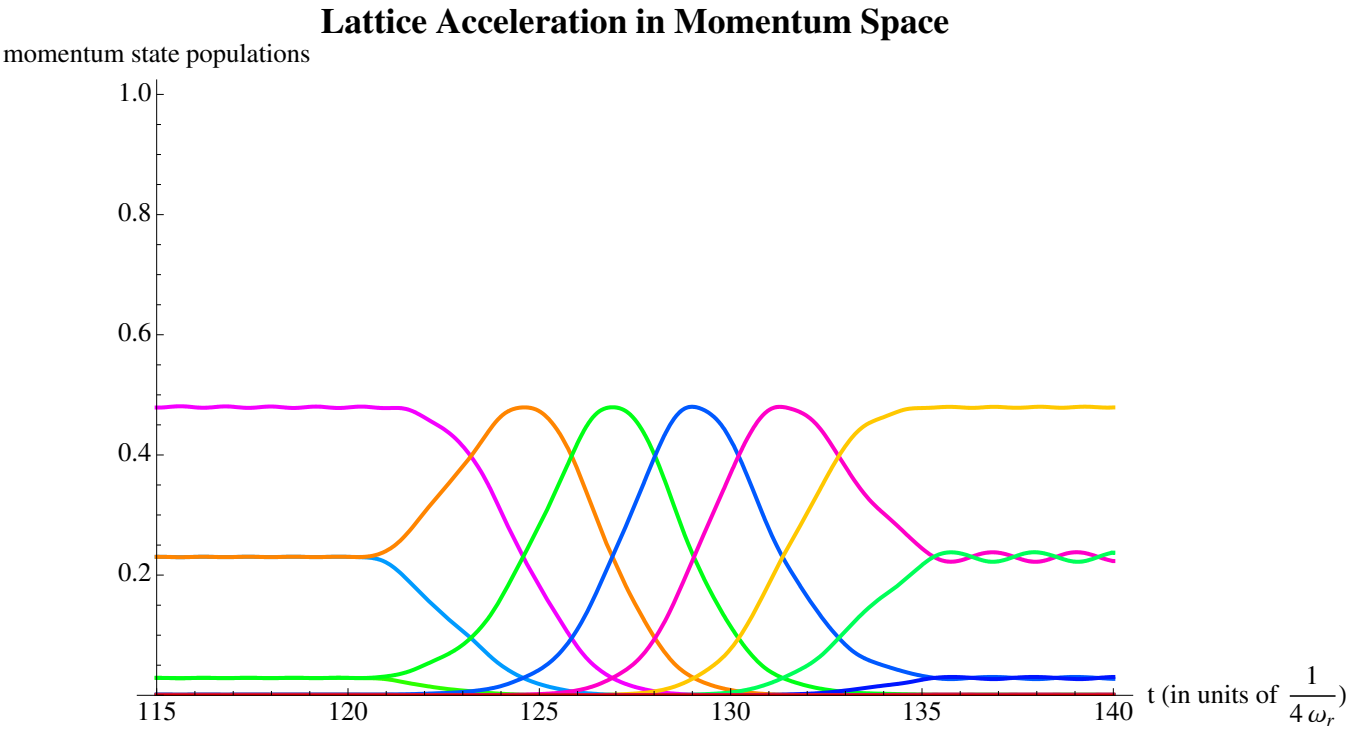

Figure 2: Here we have zoomed in on the acceleration stage from the previous figure. Notice how the Bragg transitions are blurred together, which highlights an advantage that optical lattice manipulations have over sequential two-photon Bragg transitions. The acceleration that can be achieved by sequential two-photon Bragg transitions is fundamentally limited by how large we can make the effective Rabi frequency before the two level picture breaks down. Lattice accelerations are not limited in this way and can thus transfer momentum at a much faster rate than can be achieved by sequential Bragg transitions.

where:

$$H_\alpha(\alpha) = \hbar\omega_{bragg} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & 0 & 0 & 4+2\alpha & 0 & 0 & & & & \\ & & 0 & 0 & 1+\alpha & 0 & 0 & & & \\ & & & 0 & 0 & 0 & 0 & 0 & & \\ & & & & 0 & 0 & 1-\alpha & 0 & 0 & \\ & & & & & 0 & 0 & 4-2\alpha & 0 & 0 \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{34}$$

and:

$$H_{\tilde{\Omega}}(\tilde{\Omega}) = \hbar\omega_{bragg} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & 0 & -\tilde{\Omega} & 0 & -\tilde{\Omega} & 0 & & & & \\ & & 0 & -\tilde{\Omega} & 0 & -\tilde{\Omega} & 0 & & & \\ & & & 0 & -\tilde{\Omega} & 0 & -\tilde{\Omega} & 0 & & \\ & & & & 0 & -\tilde{\Omega} & 0 & -\tilde{\Omega} & 0 & \\ & & & & & 0 & -\tilde{\Omega} & 0 & -\tilde{\Omega} & 0 \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{35}$$

Say that the eigenvectors of $H(\alpha, \tilde{\Omega})$ are given by $\vec{\Psi}_n(\alpha, \tilde{\Omega})$, where the index n runs from 0 to ∞. Let us denote the corresponding eigenvalue to each $\vec{\Psi}_n(\alpha, \tilde{\Omega})$ as $\varepsilon_n(\alpha, \tilde{\Omega})$, and we let $\Psi_{n_j}(\alpha, \tilde{\Omega})$ be the jth element of $\vec{\Psi}_n(\alpha, \tilde{\Omega})$. Now, we will consider how our eigenvalues and eigenvectors change when we increase α from some value α_0 to $\alpha_0 + 2$ while keeping $\tilde{\Omega}$ fixed. We ansatz that our new eigenvectors can be obtained by shifting our old eigenvectors to the right by 1. In other words, we propose that:

$$\Psi_{n_j}(\alpha_0 + 2, \tilde{\Omega}) = \Psi_{n_{(j-1)}}(\alpha_0, \tilde{\Omega}) \tag{36}$$

We will show that this is indeed the case by proving that $H(\alpha_0 + 2, \tilde{\Omega})\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})$ is indeed a scalar multiple of $\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})$, where $\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})$ is the vector whose jth element is $\Psi_{n_{(j-1)}}(\alpha_0, \tilde{\Omega})$. First, we note that the jth element of $H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0, \tilde{\Omega})$ is:

$$\left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0, \tilde{\Omega})\right)_j = \hbar\omega_{bragg}\left[-\tilde{\Omega}\Psi_{n_{(j-1)}}(\alpha_0, \tilde{\Omega}) - \tilde{\Omega}\Psi_{n_{(j+1)}}(\alpha_0, \tilde{\Omega})\right] \tag{37}$$

and the jth element of $H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})$ is:

$$\left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})\right)_j = \hbar\omega_{bragg}\left[-\tilde{\Omega}\Psi_{n_{(j-2)}}(\alpha_0, \tilde{\Omega}) - \tilde{\Omega}\Psi_{n_j}(\alpha_0, \tilde{\Omega})\right] \tag{38}$$

Thus, we see that:

$$\left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})\right)_j = \left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0, \tilde{\Omega})\right)_{j-1} \tag{39}$$

Now, we consider the product $H_\alpha(\alpha_0 + 2)\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})$. Using equation (34), we see that:

$$\left(H_\alpha(\alpha_0 + 2)\vec{\xi}_n(\alpha_0 + 2, \tilde{\Omega})\right)_j = \hbar\omega_{bragg}\left(j^2 - j(\alpha_0 + 2)\right)\Psi_{n_{(j-1)}}(\alpha_0, \tilde{\Omega}) \tag{40}$$

and also that:

$$\left(H_\alpha(\alpha_0)\vec{\Psi}_n(\alpha_0, \tilde{\Omega})\right)_j = \hbar\omega_{bragg}\left(j^2 - j\alpha_0\right)\Psi_{n_j}(\alpha_0, \tilde{\Omega}) \tag{41}$$

The jth element of the vector $H(\alpha_0+2,\tilde{\Omega})\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})$ is thus:

$$\left(H(\alpha_0+2,\tilde{\Omega})\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})\right)_j = \left(H_\alpha(\alpha_0+2)\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})\right)_j + \left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})\right)_j \tag{42}$$

$$= \hbar\omega_{bragg}\left(j^2 - j(\alpha_0+2)\right)\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) + \left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1} \tag{43}$$

We can express $\left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1}$ in a more meaningful form by making use of the fact that $H(\alpha_0,\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega}) = \varepsilon_n(\alpha_0,\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})$. The (j-1)th elements of each side of this equation must be equal, so:

$$\varepsilon_n(\alpha_0,\tilde{\Omega})\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) = \left(H(\alpha_0,\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1} \tag{44}$$

$$= \left(H_\alpha(\alpha_0)\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1} + \left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1} \tag{45}$$

$$= \hbar\omega_{bragg}\left((j-1)^2 - (j-1)\alpha_0\right)\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) + \left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1} \tag{46}$$

Rearranging this equation, we see that:

$$\left(H_{\tilde{\Omega}}(\tilde{\Omega})\vec{\Psi}_n(\alpha_0,\tilde{\Omega})\right)_{j-1} = \varepsilon_n(\alpha_0,\tilde{\Omega})\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) - \hbar\omega_{bragg}\left((j-1)^2 - (j-1)\alpha_0\right)\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) \tag{47}$$

Substituting this result into equation (43):

$$\left(H(\alpha_0+2,\tilde{\Omega})\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})\right)_j = \hbar\omega_{bragg}\left(j^2 - j(\alpha_0+2)\right)\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) + \varepsilon_n(\alpha_0,\tilde{\Omega})\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) \tag{48}$$

$$-\hbar\omega_{bragg}\left((j-1)^2-(j-1)\alpha_0\right)\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) \tag{49}$$

$$=\left[\varepsilon_n(\alpha_0,\tilde{\Omega})-\hbar\omega_{bragg}\left(\alpha_0+1\right)\right]\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega}) \tag{50}$$

Since this equation holds for all integers j and since by definition $\left(\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})\right)_j=\Psi_{n_{(j-1)}}(\alpha_0,\tilde{\Omega})$, we can conclude that:

$$H(\alpha_0+2,\tilde{\Omega})\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})=\left[\varepsilon_n(\alpha_0,\tilde{\Omega})-\hbar\omega_{bragg}\left(\alpha_0+1\right)\right]\vec{\xi}_n(\alpha_0+2,\tilde{\Omega}) \tag{51}$$

This indeed confirms that our new eigenvectors after increasing α by 2 are the vectors $\vec{\xi}_n(\alpha_0+2,\tilde{\Omega})$, with corresponding eigenvalues $\varepsilon_n(\alpha_0+2,\tilde{\Omega})=\varepsilon_n(\alpha_0,\tilde{\Omega})-\hbar\omega_{bragg}\left(\alpha_0+1\right)$. We can rearrange this expression for the eigenvalues in order to extract more physical meaning. Note that $\alpha_0+1=\frac{1}{4}\left[(\alpha_0+2)^2-\alpha_0^2\right]$. Thus:

$$\varepsilon_n(\alpha_0+2,\tilde{\Omega})-\varepsilon_n(\alpha_0,\tilde{\Omega})=-\frac{\hbar\omega_{bragg}}{4}\left[(\alpha_0+2)^2-\alpha_0^2\right] \tag{52}$$

Repeated application of equation (52) tells us that whenever $\alpha=2m$ for any integer m, $\varepsilon_n(\alpha,\tilde{\Omega})=-\frac{\hbar\omega_{bragg}}{4}\alpha^2+\varepsilon_n(0,\tilde{\Omega})$. We can generalize this expression to all α by adding on a correction term $p_n(\alpha,\tilde{\Omega})$ that vanishes when the condition $\alpha=2m$ holds, so that:

$$\varepsilon_n(\alpha,\tilde{\Omega})=-\frac{\hbar\omega_{bragg}}{4}\alpha^2+\varepsilon_n(0,\tilde{\Omega})+p_n(\alpha,\tilde{\Omega}) \tag{53}$$

The condition stated in equation (52) implies that $p_n(\alpha,\tilde{\Omega})$ is periodic in α such that $p_n(\alpha+2,\tilde{\Omega})=p_n(\alpha,\tilde{\Omega})$ for all α. It is convenient to use the convention such that for all α and Ω, $\varepsilon_n(\alpha,\tilde{\Omega})$ denotes the the nth eigenvalue labeled in order of increasing value. Under this convention, $\varepsilon_n(\alpha,\tilde{\Omega})$ will always be continuous, and it will be smooth everywhere except at points of degeneracy. To see why this is true, note that we can alternatively index our eigenvalues so that each is a function $\varepsilon'_n(\alpha,\tilde{\Omega})$ that is smooth at all points. We consider a

point of degeneracy such that for a given $\tilde{\Omega}$, $\varepsilon'_a(\alpha, \tilde{\Omega})$ and $\varepsilon'_b(\alpha, \tilde{\Omega})$ cross at some $\alpha = \alpha_{cross}$. Say that before the crossing $\varepsilon_n(\alpha, \tilde{\Omega}) = \varepsilon'_a(\alpha, \tilde{\Omega})$ and $\varepsilon_{(n+1)}(\alpha, \tilde{\Omega}) = \varepsilon'_b(\alpha, \tilde{\Omega})$, which implies that $\varepsilon'_b(\alpha, \tilde{\Omega}) > \varepsilon'_a(\alpha, \tilde{\Omega})$. After the crossing $\varepsilon'_b(\alpha, \tilde{\Omega}) < \varepsilon'_a(\alpha, \tilde{\Omega})$, so $\varepsilon_n(\alpha, \tilde{\Omega}) = \varepsilon'_b(\alpha, \tilde{\Omega})$ and $\varepsilon_{(n+1)}(\alpha, \tilde{\Omega}) = \varepsilon'_a(\alpha, \tilde{\Omega})$. Thus, at the crossing, $\varepsilon_n(\alpha, \tilde{\Omega})$ and $\varepsilon_{(n+1)}(\alpha, \tilde{\Omega})$ interchange the smooth energy bands that they represent. At the crossing point, it will not be the case that all partial derivatives of arbitrary order in the variables α and $\tilde{\Omega}$ are identical for the two smooth energy bands, so $\varepsilon_n(\alpha, \tilde{\Omega})$ and $\varepsilon_{(n+1)}(\alpha, \tilde{\Omega})$ will not be smooth in any neighborhood of the crossing point. An analagous conclusion would follow if we instead held α fixed and varied $\tilde{\Omega}$. Therefore, we indeed see that $\varepsilon_n(\alpha, \tilde{\Omega})$ will only be smooth in regions where it does not encounter any points of degeneracy.

The $-\frac{\hbar\omega_{bragg}}{4}\alpha^2$ term in equation (53) has a simple physical interpretation. Where $v_{Lattice}(t)$ is the velocity of the lattice in the freely falling frame, note that:

$$-\frac{\hbar\omega_{bragg}}{4}\alpha^2 = -\frac{\hbar}{4}\frac{2\hbar k^2}{m}\left(\frac{v_{Lattice}(t)}{\frac{\hbar k}{m}}\right)^2 = -\frac{1}{2}mv_{Lattice}(t)^2 \tag{54}$$

which is simply the negative of the kinetic energy of an atom traveling along a classical trajectory defined by the motion of the lattice in the freely falling frame. For interferometer geometries in which lattices act as waveguides for the atoms, the contributions to the phase shift from the $\varepsilon_0(0, \tilde{\Omega})$ and $p_0(\alpha, \tilde{\Omega})$ terms in the ground state energy and from corrections to the adiabatic approximation are negligible in many situations . If this is the case, the phase difference between the two arms of the interferometer is given by:

$$\phi_1 - \phi_2 = -\frac{1}{\hbar}\left[\int_0^T \varepsilon_0^{arm1}(t)dt - \int_0^T \varepsilon_0^{arm2}(t)dt\right] \tag{55}$$

$$= \frac{1}{\hbar}\left[\int_0^T \frac{1}{2}mv_{Lattice}^{arm1}(t)^2dt - \int_0^T \frac{1}{2}mv_{Lattice}^{arm2}(t)^2dt\right] \tag{56}$$

where T is the time elapsed during the interferometer sequence.

Observe that we could obtain this expression for the phase difference by assuming that

the lattice potential acts as a constraint that forces the atoms in each arm to traverse the classical path traveled by the lattice guiding that arm. We could then take the difference of the respective integrals of the Lagrangian over the two classical paths. Since there is no external potential outside of the lattice in the freely falling and dressed state frames, we recover equation (56). It makes sense that this approach yields the same result as our full quantum mechanical treatment when we neglect non-adiabatic corrections and the $p_0(\alpha, \tilde{\Omega})$ term, for these neglected terms embody the quantum nature of the motion (the $\varepsilon_0(0, \tilde{\Omega})$ term is identical for both arms of the interferometer under the assumption that the lattice depth does not differ between the two arms and thus only manifests in the phase shift through non-adiabatic corrections). Non-adiabatic corrections describe the tunneling of atoms in the lattice potential, while the $p_0(\alpha, \tilde{\Omega})$ term corresponds to the sloshing of atoms in the lattice as the lattice accelerates–a phenomenon which is responsible for Bloch oscillations. In contrast, the $-\frac{1}{2}mv_{Lattice}(t)^2$ term represents a purely classical quantity.

To summarize, we have shown that the eigenvalues of the lattice are the sums of a kinetic energy term, a term that depends only on the lattice depth and on which eigenvalue we are considering, and a term that is periodic in α so that it remains unchanged when we translate α by 2. Moreover, when α is changed by 2, we can obtain our new nth eigenvector by simply performing a shift operation on our old nth eigenvector, where the index n runs in order of increasing eigenvalues. We can thus conclude that if for a given $\tilde{\Omega}$ we know the eigenvalues and eigenvectors of our Hamiltonian for all α within any range of width 2, we can then find the eigenvalues and eigenvectors for arbitrary α. This result will prove to be extremely useful from a computational standpoint, since it allows us to completely describe the dynamics of the system by looking just at one range of α; and it is consistent with what we would expect from Bloch's theorem given that the lattice potential is periodic.

IV. Calculating Corrections to the Adiabatic Approximation Using the Method of Adiabatic Expansion

We will now present a method, which we will refer to as adiabatic expansion as in [12], to determine corrections of arbitrary order to the adiabatic approximation. During the course of this derivation, it will be convenient to parameterize our Hamiltonian, eigenvalues, and eigenvectors using the single variable t rather than the two variables $\alpha(t)$ and $\tilde{\Omega}(t)$. Note that much of our calculation will follow a similar outline as Griffiths' proof of the adiabatic theorem in [19].

For all t, we know that the relation $H(t)\vec{\Psi}_n(t) = \varepsilon_n(t)\vec{\Psi}_n(t)$ holds and that the eigenvectors $\vec{\Psi}_n(t)$ constitute an orthonormal set so that $\vec{\Psi}_j^\dagger(t)\vec{\Psi}_n(t) = \delta_{jn}$. Also, the eigenvectors form a complete basis for our Hilbert space, so the state vector $\Psi(t)$ can be written as a linear combination of the eigenvectors at any particular t, where in general the coefficients of each eigenvector can vary in time:

$$\vec{\Psi}(t) = \sum_{n=0}^{\infty} a_n(t)\vec{\Psi}_n(t) \tag{57}$$

Note that for each t, from a mathematical standpoint, the choice of the eigenvector $\vec{\Psi}_n(t)$ is ambiguous up to a phase. Since we will want to consider time derivatives of $\vec{\Psi}_n(t)$, we want this phase to be continuous as a function of t so that our time derivatives will be well-defined. To simplify matters further, we will choose the phase of $\vec{\Psi}_n(t)$ so that each element of $\vec{\Psi}_n(t)$ is real for all t and varies continuously with t, which we can do since $H(t)$ is a real-valued, Hermitian matrix (given that none of the elements of $H(t)$ change instantaneously, which is certainly a physically valid assumption).

To gain some intuition for the problem, we consider the situation where our Hamiltonian does not vary in time, so that we can write $H(t) = H(t_0)$ for every t. Thus, the eigenvectors at any t are given by $\vec{\Psi}_n(t) = \vec{\Psi}_n(t_0)$ with corresponding eigenvalues $\varepsilon_n(t) = \varepsilon_n(t_0)$. Using

equation (57), the Schrodinger equation tells us that:

$$\frac{\partial\vec{\Psi}(t)}{\partial t} = -\frac{i}{\hbar}H(t_0)\vec{\Psi}(t) = -\frac{i}{\hbar}H(t_0)\left(\sum_{n=0}^{\infty}a_n(t)\vec{\Psi}_n(t_0)\right) = -\frac{i}{\hbar}\left(\sum_{n=0}^{\infty}a_n(t)\varepsilon_n(t_0)\vec{\Psi}_n(t_0)\right) \tag{58}$$

In general, we see that:

$$\frac{\partial\vec{\Psi}(t)}{\partial t} = -\frac{i}{\hbar}H(t)\vec{\Psi}(t) = -\frac{i}{\hbar}H(t)\left(\sum_{n=0}^{\infty}a_n(t)\vec{\Psi}_n(t)\right) = -\frac{i}{\hbar}\sum_{n=0}^{\infty}a_n(t)\varepsilon_n(t)\vec{\Psi}_n(t) \tag{59}$$

Differentiating equation (57), we have:

$$\frac{\partial\vec{\Psi}(t)}{\partial t} = \sum_{n=0}^{\infty}\left(\dot{a}_n(t)\vec{\Psi}_n(t) + a_n(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t}\right) \tag{60}$$

Combining equations (59) and (60) yields:

$$\sum_{n=0}^{\infty}\left(\dot{a}_n(t)\vec{\Psi}_n(t) + a_n(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t}\right) = -\frac{i}{\hbar}\sum_{n=0}^{\infty}a_n(t)\varepsilon_n(t)\vec{\Psi}_n(t) \tag{61}$$

We can multiply both sides of equation (61) by $\vec{\Psi}_j^{\dagger}(t)$ on the left and make use of the orthonormality of our eigenvectors to obtain, after a slight rearrangement:

$$\dot{a}_j(t) = -\frac{i}{\hbar}a_j(t)\varepsilon_j(t) - \sum_{n=0}^{\infty}a_n(t)\vec{\Psi}_j^{\dagger}(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t} \tag{62}$$

The form of equation (62) suggests that we consider the following set of new coefficients:

$$b_n(t) = a_n(t)e^{-i\varphi_n(t)} \tag{63}$$

where:

$$\varphi_n(t) \equiv -\frac{1}{\hbar}\int_{t_0}^{t} \varepsilon_n(t')dt' \tag{64}$$

Our expansion in equation (57) can thus be rewritten as:

$$\vec{\Psi}(t) = \sum_{n=0}^{\infty} b_n(t)e^{i\varphi_n(t)}\vec{\Psi}_n(t) \tag{65}$$

and equation (62) becomes:

$$\dot{b}_j(t)e^{i\varphi_j(t)} - \frac{i}{\hbar}\varepsilon_j(t)b_j(t)e^{i\varphi_j(t)} = -\frac{i}{\hbar}\varepsilon_j(t)b_j(t)e^{i\varphi_j(t)} - \sum_{n=0}^{\infty} b_n(t)e^{i\varphi_n(t)}\vec{\Psi}_j^{\dagger}(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t} \tag{66}$$

After multiplying through by $e^{-i\varphi_j(t)}$, we find that:

$$\dot{b}_j(t) = -\sum_{n=0}^{\infty} b_n(t)\vec{\Psi}_j^{\dagger}(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t}e^{i[\varphi_n(t)-\varphi_j(t)]} \tag{67}$$

Equation (67) provides us with a relation which we could directly use to perform adiabatic expansion. But to see more clearly how the adiabatic expansion series relates to the rate at which we change the Hamiltonian in time, we will express $\dot{b}_j(t)$ in a more transparent form. Taking the time derivative of both sides of the equation $H(t)\vec{\Psi}_n(t) = \varepsilon_n(t)\vec{\Psi}_n(t)$, we obatin the relation:

$$\dot{H}(t)\vec{\Psi}_n(t) + H(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t} = \dot{\varepsilon}_n(t)\vec{\Psi}_n(t) + \varepsilon_n(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t} \tag{68}$$

Multiplying each side of this equation by $\vec{\Psi}_j^{\dagger}(t)$ on the left and using the fact that $H(t)$ is Hermitian, our relation becomes:

$$\vec{\Psi}_j^{\dagger}(t)\dot{H}(t)\vec{\Psi}_n(t) + \Big(H(t)\vec{\Psi}_j(t)\Big)^{\dagger}\frac{\partial\vec{\Psi}_n(t)}{\partial t} = \dot{\varepsilon}_n(t)\vec{\Psi}_j^{\dagger}(t)\vec{\Psi}_n(t) + \varepsilon_n(t)\vec{\Psi}_j^{\dagger}(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t} \tag{69}$$

Rearranging terms:

$$[\varepsilon_n(t) - \varepsilon_j(t)]\, \vec{\Psi}_j^\dagger(t) \frac{\partial \vec{\Psi}_n(t)}{\partial t} = \vec{\Psi}_j^\dagger(t) \dot{H}(t) \vec{\Psi}_n(t) - \dot{\varepsilon}_n(t) \delta_{jn} \tag{70}$$

As long as $\varepsilon_n(t) - \varepsilon_j(t) \neq 0$, which automatically implies the $j \neq n$, we can express $\vec{\Psi}_j^\dagger(t) \frac{\partial \vec{\Psi}_n(t)}{\partial t}$ in terms of the matrix element of $\dot{H}(t)$ and the energy difference:

$$\vec{\Psi}_j^\dagger(t) \frac{\partial \vec{\Psi}_n(t)}{\partial t} = \frac{\vec{\Psi}_j^\dagger(t) \dot{H}(t) \vec{\Psi}_n(t)}{\varepsilon_n(t) - \varepsilon_j(t)} \tag{71}$$

We can therefore write equation (67) as:

$$\dot{b}_j(t) = -\left(\sum_{n \in S_D(t)} b_n(t) \vec{\Psi}_j^\dagger(t) \frac{\partial \vec{\Psi}_n(t)}{\partial t} e^{i[\varphi_n(t) - \varphi_j(t)]} \right) - \left(\sum_{n \in S_{ND}(t)} b_n(t) \frac{\vec{\Psi}_j^\dagger(t) \dot{H}(t) \vec{\Psi}_n(t)}{\varepsilon_n(t) - \varepsilon_j(t)} e^{i[\varphi_n(t) - \varphi_j(t)]} \right) \tag{72}$$

where $S_D(t)$ is the set of all n such that $n \neq j$ and $\varepsilon_n(t) = \varepsilon_j(t)$ (since $\vec{\Psi}_j^\dagger(t) \frac{\partial \vec{\Psi}_j(t)}{\partial t} = 0$ for our situation, as verified empirically), and $S_{ND}(t)$ is the set of all n such that $\varepsilon_n(t) \neq \varepsilon_j(t)$.

Equation (72) illuminates the rationale behind the adiabatic approximation. Under the adiabatic approximation, we assume that $H(t)$ and hence also its eigenvectors vary slowly enough in time so that the righthand side of (72) can be approximated as 0. Then all the coefficients $b_j(t)$ are constant in time, meaning that the coefficients $a_j(t)$ are given by:

$$a_j(t) = a_j(t_0) e^{i\varphi_j(t)} \tag{73}$$

as is apparent from equation (63). Although the adiabatic approximation can get us quite far in our phase shift calculation, in some cases it will be necessary to consider higher order corrections. To do this, we employ the method of adiabatic expansion, which mathematically follows in the spirit of the Born approximation. To zeroth order, we take the coefficients $b_j(t)$ to be constant as dictated by the adiabatic approximation. To obtain the first order corrections to these coefficients, we substitute the zeroth order coefficients $b_j^{(0)}(t)$ into the

righthand side of equation (72) and integrate to find the first order coefficients $b_j^{(1)}(t)$. We can repeat this process recursively, substituting the coefficients $b_j^{(1)}(t)$ into equation (72) to calculate the coefficients $b_j^{(2)}(t)$ and continuing until we know our coefficients to the necessary precision. This method provides a way for us to construct a series expansion for each $b_j(t)$. We will adopt a matrix notation for the terms in this series to facilitate our discussion. The initial value of $b_j(t)$ will be represented by C_j. The first order correction to $b_j(t)$ will consist of contributions from each nonzero $b_n^{(0)}(t)$ where $n \neq j$, and we will depict the contribution from $b_n^{(0)}(t)$ as $C_{n\to j}$. So to first order, we can write:

$$b_j^{(1)}(t) = C_j + \sum_{n\neq j} C_{n\to j} \tag{74}$$

where:

$$C_{n\to j} \equiv -\int_{t_0}^{t} b_n^{(0)}(t')\vec{\Psi}_j^{\dagger}(t')\frac{\partial\vec{\Psi}_n(t')}{\partial t'}e^{i[\varphi_n(t')-\varphi_j(t')]}dt' \tag{75}$$

The second order to corrections to $b_j(t)$ will arise from the first order corrections to each $b_n(t)$ for $n \neq j$. Namely, the contribution to $b_j^{(2)}(t)$ from $b_n^{(1)}(t)$ will be:

$$-\int_{t_0}^{t} b_n^{(1)}(t')\vec{\Psi}_j^{\dagger}(t')\frac{\partial\vec{\Psi}_n(t')}{\partial t'}e^{i[\varphi_n(t')-\varphi_j(t')]}dt' = -\int_{t_0}^{t}\left(b_n^{(0)}(t') + \sum_{m\neq n} C_{m\to n}\right)\vec{\Psi}_j^{\dagger}(t')\frac{\partial\vec{\Psi}_n(t')}{\partial t'}e^{i[\varphi_n(t')-\varphi_j(t')]}dt' \tag{76}$$

$$= C_{n\to j} + \sum_{m\neq n} C_{m\to n\to j} \tag{77}$$

where:

$$C_{m\to n\to j} \equiv -\int_{t_0}^{t} C_{m\to n}\vec{\Psi}_j^{\dagger}(t')\frac{\partial\vec{\Psi}_n(t')}{\partial t'}e^{i[\varphi_n(t')-\varphi_j(t')]}dt' \tag{78}$$

under the implicit assumption that the time variable associated with $C_{m\to n}$ is appropriate

for the context in which it appears (so the time variable will be t' when $C_{m\to n}$ is inside an integral and t when it is not). Thus:

$$b_j^{(2)}(t) = C_j + \sum_{n\neq j} C_{n\to j} + \sum_{n\neq j}\sum_{m\neq n} C_{m\to n\to j} \tag{79}$$

In order to calculate corrections at arbitrary order, we define $C_{a\to b\to\cdots\to m\to n\to j}$ recursively in the natural way:

$$C_{a\to b\to\cdots\to m\to n\to j} \equiv -\int_{t_0}^{t} C_{a\to b\to\cdots\to m\to n}\vec{\Psi}_j^{\dagger}(t')\frac{\partial\vec{\Psi}_n(t')}{\partial t'}e^{i[\varphi_n(t')-\varphi_j(t')]}dt' \tag{80}$$

For arbitrary k, we can then write:

$$b_j^{(k)}(t) = C_j + \sum_{n\neq j} C_{n\to j} + \sum_{n\neq j}\sum_{m\neq n} C_{m\to n\to j} + \cdots + \sum_{n\neq j}\sum_{m\neq n}\cdots\sum_{a\neq b} C_{a\to b\to\cdots\to m\to n\to j} \tag{81}$$

where the last term includes k sums. In order for our expansion to be practical from a calculational standpoint, we want the series to converge quickly enough so that we do not have to find an unreasonable amount of terms. There are two types of terms that we would like to be able to neglect. First, where we assume that we start out in the ground state, we do not want to have to consider too many terms of the form $C_{0\to n}$ so that we only need to examine a relatively small number of eigenvectors. Terms of this form indeed decrease rapidly as n increases because the overlap of the nth eigenstate with the 0th eigenstate is essentially nonexistent for large enough n. Second, we want to be able to neglect all terms of order greater than some cutoff value. We will be able to do so as long as the time scale over which we are solving the problem is small enough so that integrating a kth order correction term against a factor of the form $\vec{\Psi}_j^{\dagger}(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t}$ yields a (k+1)th order correction term that is much smaller than the correction term of order k.

Note an important nuance in how we have phrased the above condition–we have men-

tioned nothing about a Hamiltonian that varies slowly in time. As long as our Hamiltonian does not vary at an infinitely fast rate, we can always work on a small enough time scale so that our expansion converges rapidly. To solve the problem on time scales for which the series does not converge quickly, we can simply break the problem into multiple parts. For example, say that we want to describe the evolution of our system from time 0 to time T, where T is too large for our series to converge after a finite number of terms. From a physical standpoint, the problem is that our zeroth order guess for the wavefunction (in which the coefficients $b_n(t)$ retain their initial values throughout the process) is not close enough to the actual solution. Conceptually, the way to remedy this problem is to make a better initial guess for the wavefunction. We achieve this mathematically by dividing the problem into steps. We choose a time Δt over which our series does converge rapidly and solve for the evolution of the wavefunction from time 0 to time Δt. We can then use the wavefunction at Δt as our initial guess for the wavefunction in the region from Δt to $2\Delta t$, and we can repeat this process until we reach time T. This method provides with us with a means to describe our system for a Hamiltonian that changes arbitrarily fast in time. Having a slowly varying Hamiltonian just serves to allow us to solve the problem without dividing it into as many parts (much of the time we will not have to divide the problem at all, and we will later derive conditions for when this will be the case), thus making the calculation significantly easier. Note that for large enough n (depending on the depth of the lattice), at each Bragg transition the population in the nth eigenvector will tunnel into the $(n+1)$th eigenvector. This behavior can be seen by examining our formula for population evolution on a sufficiently small time scale as detailed above, but it is perhaps more intuitive to explain this tunneling as occuring when the energy of a particular eigenvector is too large for that eigenvector to be a bound state for the lattice. Since such tunneling only happens for higher eigenvectors that are well-separated from the ground eigenvector in momentum space (given typical experimental parameters with a sufficiently deep lattice), in most cases it will negligibly effect the ground state phase evolution. However, we note that if needed, we can

fully characterize this tunneling using our adiabatic expansion prescription on a proper time scale.

To find the eigenvectors and eigenvalues that we need to calculate the terms that make a non-neglegible contribution to our expansion, we can approximate our infinite dimensional Hamiltonian matrix as a finite dimensional truncated matrix. At the end of section III., we concluded that the problem of determining our eigenvalues and eigenvectors for all α reduces to finding the eigenvalues and eigenvectors for a particular range of values of α of width 2. Examining our matrix for smaller values of α will make our calculation less cumbersome, so we can look at the range $\alpha \in [-1, 1]$ or alternatively the range $\alpha \in [0, 2]$. For α in either of these ranges, the eigenvectors with lower energies will be populated almost entirely by momentum eigenstates $|2m\hbar k\rangle$ for relatively small $|m|$. This will be the case because for α near 0, the diagonal elements $E_m(\alpha)$ of our Hamiltonian will be smallest for values of m close to 0. In the limit of $\tilde{\Omega} \to 0$, each eigenvector will be populated by only a single momentum eigenstate, where in general eigenvectors corresponding to momentum eigenstates with m closer to 0 will have lower eigenvalues. Increasing $\tilde{\Omega}$ will allow our lower eigenvectors to spread out in momentum space to a certain extent, but this will not change the fact that the lower eigenvectors will be linear combinations of momentum eigenstates corresponding to smaller values of $|m|$. As discussed previously, the eigenvectors we care about for calculational purposes will be those with eigenvalues closer to the ground state eigenvalue. We can thus consider a truncated $(2n+1)\times(2n+1)$ Hamiltonian matrix centered around $n = 0$, where we choose n to be large enough so that for the particular dynamics being described, there is negligible population in momentum eigenstates $|2m\hbar k\rangle$ for $|m| > n$.

We will now rigorously prove that this truncated matrix will provide us with all the information we need to carry out our calculation. In fact, we will derive periodicity conditions that make the process of extrapolation to values of α outside the initial range that we consider much simpler than we might have at first thought. Since knowing our eigenvalues and eigenvectors within a range of α of width 2 provides complete information about our

eigenvalues and eigenvectors as functions of α and $\tilde{\Omega}$ for all α, we can generalize our results for a particular range of α to determine $\vec{\Psi}_j^\dagger(t)\frac{\partial\vec{\Psi}_n(t)}{\partial t}e^{i[\varphi_n(t)-\varphi_j(t)]}$ for arbitrary α. To see why this is true, note that $\vec{\Psi}_j^\dagger(t)$ is just the Hermitian conjugate of one of our eigenvectors, which we already know we can find. And the phase factor $e^{i[\varphi_n(t)-\varphi_j(t)]}$ is a function of the eigenvalues, which we also know. Since we have a means to express all of our eigenvectors as functions of α and $\tilde{\Omega}$, which in turn depend on t, we have sufficient information to calculate the time derivative $\frac{\partial\vec{\Psi}_n(t)}{\partial t}$. In fact, we will prove that for given $\dot{\alpha}(t)$ and $\dot{\tilde{\Omega}}(t)$, the following symmetry exists:

$$\vec{\Psi}_j^\dagger(\alpha+2m,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})}{\partial t}=\vec{\Psi}_j^\dagger(\alpha,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t} \tag{82}$$

for any integer m. To show that this is true, we first note that where $\left(\vec{\Psi}_j(\alpha,\tilde{\Omega})\right)_k$ is the kth element of $\vec{\Psi}_j(\alpha,\tilde{\Omega})$ and $\left(\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}\right)_k$ is the kth element of $\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}$, the inner product $\vec{\Psi}_j^\dagger(\alpha,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}$ is given by (where we make use of the fact that our Hamiltonian is real-valued and hermitian to choose our eigenvectors to be real so that the hermitian conjugate of an eigenvector is identical to the eigenvector's transpose):

$$\vec{\Psi}_j^\dagger(\alpha,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}=\sum_{k=-\infty}^{\infty}\left(\vec{\Psi}_j(\alpha,\tilde{\Omega})\right)_k\left(\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}\right)_k \tag{83}$$

Analogously, we have that:

$$\vec{\Psi}_j^\dagger(\alpha+2m,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})}{\partial t}=\sum_{k=-\infty}^{\infty}\left(\vec{\Psi}_j(\alpha+2m,\tilde{\Omega})\right)_k\left(\frac{\partial\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})}{\partial t}\right)_k \tag{84}$$

It follows from our analysis in section III. that $\left(\vec{\Psi}_j(\alpha+2m,\tilde{\Omega})\right)_k=\left(\vec{\Psi}_j(\alpha,\tilde{\Omega})\right)_{(k-m)}$. This equation is tantamount to saying that $\vec{\Psi}_j(\alpha+2m,\tilde{\Omega})$ can be obtained by performing a right shift on $\vec{\Psi}_j(\alpha,\tilde{\Omega})$ m times. We now consider the term:

$$\left(\frac{\partial\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})}{\partial t}\right)_k = \frac{\partial\left(\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})\right)_k}{\partial t} = \frac{\partial\left(\vec{\Psi}_n(\alpha,\tilde{\Omega})\right)_{(k-m)}}{\partial t} = \left(\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}\right)_{(k-m)} \tag{85}$$

Plugging into equation (84), we see that:

$$\vec{\Psi}_j^\dagger(\alpha+2m,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})}{\partial t} = \sum_{k=-\infty}^{\infty}\left(\vec{\Psi}_j(\alpha,\tilde{\Omega})\right)_{(k-m)}\left(\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}\right)_{(k-m)} \tag{86}$$

Since our sum over k runs from $-\infty$ to ∞, the fact that each term is labeled by the index $k-m$ rather than k will not matter, since all possible values of this label will be summed over either way. Thus:

$$\vec{\Psi}_j^\dagger(\alpha+2m,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha+2m,\tilde{\Omega})}{\partial t} = \sum_{k=-\infty}^{\infty}\left(\vec{\Psi}_j(\alpha,\tilde{\Omega})\right)_k\left(\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}\right)_k = \vec{\Psi}_j^\dagger(\alpha,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t} \tag{87}$$

as claimed. So we have shown that $\vec{\Psi}_j^\dagger(\alpha,\tilde{\Omega})\frac{\partial\vec{\Psi}_n(\alpha,\tilde{\Omega})}{\partial t}$ is periodic in α with period 2. In light of equation (71), $\frac{\vec{\Psi}_j^\dagger(t)\dot{H}(t)\vec{\Psi}_n(t)}{\varepsilon_n(t)-\varepsilon_j(t)}$ must display this same periodicity for $\varepsilon_n(t) \neq \varepsilon_j(t)$. The denominator $\varepsilon_n(t)-\varepsilon_j(t)$ is certainly periodic in this manner, since as we can see from equation (53), each eigenvalue is periodic in α with period 2 except for the common term $-\frac{\hbar\omega_{bragg}}{4}\alpha^2$. Therefore, since we know that $\frac{\vec{\Psi}_j^\dagger(t)\dot{H}(t)\vec{\Psi}_n(t)}{\varepsilon_n(t)-\varepsilon_j(t)}$ is periodic in α with period 2, the numerator $\vec{\Psi}_j^\dagger(t)\dot{H}(t)\vec{\Psi}_n(t)$ must also be periodic in α with period 2. So we have indeed demonstrated how our truncated matrix allows us to calculate all the quantities that we need to perform adiabatic expansion. Namely, the fact that energy differences and the relevant matrix elements and inner products are periodic in α allows us to easily determine the values of these quantities for all α after employing our truncated matrix to calculate them within a particular range of α of width 2.

In a number of calculations, it will be useful for us to break $\dot{H}(t)$ into the sum of a matrix depending only on $\dot{\alpha}$ and a matrix depending only on $\dot{\tilde{\Omega}}$. Recalling how we wrote our Hamiltonian matrix as $H(t) = H_\alpha(t) + H_{\tilde{\Omega}}(t)$ in equation (33), where we note that here we write all matrices as functions of t since we are considering a situation where α and $\tilde{\Omega}$ are specific functions of t, we express $\dot{H}(t)$ as:

$$\dot{H}(t) = \dot{H}_\alpha(t) + \dot{H}_{\tilde{\Omega}}(t) \tag{88}$$

From the expressions for $H_\alpha(t)$ and $H_{\tilde{\Omega}}(t)$ in equations (34) and (35), we find that:

$$\dot{H}_\alpha(t) = \hbar\omega_{bragg} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & 0 & 0 & 2\dot{\alpha}(t) & 0 & 0 & & & & \\ & & 0 & 0 & \dot{\alpha}(t) & 0 & 0 & & & \\ & & & 0 & 0 & 0 & 0 & 0 & & \\ & & & & 0 & 0 & -\dot{\alpha}(t) & 0 & 0 & \\ & & & & & 0 & 0 & -2\dot{\alpha}(t) & 0 & 0 \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{89}$$

and

$$\dot{H}_{\tilde{\Omega}}(t) = \hbar\omega_{bragg} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & 0 & -\dot{\tilde{\Omega}}(t) & 0 & -\dot{\tilde{\Omega}}(t) & 0 & & & & \\ & & 0 & -\dot{\tilde{\Omega}}(t) & 0 & -\dot{\tilde{\Omega}}(t) & 0 & & & \\ & & & 0 & -\dot{\tilde{\Omega}}(t) & 0 & -\dot{\tilde{\Omega}}(t) & 0 & & \\ & & & & 0 & -\dot{\tilde{\Omega}}(t) & 0 & -\dot{\tilde{\Omega}}(t) & 0 & \\ & & & & & 0 & -\dot{\tilde{\Omega}}(t) & 0 & -\dot{\tilde{\Omega}}(t) & 0 \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{90}$$

We have already shown that $\vec{\Psi}_j^\dagger(t)\dot{H}(t)\vec{\Psi}_n(t)$ is periodic in α with period 2. Our proof was entirely general–it did not depend on the particular choice of $\alpha(t)$ and $\tilde{\Omega}(t)$. It follows that $\vec{\Psi}_j^\dagger(t)\dot{H}_\alpha(t)\vec{\Psi}_n(t)$ and $\vec{\Psi}_j^\dagger(t)\dot{H}_{\tilde{\Omega}}(t)\vec{\Psi}_n(t)$ must independently display such periodicity in α.

Now that we have equipped ourselves with the machinery of adiabatic expansion, we can apply it to perform experimentally relevant calculations.

V. An Example of Adiabatic Expansion: Performing an Experimentally Relevant Phase Shift Calculation

When using optical lattices as waveguides or beam splitters, many cases will arise when the two arms of the interferometer will interact with lattices of the same depth but different accelerations. Such a case is illustrated in Figure 3. The arm in which the acceleration is greater corresponds to a Hamiltonian that varies more rapidly in time, and thus we expect

the non-adiabatic correction for this arm to be greater. To calculate the phase shift for applications in precision measurement, we will need to determine the difference in the non-adiabatic corrections for the two arms.

For this example, we consider a situation where the interaction of the atoms with the lattice is divided into three distinct parts: the ramp up of the lattice, the acceleration of the lattice, and the ramp down of the lattice. For the sake of symmetry, we specify our ramps so that that decreasing ramp for the lattice depth is the reverse of the increasing ramp.

We will make our ramps adiabatic enough so that almost all of the population remains in the ground state, and we will determine the non-adiabatic correction to the phase of the ground state. Since to lowest order only the ground state is populated, the leading corrections to $b_0(t)$ will come at second order. Using the formalism in equation (81), this second order correction will be:

$$\delta b_0^{(2)}(t) = \sum_{n \neq 0} C_{0 \to n \to 0} \tag{91}$$

The largest contribution will come from $C_{0\to1\to0}$, and it is this term that we will focus on for the moment. First, since $b_0^{(0)}(t) = 1$, from equation (75) we see that:

$$b_1^{(1)}(t) = C_{0\to1} = -\int_{t_0}^{t} \frac{\vec{\Psi}_1^\dagger(t')\dot{H}(t')\vec{\Psi}_0(t')}{\varepsilon_0(t') - \varepsilon_1(t')} e^{i[\varphi_0(t') - \varphi_1(t')]} dt' \tag{92}$$

From equation (78):

$$C_{0\to1\to0} = -\int_{t_0}^{t} b_1^{(1)}(t') \frac{\vec{\Psi}_0^\dagger(t')\dot{H}(t')\vec{\Psi}_1(t')}{\varepsilon_1(t') - \varepsilon_0(t')} e^{i[\varphi_1(t') - \varphi_0(t')]} dt' \tag{93}$$

We will now introduce some notation to simplify the above expressions. Since we choose our eigenvectors to be real, $\vec{\Psi}_1^\dagger(t')\dot{H}(t')\vec{\Psi}_0(t') = \vec{\Psi}_0^\dagger(t')\dot{H}(t')\vec{\Psi}_1(t')$, and we can write $M_{10}(t') \equiv \vec{\Psi}_1^\dagger(t')\dot{H}(t')\vec{\Psi}_0(t') = \vec{\Psi}_1^\dagger(t')\dot{H}_\alpha(t')\vec{\Psi}_0(t') + \vec{\Psi}_1^\dagger(t')\dot{H}_{\tilde{\Omega}}(t')\vec{\Psi}_0(t')$. Now, let us denote:

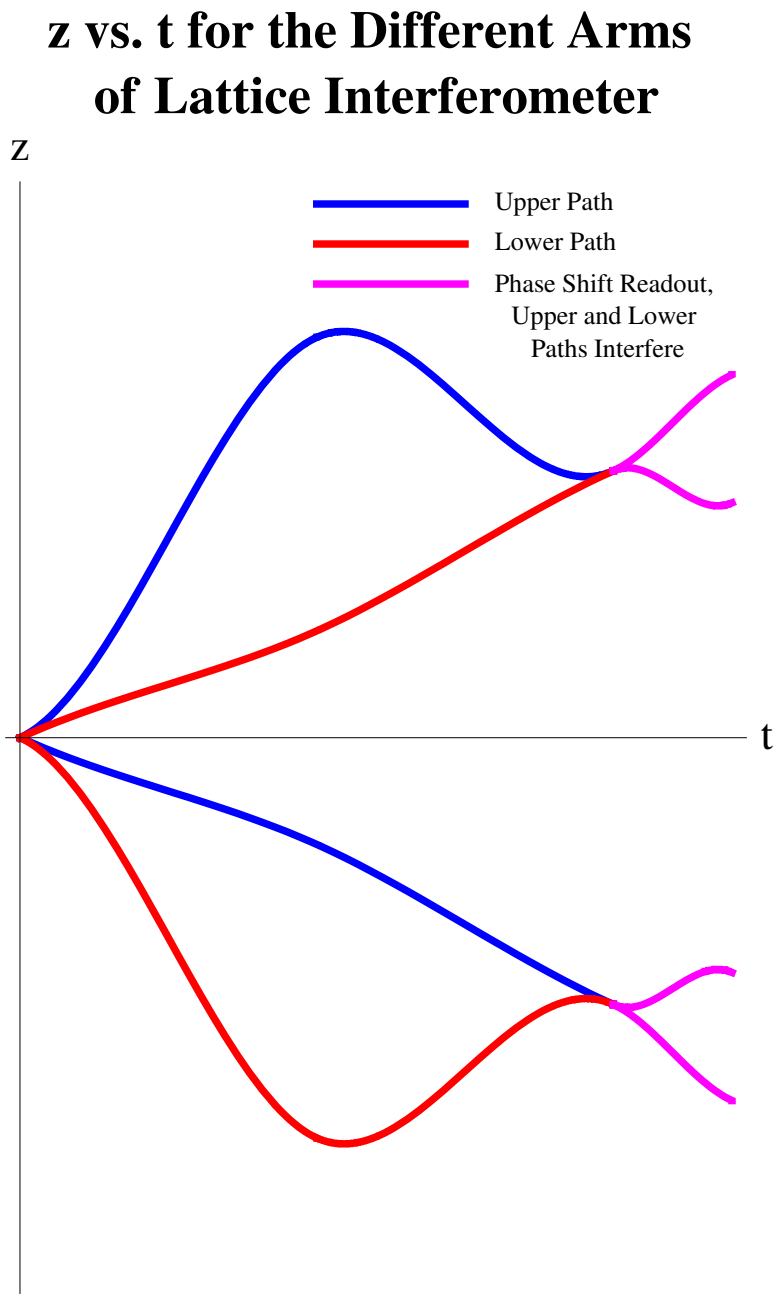

Figure 3: This figure shows a conjugate interferometer geometry that could be used to measure $\frac{\hbar}{m}$. The phase shift of the lower interferometer is subtracted from the phase shift of the upper interferometer, which suppresses the effects of laser phase noise and eliminates the gravitational phase shift up to gradients [2]. With such a scheme, a measurement of $\frac{\hbar}{m}$ to a part in 10^{15} may be possible, which could lead to the most accurate determination of the fine structure constant to date. This matter is discussed in greater detail in section VII.

$$M_{\alpha_{(10)}}(t') \equiv \frac{1}{\dot{\alpha}(t')}\vec{\Psi}_1^\dagger(t')\dot{H}_\alpha(t')\vec{\Psi}_0(t') \tag{94}$$

and:

$$M_{\tilde{\Omega}_{(10)}}(t') \equiv \frac{1}{\dot{\tilde{\Omega}}(t')}\vec{\Psi}_1^\dagger(t')\dot{H}_{\tilde{\Omega}}(t')\vec{\Psi}_0(t') \tag{95}$$

so that $M_{10}(t') = \dot{\alpha}(t')M_{\alpha_{(10)}}(t') + \dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{(10)}}(t')$. Where we denote $\Delta\varepsilon_{10}(t') \equiv \varepsilon_0(t') - \varepsilon_1(t')$ and let our initial time be $t_0 = 0$, we can write:

$$b_1^{(1)}(t) = -\int_0^t \frac{M_{10}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_0^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{96}$$

and:

$$C_{0\to1\to0} = -\int_0^t b_1^{(1)}(t')\frac{M_{10}(t')}{-\Delta\varepsilon_{10}(t')} e^{\frac{i}{\hbar}\int_0^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{97}$$

We can give specific expressions for $b_1^{(1)}(t)$ for each of the three portions of our interaction. Say that from $t = 0$ to $t = T_{ramp}$ we ramp up the lattice, from $t = T_{ramp}$ to $t = T_{ramp} + T_{accel}$ we accelerate the lattice, and from $t = T_{ramp} + T_{accel}$ to $t = 2T_{ramp} + T_{accel}$ we ramp down the lattice. During the ramp up stage, $\dot{\alpha}(t') = 0$ so that:

$$b_1^{(1)}(t) = -\int_0^t \frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{(10)}}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_0^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{98}$$

And during the acceleration stage, $\dot{\tilde{\Omega}}(t') = 0$, so:

$$b_1^{(1)}(t) = b_1^{(1)}(T_{ramp}) - \int_{T_{ramp}}^t \frac{\dot{\alpha}(t')M_{\alpha_{(10)}}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_0^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{99}$$

$$= b_1^{(1)}(T_{ramp}) - e^{-\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''}\int_{T_{ramp}}^t \frac{\dot{\alpha}(t')M_{\alpha_{(10)}}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{100}$$

where:

$$b_1^{(1)}(T_{ramp}) = -\int_0^{T_{ramp}} \frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{(10)}}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_0^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{101}$$

In the ramp down stage, the lattice acceleration is once again 0, and we obtain:

$$b_1^{(1)}(t) = b_1^{(1)}(T_{ramp}+T_{accel}) - e^{-\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''} \times \tag{102}$$

$$\times \int_{T_{ramp}+T_{accel}}^{t} \frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{(10)}}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_{T_{ramp}+T_{accel}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{103}$$

where:

$$b_1^{(1)}(T_{ramp}+T_{accel}) = b_1^{(1)}(T_{ramp}) - e^{-\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} \int_{T_{ramp}}^{T_{ramp}+T_{accel}} \frac{\dot{\alpha}(t')M_{\alpha_{(10)}}(t')}{\Delta\varepsilon_{10}(t')} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{104}$$

Now we can examine the ultimate contribution of $C_{0\to1\to0}$ to $\delta b_0^{(2)}(2T_{ramp}+T_{accel})$. Breaking up the integral in equation (97) in a manner corresponding to the three stages of our interaction, we have:

$$C_{0\to1\to0}(2T_{ramp}+T_{accel}) = -\int_0^{T_{ramp}} b_1^{(1)}(t')\frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{10}}(t')}{-\Delta\varepsilon_{10}(t')} e^{\frac{i}{\hbar}\int_0^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{105}$$

$$-e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} \int_{T_{ramp}}^{T_{ramp}+T_{accel}} b_1^{(1)}(t')\frac{\dot{\alpha}(t')M_{\alpha_{(10)}}(t')}{-\Delta\varepsilon_{10}(t')} e^{\frac{i}{\hbar}\int_{T_{ramp}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{106}$$

$$-e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''}e^{\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''}\int_{T_{ramp}+T_{accel}}^{2T_{ramp}+T_{accel}}b_1^{(1)}(t')\frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{10}}(t')}{-\Delta\varepsilon_{10}(t')}e^{\frac{i}{\hbar}\int_{T_{ramp}+T_{accel}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{107}$$

Since during the ramp up stage, $b_1^{(1)}(t)$ depends only on $\dot{\tilde{\Omega}}(t)$ but not on $\dot{\alpha}(t)$, the contribution to $C_{0\to1\to0}(2T_{ramp}+T_{accel})$ from the first term in equation (105) is common to both arms of the interferometer, because we assume that the lattice interaction processes for the two arms differ only in the magnitude of the lattice acceleration. Similarly, the portion of the third term in equation (105) containing the factor of:

$$\int_{T_{ramp}+T_{accel}}^{2T_{ramp}+T_{accel}}\left(b_1^{(1)}(t')-\left[b_1^{(1)}(T_{ramp}+T_{accel})-b_1^{(1)}(T_{ramp})\right]\right)\frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{10}}(t')}{-\Delta\varepsilon_{10}(t')}e^{\frac{i}{\hbar}\int_{T_{ramp}+T_{accel}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{108}$$

will be common to both arms. We will collectively denote these common terms as δ_{common}. The remaining terms will depend on the lattice acceleration and will thus differ between the arms. In order to more clearly illuminate the general points we are illustrating with this example, we will make the simplifying assumption that $\dot{\alpha}(t)$ is constant throughout the acceleration stage. Moreover, we assume that our lattice is deep enough so that $M_{\alpha_{(10)}}(t')$ and $\Delta\varepsilon_{10}(t'')$ are also constant during the acceleration stage, which will be an extremely accurate approximation for typical experimentally achievable situations. Note that these assumptions are certainly not necessary to carry out the calculation. They only serve to make the final result take a particularly simple form that provides physical insight into our process. In the absence of these assumptions, the calculation will be only slightly more complicated and can easily be performed. In addition, we note that the treatment given in this example can readily be generalized to the case where the lattice depth and velocity are changed simultaneously. During the acceleration stage, we will denote $\dot{\alpha}(t)$ by the constant $\dot{\alpha}$, $M_{\alpha_{(10)}}(t')$ by the constant $M_{\alpha_{(10)}}$, and $\Delta\varepsilon_{10}(t'')$ by the constant $\Delta\varepsilon_{10_{accel}}$. In Figure 4, we plot the perturbative

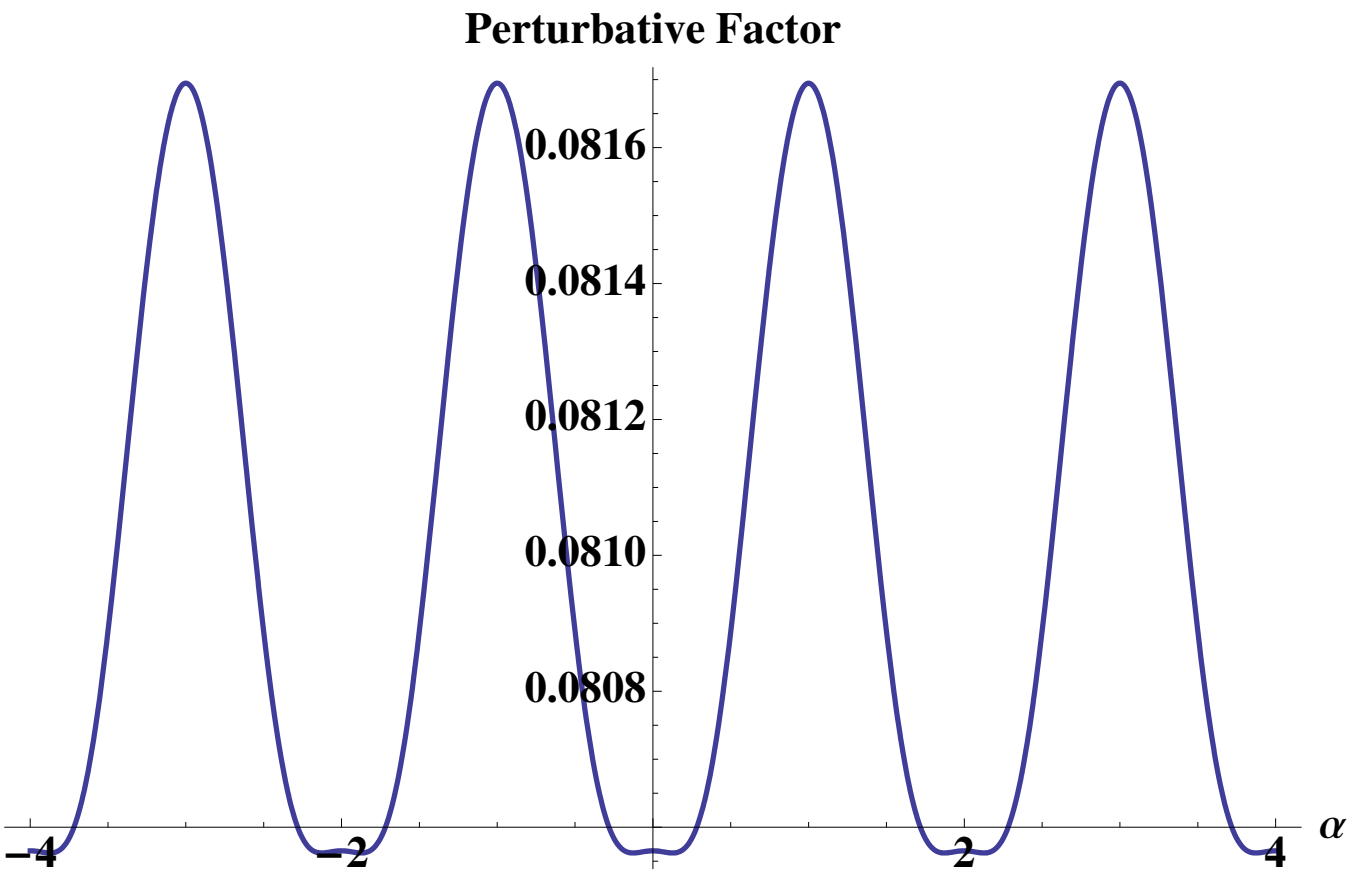

Figure 4: In this figure, we show the perturbative factor $\dot{\alpha}M_{\alpha_{(10)}}$ as a function of α in our dimensionless units (where we set $\hbar = 1$ and $\omega_{bragg} = 1$) for $\tilde{\Omega} = 1$ and $\dot{\alpha} = 0.125$. We calculated this perturbative factor using the truncated matrix method, and we see that this factor is periodic in α with period 2 as expected. Moreover, we see that $\tilde{\Omega}$ is large enough so that the variations in the perturbative factor with α are small compared to the size of the perturbative factor, which is also in accordance with our expectations.

factor $\dot{\alpha}M_{\alpha_{(10)}}$ as a function of α for typical experimental conditions, verifying that it is indeed nearly constant in α. Thus, where we define $b^{(1)}_{1_{accel}}(t) \equiv b^{(1)}_1(t) - b^{(1)}_1(T_{ramp})$ for t between T_{ramp} and $T_{ramp} + T_{accel}$, we can use equation (100) to write:

$$b^{(1)}_{1_{accel}}(t) = -e^{-\frac{i}{\hbar}\int_0^{T_{ramp}} \Delta\varepsilon_{10}(t')dt'} \int_{T_{ramp}}^{t} \frac{\dot{\alpha}M_{\alpha_{(10)}}}{\Delta\varepsilon_{10_{accel}}} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{t'} \Delta\varepsilon_{10_{accel}} dt''} dt' \tag{109}$$

$$= -\frac{\dot{\alpha}M_{\alpha_{(10)}}}{\Delta\varepsilon_{10_{accel}}} e^{-\frac{i}{\hbar}\int_0^{T_{ramp}} \Delta\varepsilon_{10}(t')dt'} \int_{T_{ramp}}^{t} e^{-\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}(t'-T_{ramp})} dt' \tag{110}$$

Evaluating the integral of the exponential, we obtain:

$$b^{(1)}_{1_{accel}}(t) = i\hbar \frac{\dot{\alpha} M_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^2} e^{-\frac{i}{\hbar}\int_0^{T_{ramp}} \Delta\varepsilon_{10}(t')dt'} \left[1 - e^{-\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}(t-T_{ramp})}\right] \tag{111}$$

We will thus have the following contribution to $C_{0\to1\to0}(2T_{ramp} + T_{accel})$ proportional to $\dot{\alpha}^2$:

$$\delta_{\dot{\alpha}^2} \equiv -e^{\frac{i}{\hbar}\int_0^{T_{ramp}} \Delta\varepsilon_{10}(t')dt'} \int_{T_{ramp}}^{T_{ramp}+T_{accel}} b^{(1)}_{1_{accel}}(t) \frac{\dot{\alpha} M_{\alpha_{(10)}}}{-\Delta\varepsilon_{10_{accel}}} e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}(t-T_{ramp})} dt \tag{112}$$

$$= i\hbar \frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^3} \int_{T_{ramp}}^{T_{ramp}+T_{accel}} \left[e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}(t-T_{ramp})} - 1\right] dt \tag{113}$$

$$= -i\hbar \frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^3} T_{accel} + \hbar^2 \frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^4} \left[e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}} T_{accel}} - 1\right] \tag{114}$$

The only other non-common term in $C_{0\to1\to0}(2T_{ramp}+T_{accel})$ is proportional to $\dot{\alpha}$, and we thus denote it as $\delta_{\dot{\alpha}}$. It is shown in Appendix B that under the assumption that the lattice depth decrease ramp is the time reversed lattice depth increase ramp, the various contributions to $\delta_{\dot{\alpha}}$ cancel so that:

$$\delta_{\dot{\alpha}} = 0 \tag{115}$$

Since $C_{0\to1\to0}$ provides the dominant contribution to $\delta b^{(2)}_0(t)$, we therefore see that:

$$\delta b^{(2)}_0(2T_{ramp} + T_{accel}) \approx \delta_{common} + \delta_{\dot{\alpha}^2} \tag{116}$$

We can extract from this equation a condition for when we can carry out adiabatic expansion without dividing the problem, as described in the previous section. For $\left|\delta b^{(2)}_0(2T_{ramp} + T_{accel})\right| \ll 1$, we can solve the problem by employing adiabatic expansion over a single region. Otherwise, we will need to divide our time range into multiple regions. To frame this condition in more concrete terms, we will adopt a convenient set of units in which all quantities are

dimensionless. These units are particularly useful when numerically solving our differential equations, where we must specify all quantities as actual numbers–and thus they are also helpful for easily allowing comparison between our analytical method and numerical results. We set $\hbar = 1$ and $\omega_{bragg} = 1$ so that all times will be in units of $\frac{1}{\omega_{bragg}}$, which is $\approx 10^{-5}$ s for Rubidium atoms. We can rewrite equation (114) as:

$$\delta_{\dot{\alpha}^2} = -i\frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^3}T_{accel} + \frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^4}\left[e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}T_{accel}} - 1\right] \tag{117}$$

with all quantities given in terms of our dimensionless units. For typical experimental parameters, the term proportional to T_{accel} in $\delta_{\dot{\alpha}^2}$ will dominate both the second term in $\delta_{\dot{\alpha}^2}$ and the δ_{common} term for acceleration times of $100\,\mu$s or greater, and so we can express the condition $\left|\delta b_0^{(2)}(2T_{ramp} + T_{accel})\right| \ll 1$ as:

$$\left|\frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^3}T_{accel}\right| \ll 1 \tag{118}$$

This condition will often hold, since in many experimentally relevant cases our acceleration time or acceleration will be small enough relative to the other parameters so that $\left|\frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^3}T_{accel}\right|$ is small. However, there may be times when we must divide the problem into multiple parts. Such a calculation will require a bit more work, but it can certainly be carried out in a systematic and relatively efficient manner. In contrast, when $\left|\frac{\dot{\alpha}^2 M^2_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^3}T_{accel}\right|$ becomes significantly greater than 1, our experience shows that numerical methods will often become unreliable and may require more computing power than is readily accessible.

We will now show how to determine the correction to the phase arising from the non-adiabatic correction $\delta b_0^{(2)}(2T_{ramp} + T_{accel})$. This procedure is essential, since in atom interferometry the principal quantity we are interested in is the phase shift between the two arms. As we recall from equation (65), the coefficient of the ground state eigenvector at time $T_{final} \equiv 2T_{ramp} + T_{accel}$ will be $b_0(T_{final})e^{i\varphi_0(T_{final})} = \left(b_0^{(0)}(T_{final}) + \delta b_0^{(2)}(T_{final})\right)e^{i\varphi_0(T_{final})}$, where $b_0^{(0)}(T_{final}) = 1$. Now, note that any complex number z can be written as $z = |z|e^{i\arg(z)}$.

We can express its complex conjugate as $z^* = |z|e^{-i\arg(z)}$. Thus, we see that $\arg(z)$ satisfies:

$$\arg(z) = \frac{1}{2i}\ln\left[\frac{z}{z^*}\right] \tag{119}$$

The phase $\phi + \delta\phi$ of the coefficient of the ground state eigenvector, where $\phi = \varphi_0(t)$ is the phase in the absence of the non-adiabatic correction and where $\Lambda \equiv |\delta_{common}|e^{i\arg(\delta_{common})} + |\delta_{\dot{\alpha}^2}|e^{i\arg(\delta_{\dot{\alpha}^2})}$, is thus:

$$\phi + \delta\phi = \frac{1}{2i}\ln\left[\frac{(1+\Lambda)\,e^{i\varphi_0(T_{final})}}{(1+\Lambda^*)\,e^{-i\varphi_0(T_{final})}}\right] \tag{120}$$

$$= \varphi_0(T_{final}) + \frac{1}{2i}\left[\ln(1+\Lambda) - \ln(1+\Lambda^*)\right] \tag{121}$$

Using the Taylor expansion $\ln(1+x) = x + O(x^2)$, we can write the correction to the phase as:

$$\delta\phi = \frac{1}{2i}\left[|\delta_{common}|e^{i\arg(\delta_{common})} - |\delta_{common}|e^{-i\arg(\delta_{common})}\right] + \frac{1}{2i}\left[|\delta_{\dot{\alpha}^2}|e^{i\arg(\delta_{\dot{\alpha}^2})} - |\delta_{\dot{\alpha}^2}|e^{-i\arg(\delta_{\dot{\alpha}^2})}\right] \tag{122}$$

$$= |\delta_{common}|\sin[\arg(\delta_{common})] + |\delta_{\dot{\alpha}^2}|\sin[\arg(\delta_{\dot{\alpha}^2})] \tag{123}$$

where we have omitted terms of second order or higher in small quantities. Note that the $|\delta_{common}|\sin[\arg(\delta_{common})]$ term will be common to both arms of the interferometer.

Equation (123) provides us with a means to calculate the leading correction to the phase shift, and comparison with numerical results for a variety of experimentally conceivable lattice depth ramps shows excellent agreement. A comparison of our adiabatic expansion method with numerical calculations is illustrated in Figure 5. Although our leading order results will often be sufficient, we note that it may sometimes be necessary to calculate higher

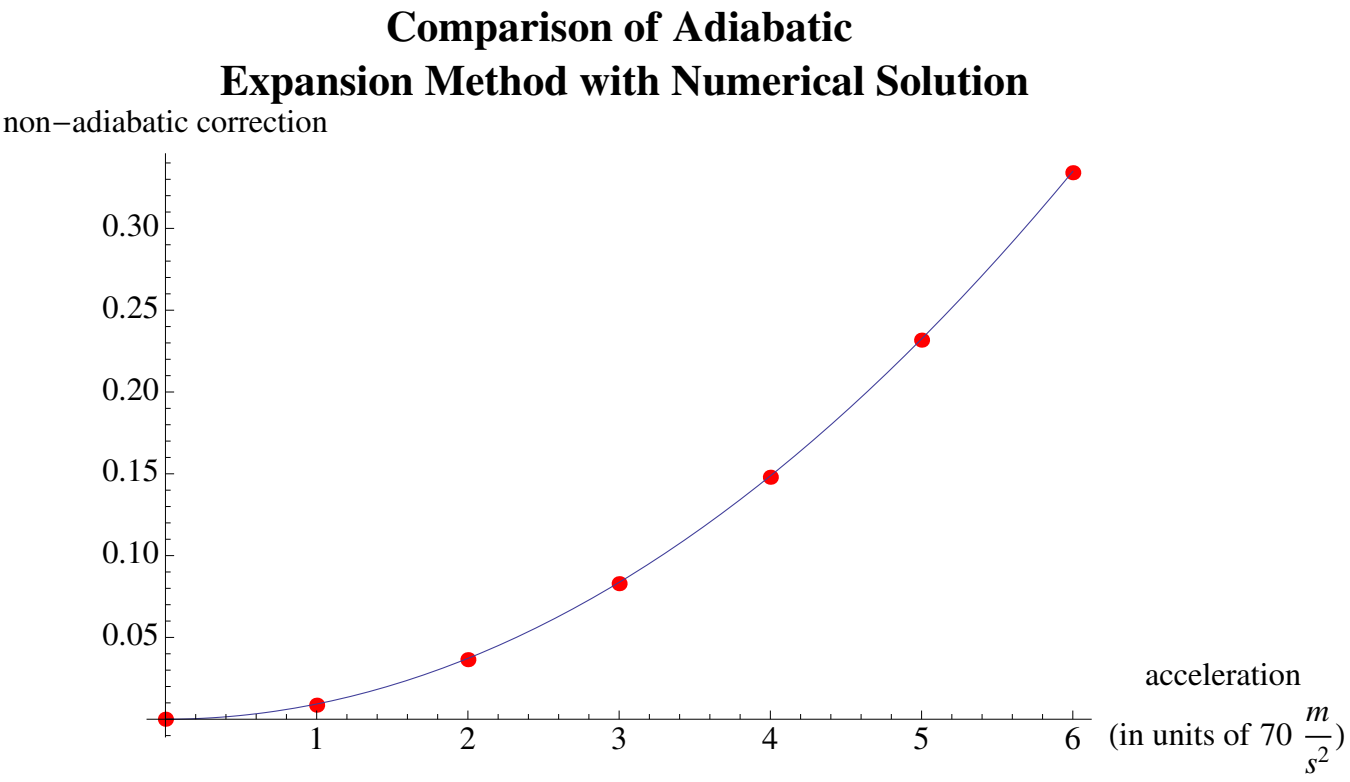

Figure 5: The above plot shows the non-adiabatic correction to the phase of an atom after acceleration by an experimentally plausible lattice beam splitter (with lattice depth ramps and acceleration time identical to those of the acceleration sequence shown in Figure 1) as calculated using a simplified adiabatic expansion method in which we keep only leading terms versus numerical simulations. We neglect corrections arising from the term δ_{common}, for these corrections are common to both arms of the interferometer to lowest order. The curve represents the prediction made by our adiabatic expansion method, while the red dots represent numerical results. Even the simple approximation used to obtain the curve agrees remarkably well with the simulations, and we note that we could easily improve this approximation by including more terms in the adiabatic expansion series. As expected, we observe that the correction scales quadratically with acceleration.

order corrections. To get an idea of what these corrections will look like, we will give an expression for the terms $C_{0\to1\to2\to1\to0}$ and $C_{0\to1\to0\to1\to0}$, which provide the dominant contribution to the fourth order correction (corrections of odd order will typically be very small since $M_{\alpha_{(jk)}}$ decreases significantly when j and k are not neighboring numbers). We will make the simplifying assumption that the matrix element $M_{\alpha_{(21)}}$ and the energy difference $\Delta\varepsilon_{21_{accel}}$ are constant to make our result more transparent, although, as before, this assumption is certainly not necessary to perform the calculation. We find that, in our dimensionless units:

$$C_{0\to1\to2\to1\to0} = \frac{\dot{\alpha}^4 M_{\alpha_{(10)}}^2 M_{\alpha_{(21)}}^2}{\left(\Delta\varepsilon_{10_{accel}}\right)^4 \left(\Delta\varepsilon_{21_{accel}}\right)^4 \left(\Delta\varepsilon_{10_{accel}} + \Delta\varepsilon_{21_{accel}}\right)} \times \tag{124}$$

$$\times \left[-i \left(\Delta\varepsilon_{21_{accel}}\right)^2 T_{accel} + \frac{\left(e^{i(\Delta\varepsilon_{10_{accel}} + \Delta\varepsilon_{21_{accel}})T_{accel}} - 1\right) \left(\Delta\varepsilon_{10_{accel}}\right)^2}{\varepsilon_{10_{accel}} + \Delta\varepsilon_{21_{accel}}} \right] + \tag{125}$$

$$+ \frac{\dot{\alpha}^4 M_{\alpha_{(10)}}^2 M_{\alpha_{(21)}}^2}{\left(\Delta\varepsilon_{10_{accel}}^5\right) \left(\Delta\varepsilon_{21_{accel}}\right)^4} \left[\Delta\varepsilon_{10_{accel}} - 2\Delta\varepsilon_{21_{accel}} + e^{i\Delta\varepsilon_{10_{accel}} T_{accel}} (2\Delta\varepsilon_{21_{accel}} - \Delta\varepsilon_{10_{accel}} (1 + i\Delta\varepsilon_{21_{accel}} T_{accel})) \right] \tag{126}$$

and:

$$C_{0\to1\to0\to1\to0} = \frac{3\dot{\alpha}^4 M_{\alpha_{(10)}}^4}{\left(\Delta\varepsilon_{10_{accel}}\right)^8} \left(1 - e^{i\Delta\varepsilon_{10_{accel}} T_{accel}}\right) + \frac{i\dot{\alpha}^4 M_{\alpha_{(10)}}^4 T_{accel}}{\left(\Delta\varepsilon_{10_{accel}}\right)^7} \left(2 + e^{i\Delta\varepsilon_{10_{accel}} T_{accel}}\right) \tag{127}$$

$$- \frac{\dot{\alpha}^4 M_{\alpha_{(10)}}^4 T_{accel}^2}{2\left(\Delta\varepsilon_{10_{accel}}\right)^6} \tag{128}$$

We have demonstrated how to carry out adiabatic expansion to perform experimentally useful calculations. We now have laid out a complete mathematical framework within which we can discuss the lattice-atom interaction, and we will turn to considerations of how to apply this knowledge to atom interferometry.

VI. Applications: LMT Beam Splitters

All sorts of atom interferometer geometries can benefit from the implementation of LMT beam splitter schemes, and optical lattices provide one of the most promising means of achieving these schemes. The cleanest type of lattice beam splitter would involve multiple initial two-photon Bragg pulses or a multi-photon Bragg pulse followed by a lattice acceleration of one of the momentum ranges or of both of the momentum ranges simultaneously [13, 14, 15]. After the initial pulse or pulses, the two arms of the interferometer will be separated in momentum space. We would ideally like this separation to be $6\hbar k$ or greater in order for the effects off-resonant lattice-atom interactions to be negligible, as we learned from numerical simulations.

The phase accumulated during the lattice acceleration stage can be calculated from the theoretical background that we have presented. Throughout our analysis, we have worked entirely in the dressed state frame, since this frame is particularly convenient for describing phase evolution in a lattice. For interferometers where we use lattices as waveguides for the atoms, we will want to perform the entire phase shift calculation in this frame. However, phase shift calculations for light-pulse atom interferometers are typically performed in the freely falling frame. Thus, for applications where lattice manipulations are used for beam splitters and mirrors in a light-pulse geometry, we want to be able to convert the phase evolution we calculate in the dressed state frame to the freely falling frame. We recall from section II. that $\hat{H}_{FF} = \hat{U}_{DS}(t)\hat{H}_{DS}\hat{U}_{DS}^{\dagger}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{DS}(t)\right)\hat{U}_{DS}^{\dagger}(t)$ where $\hat{U}_{DS} = e^{-\frac{i}{\hbar}(D_{Lab}(t)+\frac{1}{2}gt^2)\hat{p}}$. Where the state vector in the freely falling frame is $|\Psi_{FF}(t)\rangle$ and the state vector in the dressed state frame is $|\Psi_{DS}(t)\rangle$, the unitary transformation $\hat{U}_{DS}(t)$ relates the two state vectors as follows:

$$|\Psi_{FF}(t)\rangle = \hat{U}_{DS}(t)\,|\Psi_{DS}(t)\rangle \tag{129}$$

Now, say that before the lattice acceleration the momentum range that we are accelerating

is described by the state vector $|\Psi_{FF}(t_0)\rangle$ in the freely falling frame. We can then transform this state vector to the dressed state frame, describe its evolution to the final time t_f in the dressed state frame, and transform back to the freely falling frame. Where $\hat{T}_{DS}(t', t)$ is the time evolution operator the takes us from time t to time t' in the dressed state frame, we can write:

$$|\Psi_{FF}(t_f)\rangle = \hat{U}_{DS}(t_f)\hat{T}_{DS}(t_f, t_0)\hat{U}^{\dagger}_{DS}(t_0)\,|\Psi_{FF}(t_0)\rangle \tag{130}$$

We will work in momentum space in order to most readily describe evolution in the dressed state frame. In the freely falling frame, our initial momentum space wavefunction is $\vec{\Psi}_{FF}(p, t_0) = \langle p|\Psi_{FF}(t_0)\rangle$. In the dressed state frame, this wavefunction will transform to:

$$\vec{\Psi}_{DS}(p, t_0) = \langle p|\,\hat{U}^{\dagger}_{DS}(t_0)\,|\Psi_{FF}(t_0)\rangle = \langle p|\,e^{\frac{i}{\hbar}(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)\hat{p}}\,|\Psi_{FF}(t_0)\rangle \tag{131}$$

$$= e^{\frac{i}{\hbar}(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)p}\,\langle p|\Psi_{FF}(t_0)\rangle = e^{\frac{i}{\hbar}(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)p}\vec{\Psi}_{FF}(p, t_0) \tag{132}$$

Say that our momentum space wavefunction is peaked around some value p_0 at t_0. Where we let δp be the deviation from this central value, we have that:

$$\vec{\Psi}_{DS}(p_0 + \delta p, t_0) = e^{\frac{i}{\hbar}(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)(p_0+\delta p)}\vec{\Psi}_{FF}(p_0 + \delta p, t_0) \tag{133}$$

Now, say that our lattice acceleration increases the momentum of the atoms by $2n\hbar k$ and that, in the dressed state frame, a momentum eigenstate with initial momentum $p_0 + \delta p$ acquires a phase $\varphi(p_0 + \delta p)$. At time t_f, our momentum space wavefunction in the dressed state frame will be:

$$\vec{\Psi}_{DS}(p_0 + 2n\hbar k + \delta p, t_f) = e^{i\varphi(p_0+\delta p)}e^{\frac{i}{\hbar}(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)(p_0+\delta p)}\vec{\Psi}_{FF}(p_0 + \delta p, t_0) \tag{134}$$

We can now boost back into the freely falling frame:

$$\vec{\Psi}_{FF}(p_0 + 2n\hbar k + \delta p, t_f) = \langle p_0 + 2n\hbar k + \delta p | \hat{U}_{DS}(t_f) | \Psi_{DS}(t_f) \rangle \tag{135}$$

$$= \langle p_0 + 2n\hbar k + \delta p | e^{-\frac{i}{\hbar}(D_{Lab}(t_f) + \frac{1}{2}gt_f^2)\hat{p}} | \Psi_{DS}(t_f) \rangle \tag{136}$$

$$= e^{i\varphi(p_0+\delta p)} e^{-\frac{i}{\hbar}(D_{Lab}(t_f)+\frac{1}{2}gt_f^2)(p_0+2n\hbar k+\delta p)} e^{\frac{i}{\hbar}(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)(p_0+\delta p)} \vec{\Psi}_{FF}(p_0 + \delta p, t_0) \tag{137}$$

The phase accumulated during the lattice acceleration is thus:

$$\Delta\phi_{accel} = \varphi(p_0 + \delta p) - \frac{1}{\hbar}\left[(D_{Lab}(t_f) + \frac{1}{2}gt_f^2)(p_0 + 2n\hbar k + \delta p) - (D_{Lab}(t_0) + \frac{1}{2}gt_0^2)(p_0 + \delta p)\right] \tag{138}$$

We will now calculate the term $\varphi(p_0 + \delta p)$. Consider a boosted dressed state frame in which momentum $p_0 + \delta p$ in the unboosted dressed state frame corresponds to momentum 0. In essence, where $v_0 \equiv \frac{p_0}{m}$ and $\delta v \equiv \frac{\delta p}{m}$, our boosted dressed state frame travels with velocity $v_0 + \delta v$ with respect to the unboosted dressed state frame. The Hamiltonian in the boosted frame will be:

$$\hat{H}_{DS_{(p_0+\delta p)}} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt - v_0 - \delta v)\hat{p} + V_0 \sin^2(k\hat{x}) \tag{139}$$

We note that $\hat{H}_{DS} = \hat{U}_{(p_0+\delta p)}(t)\hat{H}_{DS_{(p_0+\delta p)}}\hat{U}^\dagger_{(p_0+\delta p)}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{(p_0+\delta p)}(t)\right)\hat{U}^\dagger_{(p_0+\delta p)}(t)$ for the unitary transformation:

$$\hat{U}_{(p_0+\delta p)}(t) = e^{\frac{i}{\hbar}(p_0+\delta p)\hat{x}} e^{-\frac{i}{\hbar}m\left[-(D_{Lab}(t)+\frac{1}{2}gt^2)(v_0+\delta v)+\frac{1}{2}(v_0+\delta v)^2 t\right]} \tag{140}$$

The boosted dressed state frame will allow us to use the results we derived earlier when we

discretized momentum space. Our discrete momentum basis consists of momentum eigenstates of the form $|2n\hbar k\rangle$ for integer n, so we need to work in a dressed state frame where our initial momentum is an integer multiple of $2\hbar k$ in order for this initial momentum to be in the span of our discrete basis. We make the convenient choice of a frame where we have boosted our initial momentum to 0. At time t_0, the momentum space wavefunctions in the boosted and unboosted dressed state frames are related by:

$$\vec{\Psi}_{DS_{(p_0+\delta p)}}(0,t_0) = \langle 0|\, \hat{U}^{\dagger}_{(p_0+\delta p)}(t_0)\, |\Psi_{DS}(t_0)\rangle \tag{141}$$

$$= \langle 0|\, e^{-\frac{i}{\hbar}(p_0+\delta p)\hat{x}} e^{\frac{i}{\hbar}m\left[-(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)(v_0+\delta v)+\frac{1}{2}(v_0+\delta v)^2 t_0\right]}\, |\Psi_{DS}(t_0)\rangle \tag{142}$$

$$= e^{\frac{i}{\hbar}m\left[-(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)(v_0+\delta v)+\frac{1}{2}(v_0+\delta v)^2 t_0\right]} \vec{\Psi}_{DS}(p_0+\delta p, t_0) \tag{143}$$

In section III., we learned that during the lattice acceleration, we will evolve a phase:

$$\Delta\phi_{(p_0+\delta p)} = \frac{m}{2\hbar}\int_{t_0}^{t_f} v_{Lattice}(t)^2 dt \tag{144}$$

in addition to a phase arising from the lattice depth, a phase arising from the small periodic variations in the lowest eigenvalue, and a phase arising from non-adiabatic corrections. Typically, the contribution to the final phase shift between the two arms from the phases corresponding to the lattice depth and to the periodic variations in the lowest eigenvalue will be negligible, and for the moment we will not worry about non-adiabatic corrections. In the boosted dressed state frame, the lattice velocity is $v_{Lattice}(t) = v_{Lab}(t) + gt - v_0 - \delta v$. Thus:

$$\Delta\phi_{(p_0+\delta p)} = \frac{m}{2\hbar}\int_{t_0}^{t_f} (v_{Lab}(t) + gt - v_0 - \delta v)^2 dt \tag{145}$$

$$= \frac{m}{2\hbar}\int_{t_0}^{t_f}(v_{Lab}(t)+gt)^2dt - \frac{m}{\hbar}\int_{t_0}^{t_f}(v_{Lab}(t)+gt)(v_0+\delta v)dt + \frac{m}{2\hbar}\int_{t_0}^{t_f}(v_0+\delta v)^2dt \quad (146)$$

$$= \frac{m}{2\hbar}\int_{t_0}^{t_f}(v_{Lab}(t)+gt)^2dt - \frac{m}{\hbar}\left[(D_{Lab}(t_f)+\frac{1}{2}gt_f^2)-(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)\right](v_0+\delta v)+\frac{m}{2\hbar}(v_0+\delta v)^2(t_f-t_0) \quad (147)$$

We therefore find that after the lattice acceleration:

$$\vec{\Psi}_{DS_{(p_0+\delta p)}}(2n\hbar k, t_f) = e^{i\Delta\phi_{(p_0+\delta p)}}\vec{\Psi}_{DS_{(p_0+\delta p)}}(0,t_0) \quad (148)$$

$$= e^{i\Delta\phi_{(p_0+\delta p)}}e^{\frac{i}{\hbar}m\left[-(D_{Lab}(t_0)+\frac{1}{2}gt_0^2)(v_0+\delta v)+\frac{1}{2}(v_0+\delta v)^2t_0\right]}\vec{\Psi}_{DS}(p_0+\delta p,t_0) \quad (149)$$

$$= e^{i\left[\frac{m}{2\hbar}\int_{t_0}^{t_f}(v_{Lab}(t)+gt)^2dt-\frac{m}{\hbar}(D_{Lab}(t_f)+\frac{1}{2}gt_f^2)(v_0+\delta v)+\frac{m}{2\hbar}(v_0+\delta v)^2t_f)\right]}\vec{\Psi}_{DS}(p_0+\delta p,t_0) \quad (150)$$

We can now transform back to the unboosted dressed state frame. We find that:

$$\vec{\Psi}_{DS}(p_0+2n\hbar k+\delta p,t_f) = \langle p_0+2n\hbar k+\delta p|\,\hat{U}_{(p_0+\delta p)}(t_f)\left|\Psi_{DS_{(p_0+\delta p)}}(t_f)\right\rangle \quad (151)$$

$$= \langle p_0+2n\hbar k+\delta p|\,e^{\frac{i}{\hbar}(p_0+\delta p)\hat{x}}\times \quad (152)$$

$$\times e^{-\frac{i}{\hbar}m\left[-(D_{Lab}(t_f)+\frac{1}{2}gt_f^2)(v_0+\delta v)+\frac{1}{2}(v_0+\delta v)^2t_f\right]}\left|\Psi_{DS_{(p_0+\delta p)}}(t_f)\right\rangle \quad (153)$$

$$= e^{-\frac{i}{\hbar}m\left[-(D_{Lab}(t_f)+\frac{1}{2}gt_f^2)(v_0+\delta v)+\frac{1}{2}(v_0+\delta v)^2 t_f\right]}\vec{\Psi}_{DS_{(p_0+\delta p)}}(2n\hbar k, t_f) \tag{154}$$

Substituting in for $\vec{\Psi}_{DS_{(p_0+\delta p)}}(2n\hbar k, t_f)$, we see that all dependence on $v_0 + \delta v$ cancels, and we end up with the expression:

$$\vec{\Psi}_{DS}(p_0 + 2n\hbar k + \delta p, t_f) = e^{i\frac{m}{2\hbar}\int_{t_0}^{t_f}(v_{Lab}(t)+gt)^2 dt}\vec{\Psi}_{DS}(p_0 + \delta p, t_0) \tag{155}$$

The phase accumulated in the dressed state frame during the acceleration is thus $\varphi(p_0+\delta p) = \frac{m}{2\hbar}\int_{t_0}^{t_f}(v_{Lab}(t) + gt)^2 dt +$ (non-adiabatic corrections). What this means is that as long as the lattice is close enough to being on resonance with the atoms so that they are accelerated, the phase that we evolve during an acceleration in the dressed state frame is independent of the initial velocity of the atoms up to non-adiabatic corrections. This makes the dressed state frame particularly convenient for performing calculations involving wave packets. In contrast, in the freely falling frame the accumulated phase is not independent of the initial velocity, but this dependence will cancel in our final expression for the phase shift between the two arms as long as the distance travelled by the atoms while locked into a lattice is the same for both arms, which is readily achievable. Note that velocity dependent contributions to the total phase shift can arise when we treat the problem purely in terms of dressed states (as they must, since the total phase shift is an observable quantity and must therefore be independent of the frame in which it is calculated). The key point to realize is that if the total distance travelled in the lattice is not the same for both arms, then the two arms will end up in two different dressed state frames. We will discuss this matter further in the following section. We can express the phase acquired in the freely falling frame during the lattice acceleration in the following convenient form:

$$\Delta\phi_{accel} = \frac{m}{2\hbar}\int_{t_0}^{t_f}(v_{Lab}(t) + gt)^2 dt - \frac{1}{\hbar}\left[2n\hbar k(D_{Lab}(t_f) + \frac{1}{2}gt_f^2) + (p_0 + \delta p)\Delta D\right] \tag{156}$$

up to non-adiabatic corrections, where $\Delta D \equiv (D_{Lab}(t_f) + \frac{1}{2}gt_f^2) - (D_{Lab}(t_0) + \frac{1}{2}gt_0^2)$.

VII. Applications: Atom Interferometers Using Optical Lattices as Waveguides

Light-pulse atom interferometer geometries have had tremendous success in performing many types of high-precision measurements. However, in many cases, we would like to be able to push the capabilities of atom interferometry by making more precise measurements using smaller interferometers. Atom interferometers that use optical lattices as waveguides for the atoms can make such measurements attainable. We can use a beam splitter composed of multiple Bragg pulses or a hybrid Bragg pulse/lattice acceleration scheme as described in the previous section to separate the two arms of the interferometer in momentum space. We can then control each arm independently with an optical lattice. We will once again use Bragg pulses during the π-pulse and final $\frac{\pi}{2}$-pulse stages of the interferometer sequence, with lattices acting as waveguides between these stages. Our previous analysis has developed the theoretical machinery for calculating phase shifts for these lattice interferometers. We will now examine several of the most promising applications of lattice interferometers.

Lattice interferometers can be used to make extremely precise measurements of the local gravitational acceleration g. We can derive the phase shift for such a gravimeter using equation (56). Let the velocities in the lab frame of the two arms of the interferometer be $v_{Lab}^{arm1}(t)$ and $v_{Lab}^{arm2}(t)$, respectively. The lattice velocities for the two arms in the dressed state frame will thus be:

$$v_{Lattice}^{arm1}(t) = v_{Lab}^{arm1}(t) + gt \tag{157}$$

and:

$$v_{Lattice}^{arm2}(t) = v_{Lab}^{arm2}(t) + gt \tag{158}$$

Where our interferometer sequence lasts for a time T, the phase shift will be:

$$\Delta\phi = \frac{1}{\hbar}\left[\int_0^T \frac{1}{2}m(v_{Lab}^{arm1}(t) + gt)^2 dt - \int_0^T \frac{1}{2}m(v_{Lab}^{arm2}(t) + gt)^2 dt\right] \tag{159}$$

$$= \frac{1}{\hbar}\left[\int_0^T \frac{1}{2}m(v_{Lab}^{arm1}(t)^2 - v_{Lab}^{arm2}(t)^2) dt + \int_0^T mg\Delta v(t) t dt\right] \tag{160}$$

where $\Delta v(t) \equiv v_{Lab}^{arm1}(t) - v_{Lab}^{arm2}(t)$ is the velocity difference between the two arms. We can integrate the second term in the above expression by parts, letting the distance between the two arms be $\Delta d(t) \equiv \int_0^t \Delta v(t')dt'$:

$$\int_0^T mg\Delta v(t) t dt = mg\Delta d(T)T - \int_0^T mg\Delta d(t) dt \tag{161}$$

The phase shift is thus given by:

$$\Delta\phi = -\frac{1}{\hbar}\int_0^T mg\Delta d(t) dt + \frac{1}{\hbar}\left[\int_0^T \frac{1}{2}m(v_{Lab}^{arm1}(t)^2 - v_{Lab}^{arm2}(t)^2) dt + mg\Delta d(T)T\right] \tag{162}$$

In the dressed state frame, ideal Bragg pulses simply yield contributions of $\frac{\pi}{2}$ to the overall phase shift between the arms regardless of the laser phase, and these contributions can easily be made to cancel. However, we note that in some cases, corrections to the simplified picture of an ideal Bragg pulse due to such factors as gravity gradients, detuning effects, or population loss may need to be considered. To avoid unnecessarily complicating our presentation, we will not present these corrections here. Instead, we emphasize that they are well-understood effects and refer the reader to other sources for further discussion [1, 12, 20]. The imprinting of laser phase typically associated with these pulses arises when we boost to the freely falling frame, where most phase shift calculations for light-pulse

interferometers take place. We will choose the parameters of our interferometer so that the velocities of the two arms in the lab frame are either opposite to each other or equal so that $v_{Lab}^{arm1}(t)^2 - v_{Lab}^{arm2}(t)^2 = 0$. And since we need the two arms of the interferometer to overlap at time T, it will be convenient for us to choose $\Delta d(T) = 0$. Thus, the first term in equation (162) will constitute the only contribution to $\Delta\phi$. However, our parameters will undergo small fluctuations around their desired values from shot to shot, so that the other terms in equation (162) act as a source of noise. In order to cancel the effects of this noise, we can adopt a gradiometer setup in which we have a sequence of two or more gravimeters that interact with the same lattice beams. Although fluctuations in $\Delta d(t)$ will still affect phase differences between gravimeters, which take the form $-\frac{1}{\hbar}\int_0^T m(g_1 - g_2)\Delta d(t)dt$, we can easily control the phase differences between our lattice beams well enough so that these effects are smaller than shot noise [21]. When measuring a gravity gradient, the value of g will vary due to the gradient over the range of a single gravimeter. For linear gradients, we can calculate the phase shift in the presence of a gravity gradient by assuming that the value of g corresponding to the gravimeter is equal to its value at the center of mass position for the atom (see Appendix A for a rigorous justification of this procedure). When higher order derivatives of the gravitational field become sizeable in comparison to the first derivative, this simple prescription may not suffice, and we can treat the problem perturbatively. Note also that we have ignored the effects of non-adiabatic corrections, for a wide range of experimentally feasible gravimeter geometries exist that contain sufficiently adiabatic lattice depth and velocity ramps and that make use of symmetry in such a way that the net contribution of these corrections to the phase shift will be negligible. Moreover, the phase evolution terms arising from the lattice depth and from the periodic variations in the lowest eigenvalue will cancel, provided that the atoms remain sufficiently centered in well-collimated lattice beams so that we can neglect any difference in the lattice depth between the two arms and provided that both arms undergo a nearly integral number of Bloch oscillations so that any effects of the periodic variations in the lowest eigenvalue will be common. If we want to measure

a gravitational acceleration instead of a gravitational gradient, we can use two dissimilar conjugate interferometers whose phase noise is strongly correlated as suggested in [2]. We can then use appropriate statistical methods to extract the desired signal.

A dressed state frame is defined by the velocity of the corresponding lattice and by the distance which the lattice has travelled since the beginning of the interferometer sequence. Since during much of the interferometer sequence the two arms will be addressed by two different lattices, our phase shift calculation assumes that at times the two arms will be in dressed state frames with different associated velocities. The arms begin in the same dressed state frame, and if $\Delta d(T) = 0$ they will end up in the same dressed state frame, in which case equation (162) provides the final say on the phase shift. If $\Delta d(T)$ differs slightly from 0, the two arms will end up in slightly different dressed state frames. Since we must compare phases in a common frame, it is convenient to boost both arms into the freely falling frame. The mathematical framework for doing this was laid out in section VI. In particular, we can use the result given in equation (156) for both arms. In addition to the phase difference in equation (162), we will then have an additional term in our phase shift equal to $-\frac{1}{\hbar}p_{final}\Delta d(T)$, where p_{final} is the momentum at time T of a particular momentum eigenstate in our wavepacket. When we use the interference pattern in position space to determine the phase shift, as is typically done in the lab, it follows from the discussion in [1] that p_{final} will take on the value of the center of the momentum space wavepacket. This portion of the phase shift is similar to the final term in equation (162), and like this term it will cancel in a gradiometer setup.

Where we assume that we arrange polarizations so that the four lattice beams interact in pairs, each arm will be exposed to two lattices. One lattice will be on resonance with a given arm, while the other will be highly detuned. As long as we keep this detuning large enough and/or employ geometries with sufficient symmetry between the arms, the net effect of the off-resonant lattices on the final phase shift will be negligible. In principle, we can calculate the corrections due to these off-resonant effects, but if we choose our experimental parameters

correctly this will not practically be necessary, as verified with numerical simulations. For arm 1, the detuned lattice will manifest as an additional term in our discrete Hamiltonian, given by:

$$H_{detuned}(\alpha, \tilde{\Omega}) = \hbar\omega_{bragg}\times \tag{163}$$

$$\times \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & 0 & -\tilde{\Omega}e^{-i\beta(t)} & 0 & -\tilde{\Omega}e^{i\beta(t)} & 0 & & & \\
 & & 0 & -\tilde{\Omega}e^{-i\beta(t)} & 0 & -\tilde{\Omega}e^{i\beta(t)} & 0 & & \\
 & & & 0 & -\tilde{\Omega}e^{-i\beta(t)} & 0 & -\tilde{\Omega}e^{i\beta(t)} & 0 & \\
 & & & & 0 & -\tilde{\Omega}e^{-i\beta(t)} & 0 & -\tilde{\Omega}e^{i\beta(t)} & 0 \\
 & & & & & 0 & -\tilde{\Omega}e^{-i\beta(t)} & 0 & -\tilde{\Omega}e^{i\beta(t)} \\
 & & & & & & \ddots & \ddots & \ddots
\end{pmatrix} \tag{164}$$

where $\beta(t) \equiv -\int_0^t \left[\alpha^{arm1}(t') - \alpha^{arm2}(t')\right] \omega_{bragg} dt' = -\frac{\omega_{bragg}}{v_r}\Delta d(t)$ [18]. To obtain the correction to the Hamiltonian from the detuned lattice for arm 2, we take the complex conjugate of the matrix above. For large detunings, $\beta(t)$ will vary rapidly with time so that the contribution from the detuned Hamiltonian will be small due to the rotating wave approximation. Corrections arising from the detuned Hamiltonian can be solved for perturbatively. In the presence of the detuned lattice, our equation of motion is:

$$i\hbar\frac{\partial}{\partial t}\vec{\Psi}(t) = (H(t) + H_{detuned}(t))\vec{\Psi}(t) \tag{165}$$

We take the zeroth order solution to be the solution in the absence of $H_{detuned}$, which we will denote by $\vec{\Psi}_0(t)$. $\vec{\Psi}_0(t)$ thus satisfies:

$$i\hbar\frac{\partial}{\partial t}\vec{\Psi}_0(t) = H(t)\vec{\Psi}_0(t) \tag{166}$$

Now, we will denote our first order correction by $\delta^{(1)}\vec{\Psi}(t)$. We will use a method similar to the Born approximation, plugging our nth order solution into the righthand side of equation (165) to find the (n+1)th order solution. So to first order, our equation of motion tells us:

$$i\hbar\frac{\partial}{\partial t}(\vec{\Psi}_0(t) + \delta^{(1)}\vec{\Psi}(t)) = (H(t) + H_{detuned}(t))\vec{\Psi}_0(t) \tag{167}$$

Using equation (166), this reduces to:

$$i\hbar\frac{\partial}{\partial t}\delta^{(1)}\vec{\Psi}(t) = H_{detuned}(t)\vec{\Psi}_0(t) \tag{168}$$

After integration, this gives us:

$$\delta^{(1)}\vec{\Psi}(t) = -\frac{i}{\hbar}\int_0^t H_{detuned}(t')\vec{\Psi}_0(t')dt' \tag{169}$$

Where $H_{detuned}$ rotates at frequency ω, the integral will approximately yield a suppressive factor of $\frac{1}{\omega}$, and the correction $\delta^{(1)}\vec{\Psi}(t)$ will approximately rotate with frequency ω. To obtain the second order solution, we substitute the first order solution into the equation of motion:

$$i\hbar\frac{\partial}{\partial t}(\vec{\Psi}_0(t) + \delta^{(1)}\vec{\Psi}(t) + \delta^{(2)}\vec{\Psi}(t)) = (H(t) + H_{detuned}(t))(\vec{\Psi}_0(t) + \delta^{(1)}\vec{\Psi}(t)) \tag{170}$$

Using equations (166) and (168) to rearrange this equation and integrating, we obtain:

$$\delta^{(2)}\vec{\Psi}(t) = -\frac{i}{\hbar}\int_0^t (H(t') + H_{detuned}(t'))\delta^{(1)}\vec{\Psi}(t')dt' \tag{171}$$

The integral will yield another suppressive factor of $\frac{1}{\omega}$. Ideally, the nth order correction will be suppressed by a factor of $\frac{1}{\omega^n}$. However, note that the multiplication of exponentials with opposite signs will lead to certain terms that will not always be further suppressed at higher orders. Rotation at the natural frequency at which $\vec{\Psi}_0(t)$ acquires phase due to its energy will ensure that any such terms will not become larger in higher order perturbation theory, but these terms could prevent convergence in some cases. At the very least, the first order result from the method described above will typically be relevant, and when required we can use other methods to find higher order solutions. The method of adiabatic expansion with suitably small time intervals could be used, with $\vec{\Psi}_0(t) + \delta^{(1)}\vec{\Psi}(t)$ serving as the initial guess for the wavefunction. Alternatively, we could use a Fourier expansion or another method to solve for $\delta\vec{\Psi}(t)$ so that $\vec{\Psi}_0(t) + \delta\vec{\Psi}(t)$ exactly solves our equation of motion. In this case, the differential equation of interest would be $i\hbar\frac{\partial}{\partial t}\delta\vec{\Psi}(t) = (H(t) + H_{detuned}(t))\delta\vec{\Psi}(t) + H_{detuned}(t)\vec{\Psi}_0(t)$. The point is that, if needed, we could certainly go forward with the calculation accounting for the detuned lattice. But we emphasize again that we can typically avoid situations where this will be necessary.

A lattice gravimeter can provide extraordinary levels of sensitivity. This sensitivity can be achieved over small distance scales by implementing a hold sequence in which the two arms are separated, manipulated into the same momentum eigenstate, held in place by a single lattice, and then recombined. Hold times will be limited by spontaneous emission, which decreases contrast. Modern laser technology will allow us to use detunings of hundreds or even thousands of GHz, making hold times on the order of 10 s within reach [7]. Spontaneous emission can be further decreased by using blue detuning, so that the wavefunctions of the atoms will peak at the nodes of the lattice potential. Gravimeter sensitivities using the hold method greatly exceed the sensitivities of light-pulse gravimeters. For example, for 10^7 atoms/shot and 10^{-1} shots/s, a conventional light-pulse interferometer with a 10 m

interrogation region can achieve a sensitivity of 10^{-12} g/Hz$^{1/2}$. With similar experimental parameters, a lattice interferometer with a 10 s hold time and an interrogation region of 1 cm will have a sensitivity of 10^{-13} g/Hz$^{1/2}$. If we expand the interrogation region to 1 m, we obtain a sensitivity of 10^{-15} g/Hz$^{1/2}$. This remarkable sensitivity has a plethora of potential applications. Extremely precise gravimeters and gravity gradiometers can be constructed to perform tests of general relativity, make measurements relevant to geophysical studies, and build highly compact inertial sensors. Moreover, the fact that lattice interferometers can operate with such high sensitivities over small distance scales makes them prime candidates for exploring short distance gravity. One could set up an array of lattice gravimeters to precisely map out gravitational fields over small spatial regions, as shown in Figure 6. The knowledge obtained about the local gravitational field could be useful in searching for extra dimensions as well as in studying the composition and structure of materials.

Ultra-high precision gravitational measurements are certainly among the most promising applications of lattice interferometers, but the usefulness of lattice interferometers is certainly not limited to the study of gravity. By exposing the two arms of a lattice interferometer to different electrostatic potentials, tests of atom charge neutrality with unprecedented accuracy could be achieved. The main advantage of a lattice interferometer in such a measurement is that the interrogation time can be significantly increased in comparison to the interrogation time achievable in a light-pulse geometry [22]. Moreover, the hold technique can be a powerful instrument in studying all sorts of potentials with high spatial resolution. For instance, lattice interferometers could act as highly sensitive sensors of spatial variations in magnetic or electric fields that are capable of operating over short distance scales, where the potential energy difference between the arms would result from the magnetic or electric dipole moments of the atom. Note that dissimilar conjugate interferometers employing different internal states with different dipole moments could be used to eliminate unwanted contributions to the phase shift from such factors as gravity or laser phase noise, provided that the ratio of the respective dipole moments of these states is known sufficiently well for

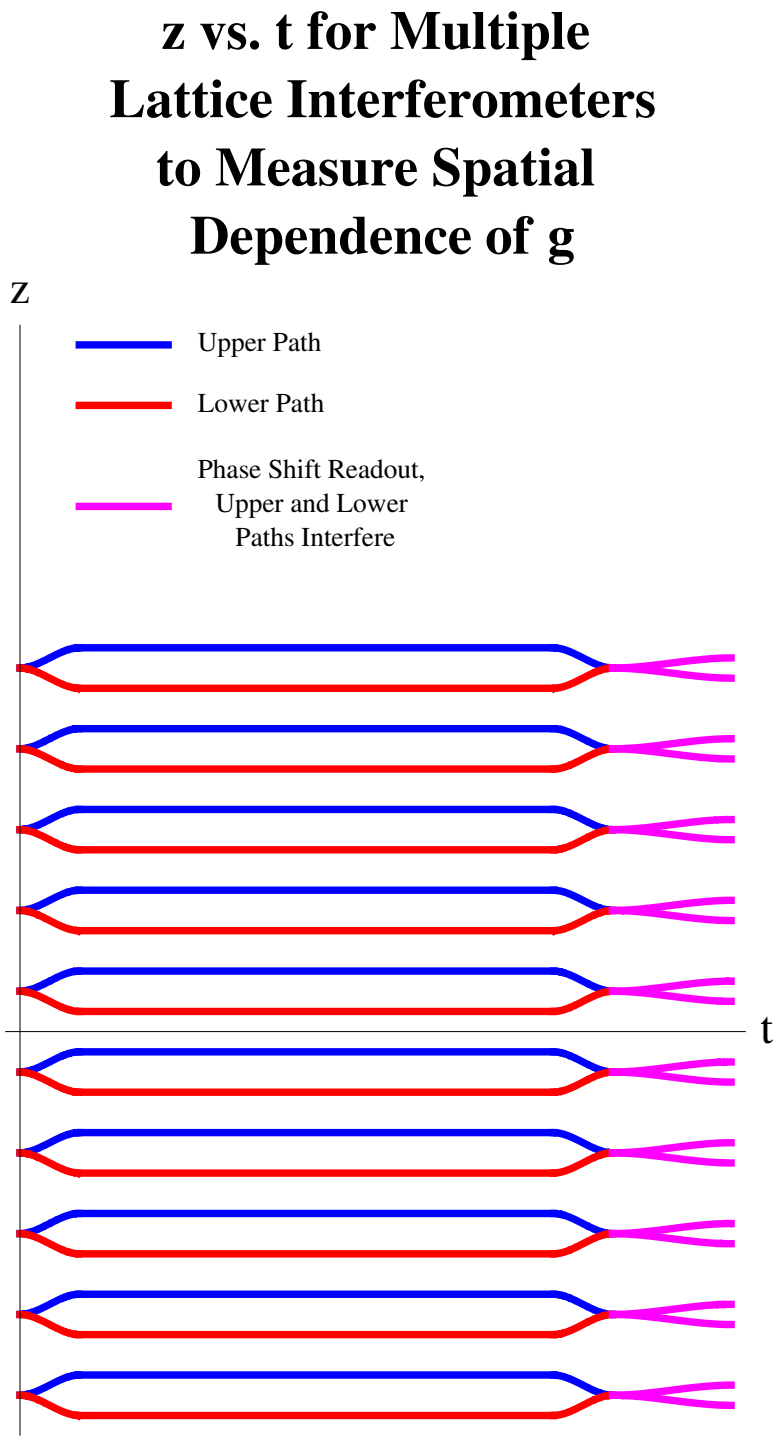

Figure 6: An array of gravimeters such as that shown above can be used to achieve measurements of a local gravitational field with high spatial resolution. After separating the atoms in each gravimeter by a small amount, we can implement a hold sequence to greatly increase sensitivity. Such an array could be used to study general relativistic effects, search for extra dimensions, examine local mass distributions, or measure Newton's constant.

the size of the field variations being measured. Conversely, by exposing the two arms of each conjugate interferometer to different controlled magnetic or electric fields, dipole moment ratios could be measured between different internal states or between different isotopes. When multiple isotopes are used, lattice interferometers have the additional advantage of ensuring that the isotopes follow identical trajectories, eliminating potential systematic effects arising from unwanted field inhomogeneities (in light-pulse atom interferometers, the fact that the isotopes have different recoil velocities ensures that they will follow slightly different trajectories). In addition, other systematic effects can be negated by varying the controlled field and neglecting portions of the phase shift that do not scale appropriately.

Equation (162) indicates that if we apply different accelerations to the two arms of the interferometer, we will see a phase shift proportional to $\frac{m}{\hbar}$ that depends on the kinetic energy difference between the arms, which can be made extremely large. A conjugate interferometer geometry (shown in Figure 3) could reduce the effects of laser phase noise and cancel the gravitational phase shift up to gradients [2]. Such a geometry could provide an extremely precise measurement of $\frac{m}{\hbar}$, as illustrated by the fact that we can achieve a phase shift of 10^{11} radians for a 5 m interrogation region and a 0.2 s interrogation time. In this situation, non-adiabatic corrections would not cancel, and the methods we have outlined for calculating these corrections would need to be applied. The calculation would be analogous to that discussed in section V. The ratio $\frac{m}{\hbar}$ is of particular interest because of its direct relation to the fine structure constant. Moreover, the fact that all terms in equation (162) are proportional to m can be exploited to provide high-precision measurements of isotope mass ratios, where we use an interferometer geometry in which the two isotopes follow identical trajectories. Isotope mass ratios could be relevant to studies of advanced models of the structure of the nucleus. Moreover, if we know the mass ratio of two isotopes sufficiently well, we can use such a geometry to measure gravitational accelerations extremely precisely, where we think of the two isotopes as providing two dissimilar conjugate interferometers. The phase noise of these two interferometers will be extremely well correlated because they

are topologically identical. And we will eliminate the need for the additional lattice beams required to the form the second, topologically distinct interferometer that would be needed if we only had a single isotope. Note that this scheme to construct dissimilar, topologically identical conjugate gravimeters would not be possible for a light-pulse geometry, for the phase shift of light-pulse gravimeters is independent of isotope mass.

Lattice interferometers can also be used to build compact and highly sensitive gyroscopes. There are multiple possible schemes in which optical lattices can enhance gyroscope sensitivity. One such scheme is to modify a typical atom based gyroscope by replacing the Raman pulses with LMT lattice beam splitters. In addition, we can implement a geometry where multiple lattices guide the atoms in a rectangle or rhombus shaped Mach-Zehnder interferometer. Such a geometry offers the potential for particularly sensitive gyroscopes and could reasonably achieve a sensitivity of 10^{-14} rad/s/Hz$^{1/2}$, making explorations of the Lense-Thirring effect attainable. Another option is to use optical lattices along multiple axes to provide complete control of the motion of the atoms in two or three dimensions (this control is only achieved in the region in which the lattices overlap, making the use of wide beams particularly advantageous). In order to achieve a Sagnac phase shift, the atoms could be guided in multiple loops, with the two arms traveling in opposite directions. Such a system could also be of interest to studies of cold atoms from a condensed matter point of view by introducing an orbital angular momentum in addition to the angular momentum of spins or vortices. Geometries in which atomic motion is controlled in multiple dimensions could also expand the possibilities for other applications of lattice interferometry (such as measurements of gravity or of electric and magnetic fields) by allowing for the measurement of potential energy differences between arbitrary paths. For instance, a compact array of three orthogonal lattice gravity gradiometers could be used to measure the nonzero divergence of the gravitational field in free space predicted by general relativity [6].

VIII. Conclusion

We have presented a detailed analytical description of the interaction between an atom and an optical lattice, using the adiabatic approximation as a starting point and then proceeding to rigorously develop a method to calculate arbitrarily small corrections using perturbative adiabatic expansion. We applied this theoretical framework to calculate the phase accumulated during a lattice acceleration in an LMT beam splitter. And we have proposed atom interferometer geometries that use optical lattices as waveguides and discussed applications of such geometries, using our theoretical methods to add rigor to this discussion. We are working toward the experimental implementation of lattice interferometers and large angle LMT lattice beam splitters, and we hope to explore the applications we have discussed. We note that the majority of unwanted systematic effects that are relevant to lattice interferometers are also shared by light-pulse interferometers and will therefore be mitigated using similar methods to those used in light-pulse interferometers. We are thus optimistic about the prospects of realizing lattice interferometers in existing apparatuses originally constructed with light-pulse geometries in mind.

Appendix A: Including Gravity Gradients in the Calculation

We will now calculate the gravimeter phase shift in the presence of a linear gravity gradient to verify the claim made in section VII. that the phase shift can be obtained by neglecting the gradient's contribution to the Hamiltonian and simply assuming that g is equal to its value at the center of mass position of the atom. We begin by looking at our problem in the lab frame, where we have added a gravity gradient term to the Hamiltonian:

$$\hat{H}_{Lab} = \frac{\hat{p}^2}{2m} + V_0 \sin^2\left[k\hat{x} - kD_{Lab}(t)\right] + mg\hat{x} + \frac{1}{2}m\lambda\hat{x}^2 \tag{172}$$

For such a gravity gradient, as a function of position in the lab frame, g is given by:

$$g(D_{Lab}(t)) = g(0) + \lambda D_{Lab}(t) \tag{173}$$

Transforming into the freely falling frame using the unitary transformation $\hat{U}_{FF}(t)$ derived in section II., we obtain the Hamiltonian:

$$\hat{H}_{FF} = \frac{\hat{p}^2}{2m} + V_0 \sin^2\left[k\hat{x} - k(D_{Lab}(t) + \frac{1}{2}gt^2)\right] + \frac{1}{2}m\lambda\left(\hat{x} - \frac{1}{2}gt^2\right)^2 \tag{174}$$

From here, we can move into the dressed state frame by applying a boost in position space of $D_{Lab}(t) + \frac{1}{2}gt^2$ as described in section II., yielding:

$$\hat{H}_{DS} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt)\hat{p} + V_0 \sin^2(k\hat{x}) + \frac{1}{2}m\lambda\left(\hat{x} + D_{Lab}(t)\right)^2 \tag{175}$$

It will be convenient to expand out the gradient term to write:

$$\hat{H}_{DS} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt)\hat{p} + V_0 \sin^2(k\hat{x}) + m\lambda D_{Lab}(t)\hat{x} + \frac{1}{2}m\lambda\left(D_{Lab}(t)\right)^2 + \frac{1}{2}m\lambda\hat{x}^2 \tag{176}$$

We will want to work in another dressed state frame that is boosted in momentum space so that there will be no $m\lambda D_{Lab}(t)\hat{x}$ term. Physically, where we define $\delta g(t) \equiv g(D_{Lab}(t)) - g(0) = \lambda D_{Lab}(t)$, this boosted dressed state frame will have velocity greater than that of the initial dressed state frame by an amount $\delta v(t) \equiv \int_0^t \delta g(t')dt'$. We thus want the boosted dressed state frame to have the Hamiltonian:

$$\hat{H}_{DS_\lambda} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt + \delta v(t))\hat{p} + V_0 \sin^2(k\hat{x}) + \frac{1}{2}m\lambda\hat{x}^2 \tag{177}$$

The unitary transformation $\hat{U}_{DS_\lambda}(t) = e^{-\frac{i}{\hbar}m\delta v(t)\hat{x}}e^{\frac{i}{\hbar}\kappa(t)}$, where $\kappa(t)$ is to be determined, will be such that $\hat{H}_{DS} = \hat{U}_{DS_\lambda}(t)\hat{H}_{DS_\lambda}\hat{U}^\dagger_{DS_\lambda}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{DS_\lambda}(t)\right)\hat{U}^\dagger_{DS_\lambda}(t)$. We will have:

$$\hat{U}_{DS_\lambda}(t)\hat{H}_{DS_\lambda}\hat{U}^\dagger_{DS_\lambda}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{DS_\lambda}(t)\right)\hat{U}^\dagger_{DS_\lambda}(t) = \tag{178}$$

.

$$= \frac{(\hat{p}+m\delta v(t))^2}{2m} - (v_{Lab}(t)+gt+\delta v(t))(\hat{p}+m\delta v(t)) + V_0\sin^2(k\hat{x}) + m\delta g(t)\hat{x} - \dot{\kappa}(t) + \frac{1}{2}m\lambda\hat{x}^2 \tag{179}$$

$$= \frac{\hat{p}^2}{2m} - (v_{Lab}(t)+gt)\hat{p} + V_0\sin^2(k\hat{x}) + m\lambda D_{Lab}(t)\hat{x} + \frac{1}{2}m\lambda\hat{x}^2 - \dot{\kappa}(t) - \frac{1}{2}m\delta v(t)^2 - m(v_{Lab}(t)+gt)\delta v(t) \tag{180}$$

We need this Hamiltonian to be equal to $\hat{H}_{DS}$ as given in equation (176), which implies that:

$$\dot{\kappa}(t) = -\frac{1}{2}m\lambda\left(D_{Lab}(t)\right)^2 - \frac{1}{2}m\delta v(t)^2 - m(v_{Lab}(t)+gt)\delta v(t) \tag{181}$$

Integrating, we find that up to an overall time-independent phase that will have no physical significance and that we therefore drop, $\kappa(t)$ is given by:

$$\kappa(t) = -\int_0^t \frac{1}{2}m\lambda\left(D_{Lab}(t')\right)^2 dt' - \int_0^t \left[\frac{1}{2}m\delta v(t')^2 + m(v_{Lab}(t')+gt')\delta v(t')\right] dt' \tag{182}$$

Having specified our unitary transformation, we can relate the state vectors in the dressed state and boosted dressed state frames for any time t:

$$|\Psi_{DS}(t)\rangle = \hat{U}_{DS_\lambda}(t)\,|\Psi_{DS_\lambda}(t)\rangle \tag{183}$$

We can describe the evolution from time 0 to time T in the dressed state frame by performing the transformation:

$$|\Psi_{DS}(T)\rangle = \hat{U}_{DS_\lambda}(T)\hat{T}_{DS_\lambda}(T,0)\hat{U}^\dagger_{DS_\lambda}(0)\,|\Psi_{DS}(0)\rangle \tag{184}$$

where $\hat{T}_{DS_\lambda}(T,0)$ takes us from time 0 to time T in the boosted dressed state frame. Note that $\delta v(0) = 0$ and $\kappa(0) = 0$, so $\hat{U}^\dagger_{DS_\lambda}(0)$ is simply the identity operator. As in section VI., we will work in momentum space, and in this case the initial momentum space wavefunctions in the two frames will be identical:

$$\vec{\Psi}_{DS_\lambda}(p,0) = \vec{\Psi}_{DS}(p,0) \tag{185}$$

We can now perform time evolution on this wavefunction in the boosted dressed state frame. If it were not for the term $V' \equiv \frac{1}{2}m\lambda\hat{x}^2$ in $\hat{H}_{DS_\lambda}$, we could apply our usual prescription for time evolution in a dressed state frame without second thought. However, the presence of V' compels us to proceed more carefully. We assume λ to be small (for example, near the earth's surface $\lambda \sim 10^{-6}\ s^{-2}$), and we will treat the term V' perturbatively. The dressed state frame is boosted in position space so that in this frame, the lattice's position always remains at the origin (although the lattice will have a nonzero velocity, making the dressed state frame somewhat counterintuitive). Therefore, assuming a deep enough lattice so that Bloch oscillations and tunneling are highly suppressed, the ground state position space wavefunction of an atom locked into the lattice will always be a corrugated Gaussian wavepacket centered at the origin regardless of the velocity and position of the lattice in the lab frame, where the corrugations arise from the fact that the wavefunction will typically be spread over hundreds or thousands of peaks and dips in the lattice. To lowest order (neglecting any effect of the perturbation on Bloch oscillations or tunneling between different lattice sites), the perturbation will lead to a position dependent energy shift, which will vary across the occupied lattice sites; and the energy scale of this shift will be on the order of magnitude of the expectation value of V' in the unperturbed ground state, where we note that the behavior of V' outside of the region populated by the atomic wavefunction will have

no bearing on the problem. Two phenomena combine to make the energy shift exceedingly small: the small size of the constant λ and the small spatial spread of the atom's wavefunction in position space (typically ~ 1 mm or smaller). The latter of these facts allows us to treat V' perturbatively while not doing so for the $m\lambda D_{Lab}(t)\hat{x}$ term, since V' is quadratic in $\hat{x}$. Besides being extremely small, any correction to the wavefunction's evolution arising from V' will be almost entirely common to both arms of the interferometer, and we can typically neglect it even without relying on the additional cancellation that comes from a gradiometer setup. This common mode cancellation occurs because the wavefunctions for the two arms are nearly identical in their respective dressed state frames up to a shift in momentum space–which could be boosted away and will not affect the dynamics of the wavefunctions' shapes–and up to an overall phase, because, as stated before, we assume that the lattice is sufficiently deep so as to highly suppress Bloch oscillations and tunneling. Any coupling of the perturbation to Bloch oscillations or tunneling will therefore be a much smaller effect than the energy shift that arises from lowest order perturbation theory. Typically, the presence of a small gradient will alter non-adiabatic corrections negligibly, so we can safely proceed to calculate phase evolution in the boosted dressed state frame using the adiabatic approximation. We note that in the presence of potentials that vary more rapidly over the spatial extent of the atom's wavefunction, there may be times when it is necessary to carry out a perturbative calculation, which could be performed numerically.

For an optical lattice potential, momentum is transferred in units of $2\hbar k$. So where we assume a total momentum transfer of $2n\hbar k$ for integer n after the interferometer sequence is complete, our momentum space wave function in the boosted dressed space frame will be such that:

$$\vec{\Psi}_{DS_\lambda}(p + 2n\hbar k, T) = e^{i\phi_{grad}}\vec{\Psi}_{DS_\lambda}(p, 0) = e^{i\phi_{grad}}\vec{\Psi}_{DS}(p, 0) \tag{186}$$

where:

$$\phi_{grad} = \frac{m}{2\hbar}\int_0^T (v_{Lab}(t) + gt + \delta v(t))^2 dt = \tag{187}$$

$$= \frac{m}{2\hbar}\left[\int_0^T (v_{Lab}(t) + gt)^2 dt + 2\int_0^T (v_{Lab}(t) + gt)\delta v(t) dt + \int_0^T \delta v(t)^2 dt\right] \tag{188}$$

We can now transform back into the unboosted dressed state frame:

$$\vec{\Psi}_{DS}(p + 2n\hbar k, T) = \langle p + 2n\hbar k| \hat{U}_{DS_\lambda}(T) |\Psi_{DS_\lambda}(T)\rangle \tag{189}$$

$$= \langle p + 2n\hbar k| e^{-\frac{i}{\hbar}m\delta v(T)\hat{x}} e^{\frac{i}{\hbar}\kappa(T)} |\Psi_{DS_\lambda}(T)\rangle \tag{190}$$

$$= e^{\frac{i}{\hbar}\kappa(T)} \vec{\Psi}_{DS_\lambda}(p + 2n\hbar k + m\delta v(T), T) \tag{191}$$

$$= e^{i\left[\frac{1}{\hbar}\kappa(T) + \phi_{grad}\right]} \vec{\Psi}_{DS}(p + m\delta v(T), 0) \tag{192}$$

Adding $\frac{1}{\hbar}\kappa(T)$ and ϕ_{grad} using equations (182) and (187), we obtain our final expression for phase evolution in the presence of a gravity gradient:

$$\vec{\Psi}_{DS}(p + 2n\hbar k, T) = e^{i\phi_\lambda} \vec{\Psi}_{DS}(p + m\delta v(T), 0) \tag{193}$$

for:

$$\phi_\lambda = \frac{1}{\hbar}\left[\int_0^T \frac{1}{2} m (v_{Lab}(t) + gt)^2 dt - \int_0^T \frac{1}{2} m\lambda \left(D_{Lab}(t)\right)^2 dt\right] \tag{194}$$

Now, we note that we will ultimately be measuring interference patterns in position space, so we will consider the position space wavefunction, which can be obtained from the momentum space wavefunction by applying an inverse Fourier transformation:

$$\vec{\Psi}_{DS}(x,T) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \vec{\Psi}_{DS}(p + 2n\hbar k, T) e^{\frac{i}{\hbar}(p+2n\hbar k)x} dp \tag{195}$$

$$= e^{i\phi_\lambda} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \vec{\Psi}_{DS}(p + m\delta v(T), 0) e^{\frac{i}{\hbar}(p+2n\hbar k)x} dp \tag{196}$$

Changing integration variables to $q = p + m\delta v(T)$, our expression becomes:

$$\vec{\Psi}_{DS}(x,T) = e^{i\phi_\lambda} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \vec{\Psi}_{DS}(q, 0) e^{\frac{i}{\hbar}(q - m\delta v(T) + 2n\hbar k)x} dq \tag{197}$$

Noting that our initial position space wavefunction is $\vec{\Psi}_{DS}(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \vec{\Psi}_{DS}(q,0) e^{\frac{i}{\hbar}qx} dq$, we find that:

$$\vec{\Psi}_{DS}(x,T) = e^{i\phi_\lambda} e^{\frac{i}{\hbar}(-m\delta v(T) + 2n\hbar k)x} \vec{\Psi}_{DS}(x,0) \tag{198}$$

The phase term $2nkx$ will be common to both arms of our interferometer, but the phase term $\delta\phi(x) \equiv -\frac{m}{\hbar}\delta v(T)x$ will not be, and it will lead to a position dependent phase shift of:

$$\delta\phi^{arm1}(x) - \delta\phi^{arm2}(x) = -\frac{m}{\hbar} \left[\delta v^{arm1}(T) - \delta v^{arm2}(T)\right] x \tag{199}$$

Recalling that $\delta v(T) = \int_0^T \lambda D_{Lab}(t) dt$, and noting that λx is equal to the difference $g(x) - g(0)$ of the gravitational acceleration at position x versus position 0, we can rewrite our above expression in an enlightening form:

$$\delta\phi^{arm1}(x) - \delta\phi^{arm2}(x) = -\int_0^T \frac{m}{\hbar} \left(g(x) - g(0)\right) \left[D_{Lab}^{arm1}(t) - D_{Lab}^{arm2}(t)\right] dt \tag{200}$$

$$= -\int_0^T \frac{m}{\hbar} \left(g(x) - g(0)\right) \Delta d(t) dt \tag{201}$$

This position dependent phase shift simply implies that for a given x coordinate, the portion

of the phase shift analogous to the first term in equation (162) can be obtained by replacing $g(0)$ with $g(x)$. In a cloud of atoms with a nonzero spread in initial position, the form of our position dependent phase shift ensures that the phase shift corresponding to a particular x will be the same for all atoms in the cloud. This fact opens up the possibility of achieving a compact, high resolution gravity gradiometer using only a single initial cloud, where the value of the gradient can be extracted from measuring the spatial dependence of the phase shift using a high quality camera. The tremendous inherent sensitivity of lattice interferometry will ensure that extremely precise measurements can be achieved even with a small baseline over which to measure the gradient.

We will now consider the contribution to the phase shift arising from ϕ_λ. To compute the phase shift between the two arms, we take the difference of ϕ_λ^{arm1} and ϕ_λ^{arm2}. Note that the contribution arising from the first term in equation (194) will be equal to the phase shift in the absence in the absence of a gradient as given in equation (159). The change in the phase shift arising from the gradient is therefore equal to:

$$\Delta\phi_{gradient} = -\frac{1}{\hbar}\int_0^T \frac{1}{2}m\lambda\left[\left(D_{Lab}^{arm1}(t)\right)^2 - \left(D_{Lab}^{arm2}(t)\right)^2\right]dt \tag{202}$$

In the simple case where the lab frame velocities of the two arms are either opposite or both equal to 0 throughout the duration of the interferometer sequence, as in a standard gravimeter geometry with a hold sequence, $D_{Lab}^{arm1}(t)$ and $D_{Lab}^{arm2}(t)$ will always be opposite. Thus, $\Delta\phi_{gradient}$ will be 0. For this particular case, the point halfway between $D_{Lab}^{arm1}(t)$ and $D_{Lab}^{arm2}(t)$), which we denote as $x_{half}(t)$, remains at the origin. More generally, we can have a geometry where $x_{half}(t)$ moves throughout the interferometer sequence. By definition, $D_{Lab}^{arm1}(t) - x_{half}(t) = x_{half}(t) - D_{Lab}^{arm2}(t)$, and so where we define $D_{half}(t) \equiv D_{Lab}^{arm1}(t) - x_{half}(t)$, $D_{Lab}^{arm2}(t) = x_{half}(t) - D_{half}(t)$. Therefore, we can rewrite equation (202) as:

$$\Delta\phi_{gradient} = -\frac{1}{\hbar}\int_0^T \frac{1}{2}m\lambda\left[(x_{half}(t)+D_{half}(t))^2-(x_{half}(t)-D_{half}(t))^2\right]dt \tag{203}$$

$$= -\frac{1}{\hbar}\int_0^T m\lambda x_{half}(t)(2D_{half}(t))dt \tag{204}$$

Observe that $2D_{half}(t) = \Delta d(t)$ as defined in section VII. Combining equation (204) with equation (162) and neglecting the terms that act only as a source of noise, we find that the phase shift due to ϕ_λ is:

$$\Delta\phi = -\frac{1}{\hbar}\int_0^T m\left[g(0)+\lambda x_{half}(t)\right]\Delta d(t)dt = -\frac{1}{\hbar}\int_0^T mg(x_{half}(t))\Delta d(t)dt \tag{205}$$

Note that the center of mass position of the atomic wavepacket at time t is given by the sum center of mass position at time 0 and $x_{half}(t)$:

$$x_{CM}(t) = x_{CM}(0) + x_{half}(t) \tag{206}$$

After spatial averaging, the position dependent phase term $\delta\phi^{arm1}(x) - \delta\phi^{arm2}(x)$ yields a contribution of $-\int_0^T \frac{m}{\hbar}\left(g(x_{CM}(0)) - g(0)\right)\Delta d(t)dt$. Combining this term with equation (205) and making use of the linearity of g in x, we find that the total spatially averaged phase shift in the presence of a gravity gradient is:

$$\Delta\phi_{avg} = -\frac{1}{\hbar}\int_0^T m\left[g(x_{CM}(0))+\lambda x_{half}(t)\right]\Delta d(t)dt = -\frac{1}{\hbar}\int_0^T m\left[g(0)+\lambda(x_{CM}(0)+x_{half}(t))\right]\Delta d(t)dt \tag{207}$$

$$= -\frac{1}{\hbar}\int_0^T mg(x_{CM}(t))\Delta d(t)dt \tag{208}$$

which confirms our claim at the beginning of this appendix.

We can extend our treatment of a linearly varying gravitational field to describe a gravitational field that varies spatially in an arbitrary manner. In this case, we will replace the potential $\frac{1}{2}\lambda\hat{x}^2$ by the general potential $V(\hat{x})$ such that $V(0) = \frac{dV(0)}{dx} = 0$, which we assume to be small enough so that we can treat the problem perturbatively. In the lab frame, our Hamiltonian is then:

$$\hat{H}_{Lab} = \frac{\hat{p}^2}{2m} + V_0 \sin^2\left[k\hat{x} - kD_{Lab}(t)\right] + mg\hat{x} + V(\hat{x}) \tag{209}$$

As a function of position in the lab frame, the gravitational acceleration will be:

$$g(D_{Lab}(t)) = g(0) + \frac{dV(D_{Lab}(t))}{dx} \tag{210}$$

After transforming into the dressed state frame as in the case of the linear gradient, we obtain the Hamiltonian:

$$\hat{H}_{DS} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt)\hat{p} + V_0 \sin^2(k\hat{x}) + V(\hat{x} + D_{Lab}(t)) \tag{211}$$

From here, it will be useful to Taylor expand $V(\hat{x} + D_{Lab}(t))$ around $D_{Lab}(t)$, writing:

$$V(\hat{x} + D_{Lab}(t)) = V(D_{Lab}(t)) + \frac{dV(D_{Lab}(t))}{dx}\hat{x} + \sum_{n=2}^{\infty} \frac{1}{n!}\frac{d^n V(D_{Lab}(t))}{dx^n}\hat{x}^n \tag{212}$$

We can now rewrite our dressed state Hamiltonian as:

$$\hat{H}_{DS} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt)\hat{p} + V_0 \sin^2(k\hat{x}) + V(D_{Lab}(t)) + \frac{dV(D_{Lab}(t))}{dx}\hat{x} + \sum_{n=2}^{\infty} \frac{1}{n!}\frac{d^n V(D_{Lab}(t))}{dx^n}\hat{x}^n \tag{213}$$

As in the case of the linear gravity gradient, we will work in a boosted dressed state frame

such that the term linear in $\hat{x}$ no longer appears in the Hamiltonian. In our more general case, we will have $\delta g(t) \equiv g(D_{Lab}(t)) - g(0) = \frac{dV(D_{Lab}(t))}{dx}$, with the corresponding velocity boost of $\delta v(t) \equiv \int_0^t \delta g(t')dt'$ as before. Our desired boosted dressed state Hamiltonian will then be:

$$\hat{H}_{DS_{boost}} = \frac{\hat{p}^2}{2m} - (v_{Lab}(t) + gt + \delta v(t))\hat{p} + V_0 \sin^2(k\hat{x}) + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{d^n V(D_{Lab}(t))}{dx^n} \hat{x}^n \tag{214}$$

We will again use a unitary transformation of the form $\hat{U}_{DS_{boost}}(t) = e^{-\frac{i}{\hbar} m\delta v(t)\hat{x}} e^{\frac{i}{\hbar}\kappa(t)}$, and the relation $\hat{H}_{DS} = \hat{U}_{DS_{boost}}(t)\hat{H}_{DS_{boost}}\hat{U}^{\dagger}_{DS_{boost}}(t) + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_{DS_{boost}}(t)\right)\hat{U}^{\dagger}_{DS_{boost}}(t)$ will determine that:

$$\kappa(t) = -\int_0^t V(D_{Lab}(t'))dt' - \int_0^t \left[\frac{1}{2} m\delta v(t')^2 + m(v_{Lab}(t') + gt')\delta v(t')\right] dt' \tag{215}$$

in analogy to the case of the linear gravity gradient. Now, we have that:

$$|\Psi_{DS}(T)\rangle = \hat{U}_{DS_{boost}}(T)\hat{T}_{DS_{boost}}(T,0)\hat{U}^{\dagger}_{DS_{boost}}(0)\,|\Psi_{DS}(0)\rangle \tag{216}$$

As in our previous calculation, $\hat{U}^{\dagger}_{DS_{boost}}(0)$ will be the identity operator so that $\vec{\Psi}_{DS_{boost}}(p,0) = \vec{\Psi}_{DS}(p,0)$. As before, we perform time evolution in the boosted dressed state frame under the adiabatic approximation, noting that we will once again have a position dependent energy shift arising from the perturbation $V' \equiv \sum_{n=2}^{\infty} \frac{1}{n!} \frac{d^n V(D_{Lab}(t))}{dx^n} \hat{x}^n$. The net effect of this energy shift on the phase difference between the arms of the interferometer will be negligible in many cases, but in some situations it may be necessary to take into account. In general, smaller higher order derivatives of the gravitational potential and a smaller separation between the two arms will decrease the net effect of the energy shift. Moreover, the energy shift can be made significantly smaller by decreasing the spatial spread of the atomic wavepackets in the

axis along which we are performing our gravitational measurements, since the expectation value of all the terms in V' in our ground state wavefunction will be quadratic or of higher order in this spread. We note that in order to characterize the gravitational field with a high spatial resolution, an ideal option is to use an array of closely spaced lattice gravimeters with small separations between their respective arms, as illustrated in Figure 6. We will proceed with our phase shift calculation neglecting any contributions from the energy shift arising from our perturbation, where we note that any such contributions can be determined through perturbative calculation if necessary.

In direct analogy to the case of the linear gradient, we will obtain the result:

$$\vec{\Psi}_{DS}(p + 2n\hbar k, T) = e^{i\phi_\lambda}\vec{\Psi}_{DS}(p + m\delta v(T), 0) \tag{217}$$

where:

$$\phi_\lambda = \frac{1}{\hbar}\left[\int_0^T \frac{1}{2}m(v_{Lab}(t) + gt)^2 dt - \int_0^T V(D_{Lab}(t))dt\right] \tag{218}$$

And as before, we will have a position dependent phase shift of $\delta\phi^{arm1}(x) - \delta\phi^{arm2}(x) = -\frac{m}{\hbar}\left[\delta v^{arm1}(T) - \delta v^{arm2}(T)\right]x$. The phase shift resulting from ϕ_λ proves to be a generalization of the analagous phase shift for the linear gradient case:

$$\Delta\phi = -\frac{1}{\hbar}\int_0^T mg(0)\Delta d(t)dt - \frac{1}{\hbar}\int_0^T \left[V\left(D_{Lab}^{arm1}(t)\right) - V\left(D_{Lab}^{arm2}(t)\right)\right]dt \tag{219}$$

As in our discussion at the end of section III., our result is exactly what we would expect from naively treating the atoms as classical particles traveling along trajectories determined by the lattices and integrating the Lagrangian over these trajectories. This correspondence is the result of the fact that we can typically choose the parameters of the interferometer sequence so that the quantum nature of the problem–as embodied in non-adiabatic corrections and perturbative energy shifts (the latter of which are due to the finite spatial spread of the atomic wavepackets)–makes a negligible overall contribution to the phase

shift. In particular, we can choose an interferometer geometry with sufficient symmetry so that non-adiabatic corrections cancel to an extremely large degree, and in many cases we can choose the spatial spread of the wavepackets to be sufficiently small so that the net contribution of the energy shifts to the phase shift is negligible. But we emphasize that the phase shift calculation is inherently quantum mechanical in nature, and it is only appropriate to discuss any sort of classical correspondence after carefully carrying out this calculation making proper use of quantum mechanics.

Appendix B: Derivation of the Cancellation of Lowest Order Non-Adiabatic Corrections Proportional to the Lattice Acceleration

In this appendix, we will derive the result stated in equation (115) of section V., which states that the contribution to $C_{0\to 1\to 0}(2T_{ramp}+T_{accel})$ that is proportional to $\dot{\alpha}$ is $\delta_{\dot{\alpha}}=0$, under the assumption that the lattice depth decrease ramp is the time reversed lattice depth increase ramp. We recall from section V. that the dominant contribution to the second order non-adiabatic correction to $b_0(t)$ is given by $\delta_{\dot{\alpha}^2}+\delta_{\dot{\alpha}}$, where we neglect corrections that are common to both arms of the interferometer. It follows from equation (105) and the subsequent discussion that:

$$\delta_{\dot{\alpha}} = -e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''}\int_{T_{ramp}}^{T_{ramp}+T_{accel}} b_1^{(1)}(T_{ramp})\frac{\dot{\alpha}(t')M_{\alpha_{(10)}}(t')}{-\Delta\varepsilon_{10}(t')}e^{\frac{i}{\hbar}\int_{T_{ramp}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \quad (220)$$

$$-e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''}e^{\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''}\times \quad (221)$$

$$\times \int_{T_{ramp}+T_{accel}}^{2T_{ramp}+T_{accel}} b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel}) \frac{\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{10}}(t')}{-\Delta\varepsilon_{10}(t')} e^{\frac{i}{\hbar}\int_{T_{ramp}+T_{accel}}^{t'}\Delta\varepsilon_{10}(t'')dt''}dt' \tag{222}$$

Let us denote the first and second terms in the above expression for $\delta_{\dot{\alpha}}$ by $T1$ and $T2$, respectively. After comparison with equation (109), it is apparent that:

$$T1 = -b^{(1)}_{1}(T_{ramp})\left[b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel})\right]^{*} \tag{223}$$

We can rearrange our equation for $T2$ by making the following subsitutions: $\tau = t' - (T_{ramp} + T_{accel})$ and $\tau' = t'' - (T_{ramp} + T_{accel})$. Since we assume that the lattice depth decrease ramp is the time reversed lattice depth increase ramp, for $0 \leq \tau \leq T_{ramp}$ we will have the relations $\Delta\varepsilon_{10}(t') = \Delta\varepsilon_{10}(T_{ramp} - \tau)$ and $\dot{\tilde{\Omega}}(t')M_{\tilde{\Omega}_{10}}(t') = -\dot{\tilde{\Omega}}(T_{ramp} - \tau)M_{\tilde{\Omega}_{10}}(T_{ramp} - \tau)$ (where we have a negative sign because the direction of change in the lattice depth reverses from the upward ramp to the downward ramp). Substitution yields:

$$T2 = -e^{\frac{i}{\hbar}\int_{0}^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel}) e^{\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''} \times \tag{224}$$

$$\times \int_{0}^{T_{ramp}} \frac{\dot{\tilde{\Omega}}(T_{ramp}-\tau)M_{\tilde{\Omega}_{10}}(T_{ramp}-\tau)}{\Delta\varepsilon_{10}(T_{ramp}-\tau)} e^{\frac{i}{\hbar}\int_{0}^{\tau}\Delta\varepsilon_{10}(T_{ramp}-\tau')d\tau'}d\tau \tag{225}$$

We can now make the additional substitutions $u = T_{ramp} - \tau$ and $u' = T_{ramp} - \tau'$, and we can rewrite $T2$ as:

$$T2 = -e^{\frac{i}{\hbar}\int_{0}^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel}) e^{\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''} \times \tag{226}$$

$$\times \int_{T_{ramp}}^{0} \frac{\dot{\tilde{\Omega}}(u)M_{\tilde{\Omega}_{10}}(u)}{\Delta\varepsilon_{10}(u)} e^{\frac{i}{\hbar}\int_{T_{ramp}}^{u}\Delta\varepsilon_{10}(u')(-du')}(-du) \tag{227}$$

$$= -e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel}) e^{\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''} \times \tag{228}$$

$$\times \int_0^{T_{ramp}} \frac{\dot{\tilde{\Omega}}(u) M_{\tilde{\Omega}_{10}}(u)}{\Delta\varepsilon_{10}(u)} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{u}\Delta\varepsilon_{10}(u')du'} du \tag{229}$$

where we note that $du = -d\tau$ and that $du' = -d\tau'$. Recalling equation (111), we see that:

$$b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel}) = -i\hbar \frac{\dot{\alpha} M_{\alpha_{(10)}}}{\left(\Delta\varepsilon_{10_{accel}}\right)^2} e^{-\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t')dt'} \left(e^{-\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}T_{accel}} - 1\right) \tag{230}$$

Moreover, because we assume that $\Delta\varepsilon_{10}(t)$ takes on the constant value $\Delta\varepsilon_{10_{accel}}$ during the lattice acceleration stage, as discussed in section V., we can write $e^{\frac{i}{\hbar}\int_{T_{ramp}}^{T_{ramp}+T_{accel}}\Delta\varepsilon_{10}(t'')dt''} = e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}T_{accel}}$. We can thus regroup the factors in $T2$ to obtain the following result:

$$T2 = -e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')dt''} \left[-i\hbar \frac{\dot{\alpha} M_{\alpha_{(10)}}}{\left(\Delta\varepsilon_{10_{accel}}\right)^2} e^{-\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t')dt'} \left(1 - e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}T_{accel}}\right)\right] \times \tag{231}$$

$$\times \int_0^{T_{ramp}} \frac{\dot{\tilde{\Omega}}(u) M_{\tilde{\Omega}_{10}}(u)}{\Delta\varepsilon_{10}(u)} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{u}\Delta\varepsilon_{10}(u')du'} du \tag{232}$$

$$= \left[i\hbar \frac{\dot{\alpha} M_{\alpha_{(10)}}}{\left(\Delta\varepsilon_{10_{accel}}\right)^2} e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')} \left(1 - e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}T_{accel}}\right)\right] \times \tag{233}$$

$$\times \int_0^{T_{ramp}} \frac{\dot{\tilde{\Omega}}(u) M_{\tilde{\Omega}_{10}}(u)}{\Delta\varepsilon_{10}(u)} e^{-\frac{i}{\hbar}\int_0^{u}\Delta\varepsilon_{10}(u')du'} du \tag{234}$$

where we have noted that $e^{-\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t')dt'} e^{-\frac{i}{\hbar}\int_{T_{ramp}}^{u}\Delta\varepsilon_{10}(u')du'} = e^{-\frac{i}{\hbar}\int_0^{u}\Delta\varepsilon_{10}(u')du'}$ since t' is just a dummy variable that can be relabeled as u'. We can easily see that:

$$\left[i\hbar\frac{\dot{\alpha}M_{\alpha_{(10)}}}{(\Delta\varepsilon_{10_{accel}})^2}e^{\frac{i}{\hbar}\int_0^{T_{ramp}}\Delta\varepsilon_{10}(t'')}\left(1-e^{\frac{i}{\hbar}\Delta\varepsilon_{10_{accel}}T_{accel}}\right)\right] = -\left[b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel})\right]^* \quad (235)$$

Furthermore, examination of equation (98) tells us that $\int_0^{T_{ramp}}\frac{\dot{\tilde{\Omega}}(u)M_{\tilde{\Omega}_{10}}(u)}{\Delta\varepsilon_{10}(u)}e^{-\frac{i}{\hbar}\int_0^u\Delta\varepsilon_{10}(u')du'}du = -b_1^{(1)}(T_{ramp})$. We therefore find that:

$$T2 = b_1^{(1)}(T_{ramp})\left[b^{(1)}_{1_{accel}}(T_{ramp}+T_{accel})\right]^* = -T1 \quad (236)$$

and hence that:

$$\delta_{\dot{\alpha}} = T1 + T2 = 0 \quad (237)$$

as desired.

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