QUANTUM TRAJECTORIES AND FEEDBACK

By

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A thesis submitted to the University of Queensland
for the degree of Doctor of Philosophy
Department of physics
November 1994

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Howard Mark Wiseman

Never speak more clearly than you think — *Neils Bohr* [1]

[1] quoted on p. 178 of the biography by Pais [107]

Acknowledgments

Primary acknowledgments are due to my thesis supervisor, Dr. Gerard Milburn. He introduced me to the captivating and rewarding area of the quantum theory of feedback as my thesis supervisor in my Honours year. Even earlier than that, his second year lectures in quantum mechanics were my first formal introduction to that subject, and confirmed my desire to become a quantum physicist. Over the course of my PhD, he has always been approachable and helpful. In particular, I value the way he kept a broad view of the problems. When I could see no way to generalize some result, he was confident that it could be done, and usually this was the case. I am grateful for the general advice and encouragement he gave me, especially in publishing alone and applying for post-Doctoral positions. I also express my gratitude to my associate supervisor Prof. Peter Drummond, for his physical insight. We had differences of opinion on more than one occasion, and almost invariably he turned out to be correct.

Outside this department, I must thank Prof. Howard Carmichael for his inspiring research seminar at the Australian National University Summer School of January 1992 on the topic of quantum trajectories. At that time, I was searching for a PhD topic, and quickly became fascinated by this new area. I value the communication we have had since then. Correspondence from Prof. Crispin Gardiner and Bill Munro has been valuable as well. Their disagreement with my work forced me to try to express it more clearly. This thesis also owes much to my contact with the experimental and theoretical quantum optics group at ANU, especially Matthew Taubman and Dr. Hans-A. Bachor. I thank them for suggesting and testing new applications for quantum feedback theory, for showing me the experimental side of quantum optics, and for their patience in the face of our changing opinions. I have also had useful discussions with Drs. Barry Sanders and Jim Cresser from Macquarie University.

To my colleagues in the Department, especially the postgraduates in quantum optics and laser physics, I give thanks for stimulating intellectual and social interactions. In particular, I appreciated Sigurd Dyrting for his analytical skills, and Drs. Michael Gagen and Kingsley Jones for their imaginative ideas. Rodney McDuff and Paul Kinsler deserve mention for enabling me to use the Department's computing resources. Bruce Wielinga and Ariel Liebman were, along with Sigurd Dyrting, valued room mates and companions. Finally, I thank my wife Nadine for bearing with me throughout this thesis while she pursued her own research.

List of Publications

In this chronological list of refereed publications, those on which this thesis is based are asterisked.

1. H.M. Wiseman and G.J. Milburn, "Noise reduction in a laser by nonlinear damping", Phys. Rev. A **44**, 7815 (1991).
2. H.M. Wiseman and G.J. Milburn, "Reduction in laser-intensity fluctuations by a feedback-controlled output mirror", Phys. Rev. A **46**, 2853 (1992).
3. * H.M. Wiseman and G.J. Milburn, "Quantum theory of field-quadrature measurements", Phys. Rev. A **47**, 642 (1993).
4. * H.M. Wiseman and G.J. Milburn, "Quantum theory of optical feedback via homodyne detection", Phys. Rev. Lett. **70**, 548 (1993).
5. * H.M. Wiseman and G.J. Milburn, "Interpretation of quantum jump and diffusion processes illustrated on the Bloch sphere" Phys. Rev. A **47**, 1652 (1993).
6. * H.M. Wiseman, "Stochastic quantum dynamics of a continuously monitored laser" Phys. Rev. A **47**, 5180 (1993).
7. M.J. Gagen, H.M. Wiseman, and G.J. Milburn, "Continuous position measurement and the quantum Zeno effect", Phys. Rev. A **48**, 132 (1993).
8. * H.M. Wiseman and G.J. Milburn, "Squeezing via feedback" Phys. Rev. A **49**, 1350 (1994).
9. * H.M. Wiseman, "Quantum theory of continuous feedback" Phys. Rev. A **49**, 2133 (1994).
10. * H.M. Wiseman and G.J. Milburn, "All-optical versus electro-optical feedback" Phys. Rev. A **49**, 4110 (1994).
11. * H.M. Wiseman and G.J. Milburn, Reply to L. Diosi and N. Gisin, Comment on "Quantum theory of optical feedback via homodyne detection", Phys. Rev. Lett. **72**, 4054 (1994)
12. * H.M. Wiseman, "Using feedback to eliminate back-action in quantum measurements", to be published in Phys. Rev. A.
13. * H.M. Wiseman, M.S. Taubman, and H-A. Bachor, "Feedback-enhanced squeezing in secon harmonic generation", submitted to Phys. Rev. A
14. * H.M. Wiseman "$SU(2)$ distribution functions and measurement of the fluorescence of a two-level atom", submitted to Quantum Optics

Abstract

A quantum trajectory describes the evolution of a continuously monitored quantum system conditioned on the measurement record. In this thesis I consider quantum trajectories for optical measurement schemes, such as direct and homodyne detection. Those for heterodyne detection and detection in the presence of white noise are derived for the first time. The quantum trajectories for various systems are investigated analytically and numerically, and their interpretation discussed. The principle rôle for quantum trajectories in this thesis is to treat feedback, where the detected photocurrent is used to control the system dynamics. In the Markovian limit, a master equation incorporating the feedback can be derived. In this derivation I use a new approach to stochastic differential equations. I also treat feedback using quantum Langevin equations, which do not involve quantum measurement theory. This is necessary for all-optical feedback, which I compare and contrast with electro-optical feedback. Applications for quantum feedback include noise reduction, and removing the back-action in continuous quantum measurements (a new concept).

List of Symbols

The following list is neither exhaustive nor exclusive, but may be helpful. It generally contains only those symbols used in more than one chapter.

a	annihilation operator for an optical mode
$\mathcal{A}$	the anticommutating superoperator: $\mathcal{A}[r]\rho = \frac{1}{2}\{r^\dagger r, \rho\}$
b	annihilation operator for a traveling field, obeying $[b(t), b^\dagger(t')] = \delta(t-t')$
c	system operator coupled to a bath by dipole interaction
c	as a subscript indicates a conditioned state
$\mathcal{D}$	$= \mathcal{J} - \mathcal{A}$; the irreversible evolution generator
dB	$= b\,dt$; bose bath operator analog of infinitesimal Wiener increment
dN	infinitesimal increment in a point process
dV	$= \zeta dt$; infinitesimal increment in a complex Wiener process
dW	$= \xi dt$; infinitesimal increment in a real Wiener process
$\mathrm{E}[\cdots]$	expectation value of $\cdots$
F	Hamiltonian multiplied by the homodyne photocurrent to give feedback
g	a coupling constant
$\mathcal{G}$	the nonlinear superoperator: $\mathcal{G}[r]\rho = \mathcal{J}[r]\rho/\mathrm{Tr}[r^\dagger r\rho] - \rho$
h	a feedback loop response function
H	a Hamiltonian
$\mathcal{H}$	the nonlinear superoperator: $\mathcal{H}[r]\rho = r\rho + \rho r^\dagger - \mathrm{Tr}[(r+r^\dagger)\rho]\rho$
I	a photocurrent
$\Im$	the imaginary part of
$\mathcal{J}$	the jump superoperator: $\mathcal{J}[r]\rho = r\rho r^\dagger$
$\mathcal{K}$	a Liouville superoperator controlled by feedback
L	$= 2N + 1 + M + M^*$; quadrature variance of a white noise bath
$\mathcal{L}$	a Liouville superoperator
$m, \lvert m\rangle$	a photon number, eigenstate
M	complex amplitude variance of a white noise bath
$n, \lvert n\rangle$	a photon number, eigenstate
N	intensity of a white noise bath
$\mathcal{O}$	an operation
p	a probability distribution
$\mathrm{Pr}[\cdots]$	probability for $\cdots$
P	Glauber-Sudarshan P function

Q	Husimi Q function
r	an arbitrary operator
R	a density operator for two interacting systems
$\Re$	the real part of
s	an arbitrary system operator
S	a spectrum
T	duration of measurement
$\mathrm{Tr}[\cdots]$	Trace of $\cdots$
U	a unitary operator
U, u	normally ordered variance
V	an Hermitian operator
V, v	true variance
W	an effect for an operation: $\mathrm{Tr}[\mathcal{O}\rho] = \mathrm{Tr}[\rho W]$
W	Wigner W function
x	$= c + c^\dagger$; x quadrature for a system dipole
y	$= -\mathrm{i}c + \mathrm{i}c^\dagger$; y quadrature for a system dipole
Z	Hamiltonian multiplied by direct photocurrent to give feedback
α	a measurement result
$\alpha, \|\alpha\rangle$	a coherent amplitude, state
β	a traveling coherent field amplitude
γ	a local oscillator coherent amplitude
γ	a damping rate
Γ	a rate for some derived irreversible process
Δ	a detuning
ζ	Gaussian complex white noise obeying $\mathrm{E}[\zeta^*(t)\zeta(t')] = \delta(t-t')$
η	an efficiency
κ	a cavity damping rate
λ	feedback gain parameter
ν	vacuum annihilation operator for traveling field
ξ	Gaussian white noise, especially of x quadrature, obeying $\mathrm{E}[\xi(t)\xi(t')] = \delta(t-t')$
ρ	a density operator = a state matrix
σ	a Pauli matrix
τ	feedback loop time delay
υ	Gaussian white noise of y quadrature
ϕ, φ	an optical phase
χ	an optical nonlinearity strength
ψ	a state vector
Ψ	a state vector for two or more interacting systems
ω	a frequency
Ω	a measurement operator
$\tilde{}$	indicates an unnormalized conditioned state
$\tilde{}$	indicates a Fourier transform in time

List of Abbreviations

BAE	Back-Action Evading
DPO	Degenerate Parametric Oscillator
FPE	Fokker-Planck Equation
QLE	Quantum Langevin Equation
QND	Quantum Non-Demolition
QPP	Quantum Point Process
SDE	Stochastic Differential Equation
SME	Stochastic Master Equation
SSE	Stochastic Schrödinger Equation
TLA	Three Letter Acronym

Précis: Quantum Feedback

The following is a very brief technical summary of the most important results of this thesis [1].

Consider an optical field in one dimension with canonical commutation relations

$$[b(z,t), b^\dagger(z',t)] = v\delta(z-z'), \tag{0.1}$$

propagating at speed v. Let it be coupled to a system at position $z=0$ by the Hamiltonian

$$V_{\rm dipole}(t) = {\rm i}[b^\dagger(0,t)c(t) - c^\dagger(t)b(0,t)], \tag{0.2}$$

where a rotating-wave approximation has been used [2]. Let the output photocurrent be fed back to control the system via

$$V_{\rm feedback}(t) = Z(t)b^\dagger(v\tau,t)b(v\tau,t). \tag{0.3}$$

For a vacuum input field [zero eigenstate of $b(z,t)$ for $z<0$], the explicit increment in an arbitrary system operator is

$$\begin{aligned} ds &= {\rm i}[H,s]dt - [s,c^\dagger]\left(\tfrac{1}{2}c + b_0\right)dt + \left(\tfrac{1}{2}c^\dagger + b_0^\dagger\right)[s,c]dt \\ &\quad + [c^\dagger(t-\tau) + b_0^\dagger(t-\tau)]\left(e^{{\rm i}Z}se^{-{\rm i}Z} - s\right)[c(t-\tau) + b_0(t-\tau)]dt, \end{aligned} \tag{0.4}$$

where $b_0(t) \equiv b(0^-,t)$ commutes with $s(t)$ and obeys

$$b_0(t)b_0^\dagger(t)dt = 1, \tag{0.5}$$

with other such moments vanishing [2]. For finite delay τ, the output field operator $b_0(t-\tau)+c(t-\tau)$ commutes with $s(t)$, but for $\tau=0$ it does not. In the latter case, one obtains the corresponding feedback master equation

$$\dot\rho = -{\rm i}[H,\rho] + e^{-{\rm i}Z}c\rho c^\dagger e^{{\rm i}Z} - \tfrac{1}{2}\{c^\dagger c, \rho\}. \tag{0.6}$$

This has an obvious interpretation in terms of the quantum jumps associated with photodetection in the theory of quantum trajectories [3].

[1] H.M. Wiseman, *Quantum Trajectories and Feedback* (Physics Department, University of Queensland, 1994)
[2] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[3] H.J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag, Berlin, 1993)

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Chapter 1

Introduction

The purpose of this introduction is to introduce the two central topics of this thesis, quantum trajectories and feedback, and to outline the organization of this thesis. I introduce quantum trajectories by first describing classical trajectories, and arguing that the essential difficulty in defining a quantum analog is entanglement. This difficulty can be resolved by invoking quantum measurement theory. The idea of feeding back a quantum measurement result then follows naturally. There is an alternative approach to quantum-limited feedback which treats the feedback apparatus as a physical quantum system. In order to reconcile these two approaches in principle, I briefly discuss quantum metaphysics in Sec. 1.2. A simple example of quantum feedback, utilizing the Stern-Gerlach device, is treated in both ways in Sec. 1.3. The idea of environmental decoherence is introduced in discussing why the measurement and no-measurement treatments are not experimentally distinguishable in practice. Sec. 1.4 is the outline for the remainder of the thesis.

1.1 Quantum and Classical Trajectories

1.1.1 Classical Trajectories

The concept of a trajectory presents no difficulties in classical mechanics; it can be defined to be the path followed by a point in phase space. For closed systems, the trajectory will be given by Hamilton's equations. Such a trajectory is deterministic and reversible in time. If two systems interact, each system will have a well-defined trajectory in its own phase space. For large numbers of interacting systems, it may be impractical to try to solve Hamilton's equations. Often, however, it is possible to make assumptions which allow an approximate solution to be found. If one is only interested in one system, then one may ignore the detailed dynamics of the other systems by treating them statistically. That is to say, the other systems are treated as a bath or reservoir. This is possible if the system is weakly coupled to the bath, and if the dynamics are such as to distribute the information about the system among the many bath degrees of freedom. The result of these approximations is a trajectory for the open system alone, which may now be irreversible and stochastic. Sometimes, it is possible to make the further approximation that information is dissipated so rapidly in the bath that the system dynamics becomes Markovian. That is to say, the rate of change of the system depends only on its current state.

An example which illustrates these features is one which will feature heavily in this thesis: a damped optical cavity. Let the cavity have a frequency ω_0 and an intensity damping rate $\kappa \ll \omega_0$.

Represent the field by a complex amplitude α, so that $\hbar\omega_0|\alpha|^2$ is the energy of the cavity. Note that the presence of Planck's constant here is merely a convenient normalization. Approximating the external field modes as a thermal reservoir at temperature T, it is possible to derive the following stochastic Markovian equation for α:

$$\dot{\alpha}(t) = -\left(\mathrm{i}\omega_0 + \frac{\kappa}{2}\right)\alpha(t) + \sqrt{\kappa N}\zeta(t). \tag{1.1}$$

Here, $N = 1/[\exp(\hbar\omega_0/k_B T) - 1]$, and $\zeta(t)$ represents complex Gaussian white noise [50]. Physically, $\zeta(t)$ arises from the fluctuating amplitude of the field incident on the cavity. This equation (which can be called a Langevin equation [50]) describes a continuous but noisy trajectory for the system.

Because of the stochasticity, different trajectories can follow from the same initial conditions. Thus, it is useful to consider an ensemble of trajectories obeying Eq. (1.1). This ensemble can be represented by a distribution function $p(\alpha, t)$, which can be shown to obey [50]

$$\dot{p}(\alpha, \alpha^*; t) = \left[\partial_\alpha\left(\mathrm{i}\omega_0 + \frac{\kappa}{2}\right)\alpha + \partial_{\alpha^*}\left(-\mathrm{i}\omega_0 + \frac{\kappa}{2}\right)\alpha^* + \kappa N\partial_\alpha\partial_{\alpha^*}\right]p(\alpha, \alpha^*; t), \tag{1.2}$$

where $\partial_\alpha = \partial/\partial\alpha$ *et cetera*. In classical mechanics, it is possible to take either the evolution equation for the distribution function (which happens to be a Fokker-Planck equation in this case), or the stochastic trajectory equation (1.1), to be the fundamental description of the irreversible process. That is because there is a one-to-one correspondence between the two. However, the stochastic trajectory description seems more pleasing, because it enables open systems to be described by the same formal structure as is natural for closed systems (although see Ref. [115] for a dissenting view).

1.1.2 Quantum Entanglement

Quantum mechanics is in some ways quite similar to classical mechanics, and in other ways very different. The aspect presented by a particular quantum system depends strongly on the way the dynamics is expressed. For instance, Heisenberg's equations of motion for operators are often formally identical to Hamilton's equations for the corresponding classical variables. Such equivalence tends to obscure the peculiar features of quantum mechanics. It is probably partly for this reason that traditional introductions to quantum mechanics use the Schrödinger rather than the Heisenberg picture, and I will do so also. The fundamental equation of motion, analogous to Hamilton's equations, is then Schrödinger's equation for the system wavefunction. Like Hamilton's equations, it is deterministic and reversible. An important difference is that the wavefunction is a function in configuration space, not phase space. This is related to Heisenberg's uncertainty principle for position and momentum, which is a significant departure from classical mechanics.

If the uncertainty principle for the position and momentum of a particle were the only difference between classical and quantum mechanics, then it is doubtful that quantum mechanics would be regarded as really distinct from classical mechanics. Schrödinger's wave equation for a single particle can, after all, be written in a form similar to the Hamilton-Jacobi equation [66]. In fact, the only difference is an extra term in the Hamiltonian, the so-called quantum potential of Bohm [16]. For a single particle, the quantum potential takes the form of a real potential in 3-space, which is a (possibly unbounded) function of the modulus of the wavefunction and its derivatives. This gives the simple interpretation of the wavefunction as a field with the same status as Maxwell's fields, producing a "quantum force" on a particle, enabling it to tunnel through barriers and to do other such "quantum" acts. In this picture, particles really have well-defined position and momenta, and Heisenberg's uncertainty principle appears due to the inherent clumsiness of a macroscopic observer.

The reason that this superficially satisfying interpretation of Bohm does not have wider currency is a good one: it fails for more than one particle. That is not to say that the theory fails, but rather that it fails to be satisfying. The wavefunction for two particles exists not in 3-space, but in 6-dimensional configuration space. Hence the quantum potential cannot be regarded as a potential like the electric potential. The underlying reason why Bohm's idea fails is the aspect of quantum mechanics much stranger than mere uncertainty: entanglement. Entanglement occurs whenever quantum systems interact. The total system may be described by a joint wavefunction, but it is not possible in general to assign a separate wavefunction for each subsystem. Thus, the concept of a quantum trajectory as the evolution of a wavefunction in a system's configuration space breaks down as soon as the system interacts with another system. This was recognized by Schrödinger, and caused him to abandon his initial realistic interpretation for the wavefunction [122]. It was also the major source of Einstein's disquiet about quantum mechanics [45]

1.1.3 Quantum Trajectories

Although an interacting system cannot in general be assigned a wavefunction, it can be assigned a more general state, a density operator ρ. In order to leave behind connotations of wave-particle duality, the wavefunction will in general be called the *state vector*, and the density operator the *state matrix*, and configuration space will be replaced by the more general Hilbert space. The state matrix represents an ensemble of state vectors, much as a classical probability distribution represents an ensemble of points in phase space (although there is an important difference which will be discussed soon). As in the classical case, it is only possible to derive an equation of motion for the state matrix of an open system if the other systems with which it interacts may be treated as a bath. If the bath quickly dissipates the information about the system, then a Markovian evolution equation can be derived, called a *master equation*. It is deterministic because it describes an ensemble, and is irreversible. The quantum analog to the classical Fokker-Planck equation (1.2) for a damped cavity is the master equation

$$\dot{\rho} = -i\omega_0[a^\dagger a, \rho] + \frac{\kappa}{2}(N+1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \frac{\kappa}{2}N(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger), \qquad (1.3)$$

where a is the annihilation operator for the cavity mode, and N is as before.

There is obviously a strong analogy between the classical equation of motion for a distribution function in phase space, and the quantum master equation. What then is the quantum analog of the individual trajectories which make up the classical ensemble? As noted above, it is possible to write down Heisenberg equations of motion which often appear identical to the classical Langevin equations. These can be known as quantum Langevin equations [52]. However, this only gives the appearance of a trajectory in the system's Hilbert space; in fact, the quantum Langevin equations include operators from other Hilbert spaces because the entanglement is still there. A better quantum analog would be a stochastic equation of motion for the system's state vector, since this does exist only in the system's Hilbert space. If this could be done, then it would seem that the master equation for the state matrix could then be regarded as merely an ensemble of quantum trajectories representing the actual stochastic evolution of individual systems.

Such quantum trajectories can be defined, and have attracted a great deal of interest lately [21, 37, 54, 58, 135, 151]. The term "quantum trajectory" in this context was coined by Carmichael [21], who calls the ensemble of quantum trajectories an "unraveling" of the master equation. Unlike the classical case, there are many different unravelings for the same master equation. The nature of the trajectories in these distinct ensembles may be quite different, some being jump-like and some diffusive. The reason for this is that there is in general no unique way to decompose a state matrix into an ensemble of state vectors . This is another characteristic feature of quantum mechanics. The existence of different unravelings is a problem for the interpretation of the trajectories as representing actual stochastic evolution of individual systems. Since there is no reason to favour any one ensemble over any other, it is impossible to attach any fundamental significance to a particular ensemble of quantum trajectories. Physically, the naïve interpretation of the master equation fails because the system state is still entangled with its environment (the bath), so there is no way that it can be unambigously assigned a state vector.

1.1.4 Quantum Theory of Measurement

So far, the present discussion of quantum theory has been limited to quantum mechanics. By quantum *mechanics*, I mean the physical part of quantum theory, consisting of evolution which can be generated by a quantum Hamiltonian. This includes the irreversible, nonunitary evolution which results from a system interacting with a bath, because it is an approximation derived within quantum mechanics. However, there is another side to quantum theory: changes in the system state which are fundamentally nonunitary and even nonlinear. I am of course referring to measurement, which occupies a privileged position in quantum theory. Measurement offers a way to disentangle systems which have become entangled. Specifically, if two systems interact and become entangled, and then a measurement is made on one system, projecting it into a pure state, then the second system will also be projected into a pure state. In the context of a system interacting with a bath, this means that a perfect measurement of the bath after it has interacted with the system will project the system into a pure state. This gives an interpretation for some of the ensembles of quantum trajectories mentioned above. Different ensembles can result from different choices of measurement on the bath. It is this freedom of choice in measurements which gives rise to the violation of Bell's inequalities [10], and so it is not surprising that it also disallows a simple interpretation of master equation evolution as a unique ensemble of quantum trajectories.

In this point of view, **a quantum trajectory is the evolution of a system conditioned on the results of measurements made on that system.** Because of entanglement, a measurement of the bath is a measurement of the system, even if it is far removed in space. The change in the state of the system due to the measurement is not a physical process; it is the change in the observer's state of knowledge about the system based on the receipt of new information. Now if the concept of a quantum trajectory is to be useful, it should apply to laboratory experiments, where measurements are not perfectly efficient, and there are often sources of irreversibility which are not accessible to measurement at all. Thus it is too limiting to restrict quantum trajectories to those cases when the knowledge about the system is perfect, and it can be described by a state vector. The definition I am using for a quantum trajectory includes any sort of quantum evolution, ranging from the Schrödinger equation for closed systems, to the master equation for open (but unobserved) systems, to the stochastic, nonlinear, nonunitary stochastic Schrödinger equations and stochastic master equations of open systems subject to observation. To understand the nature of such quantum trajectories is the first aim of this thesis.

1.1.5 Quantum Theory of Feedback

If one has an evolution equation which describes the change in the state of an open quantum system, conditioned on a measurement result, then one can consider feeding back that measurement result. That is to say, one can use the measurement result to alter the future dynamics of the system. Quantum-limited feedback in open systems is an area of study which has arisen within the past decade, independent of quantum trajectories. That it was not considered earlier is probably due to the fact that the relevant technology was not sufficiently quantum-limited to make a quantum theory of feedback necessary. The first attempts to include the effect of quantum noise [155, 73, 126] did so in an approximate way, so that the quantum fluctuations were indistinguishable from the classical fluctuations in the anti-quantum theory of stochastic electrodynamics [99]. In the case of traveling wave feedback as in Ref. [126], this was recognized as a necessary approximation, but for feedback onto a cavity it is possible to deal with the quantum noise exactly. The second aim of this thesis is to give this exact theory of quantum-limited feedback. This is done using quantum trajectories, and also quantum Langevin equations. The latter method is necessary to show the relationship with the early, approximate theories. Also, I will investigate the action of feedback on some simple systems, with practical applications.

A quantum theory of feedback using quantum trajectories explicitly includes a measurement step. Such a step is not part of quantum mechanics, in the sense defined above. Therefore, one might naïvely expect that the resulting feedback dynamics could not be derived within quantum mechanics. If this were the case, a feedback experiment could conceivably allow the universal validity of quantum mechanics to be tested. That is to say, the results of an experiment could differ depending on whether the feedback mechanism was simply a physical system, or whether it had the special property which defines a measurement apparatus. This special property could be the size (in some sense) of the apparatus involved or whether or not it was observed by a conscious observer [143, 111]. The latter possibility would be particularly alarming for an experimentalist. Fortunately, however, such an experiment is impossible. The predictions of the feedback theory using quantum trajectories are identical to those of a purely mechanical theory of quantum feedback. This latter theory is the one referred to above, using quantum Langevin equations. To understand why the two descriptions of feedback should be expected to be equivalent, it is helpful to discuss briefly the metaphysics of quantum measurement theory.

1.2 Reality and Perception

There are two basic attitudes to a quantum measurement: to consider it a physical event, or to consider it a mental event. From the former attitude would follow a prediction that the validity of (unitary) quantum mechanics would break down at some point, and that this could be experimentally distinguished. The hypothetical feedback experiment discussed above is one such experiment. As yet, there is no significant experimental evidence, nor any compelling theory, to support this point of view. Of course, this situation may change in the future, but the simpler solution (which one is obliged to take by Occam's razor) is to take the second attitude. This choice is extremely well supported by both theory and experiment, because it requires only one postulate extra to quantum mechanics: that measure in Hilbert space is perceived as probability. Thus observation is still an essential part of this interpretation of quantum physics, but it is denied any physical status. I will not attempt to give a name to this position. In terms of its predictions, it is identical with the standard

Copenhagen interpretation [14, 15, 75, 76]. In spirit, it is close to the Relative State interpretation of Everett [46] (which unfortunately has become more widely known under the misnomer "Many Worlds interpretation" due to the efforts of misguided followers of Everett [40]).

The best explanation for this point of view which I have come across was given as long ago as 1939, by London and Bauer [91]. For the reader's convenience, a large part of the relevant section of their article is reproduced in Appendix A, with additional comments. They consider an entangled state involving an object x, an apparatus y, and an observer z as subsystems, with wavefunction

$$\Psi(x, y, z) = \sum_k \psi_k u_k(x) v_k(y) w_k(z), \tag{1.4}$$

where u_k, v_k, w_k are wavefunctions and ψ_k is merely a complex amplitude. Evidently, no measurement has taken place here; only unitary quantum mechanics has been used. Regarding the observer, London and Bauer say

> The observer has a completely different impression. [...] He possesses a characteristic and quite familiar faculty which we can call the "faculty of introspection." He can keep track from moment to moment of his own state. By virtue of this "immanent knowledge" he attributes to himself the right to create his own objectivity — that is, to cut the chain of statistical correlations summarized in Eq. (1.4) by declaring "I am in state w_k" or more simply, "I see [the state of the apparatus is v_k]," or even directly, "[the state of the object is u_k]."

The central point here is that there is no contradiction between the physical reality of the wavefunction (1.4), and the perception of an observer who is part of that wavefunction. For another expression of this point of view, which also deals with the rôle of the environment, see Zeh [156].

It could be argued that the apparent conflict between perception and reality is much older than quantum theory. Schrödinger himself [123, 124] was impressed by the insight of one of the founders of science, Democritus of Abdera [141]. One of the few surviving fragments of his writings is a dialog between the intellect and the senses:

> *Intellect:* Ostensibly there is colour, ostensibly sweetness, ostensibly bitterness; in reality there are only atoms and the void.
>
> *Senses:* Wretched intellect, do you hope to defeat us while from us you obtain your evidence? Your victory is your defeat.

Evidently Democritus recognized that although the intellect can deny the physical reality of the senses, the senses nevertheless are necessary for the intellect to understand this reality. To me, it is not at all obvious which, if either, opinion (the intellect or the senses) is more "down to earth". The essence of the philosophy of quantum theory which I have presented is to embrace both attitudes. At times I will side with the intellect and insist on an "atoms and the void" treatment (using only quantum mechanics) of a feedback loop. At other times, I will side with the senses and treat observation and observed results as part of reality. The two views are complementary, in the meaning of Bohr [15].

1.3 Stern-Gerlach Device with Feedback

1.3.1 Measurement Approach

In the preceding section, I claimed that quantum mechanics will not be observed to be violated in an experiment such as a feedback experiment, even if it uses a measurement device. If one accepts the philosophy of quantum theory which I presented, then this statement follows automatically. However, it is not immediately obvious how it is possible for quantum mechanics to deal with measurement and feedback. To see how it is possible, consider the simplest possible measurement plus feedback device, based on the Stern-Gerlach apparatus. It is most easily described using the measurement theory approach. The Stern-Gerlach device consists of a particle beam traversing a region with a spatially non-uniform transverse magnetic field. The effect of this nonuniformity is to cause a transverse magnetic force on the particle, which will depend on the particle's magnetic moment. Assume for simplicity that the particle has two spin states, called up (u) and down (d). Then the differing force will cause particles in state u to move in one direction (say upwards for simplicity), while those in state d will move in the opposite direction. Let the state of the incoming particles (at time t_0) be a superposition of up and down spins

$$|\psi(t_0)\rangle = c_u|u\rangle + c_d|d\rangle. \tag{1.5}$$

Then the Stern-Gerlach device effects a measurement of the spin of the particle, and the superposition becomes a mixture at time t_1

$$\rho(t_1) = |c_u|^2|u\rangle\langle u| + |c_d|^2|d\rangle\langle d|. \tag{1.6}$$

The weighting factor $|c_u|^2$ represents the probability that the spin is measured to be up, and similarly for $|c_d|^2$.

Now, consider feedback in this model. A conceivable purpose of the feedback would be to convert the superposition (1.5) of spin states into one state, say the up state. This is not possible by Hamiltonian evolution in the spin space, because it represents nonunitary evolution. However, it is possible with measurement plus feedback. If the particle is detected in the up state, it is left alone; if in the down state, it is subject to another magnetic field causing rotation of the spin direction from down to up. This can be modeled by the Hamiltonian

$$H = \lambda\left(|u\rangle\langle d| + |d\rangle\langle u|\right), \tag{1.7}$$

acting for the time $t_2 - t_1 = \pi/2\lambda$. This Hamiltonian acts only if the state d is obtained, so the state of the system following the feedback is

$$\rho(t_2) = |c_u|^2|u\rangle\langle u| + |c_d|^2 U_d|d\rangle\langle d|U_d^\dagger = |u\rangle\langle u|, \tag{1.8}$$

where

$$U_d = \exp\left[-\mathrm{i}(\pi/2)\left(|u\rangle\langle d| + |d\rangle\langle u|\right)\right] = |u\rangle\langle d| + |d\rangle\langle u|. \tag{1.9}$$

1.3.2 No-Measurement Approach

As an alternative to the preceding subsection, I will now give a description of the Stern-Gerlach device with feedback without using quantum measurement. The first stage of the device simply produces a correlation between the spin of the system and the position of the wavepacket of the particle. This can be produced by Hamiltonian evolution, and is expressed by the entangled wavefunction

$$|\Psi(t_1)\rangle = c_u|u\rangle|\phi_u\rangle + c_d|d\rangle|\phi_d\rangle, \tag{1.10}$$

where ϕ_u and ϕ_d denote wavepackets which can assumed to be orthogonal. [This should be compared to Eq. (A.1), where the position of the particle is playing the rôle of the apparatus.] The reduced state of the spin system is given by the mixture (1.6). However, in this case the weightings have not been given a probability interpretation.

The feedback is produced by having a suitable magnetic field in the path of the d wavepacket. The spatial restriction of the magnetic field to the area of the ϕ_d wavepacket can be simply modeled by a projector in the Hamiltonian for the second stage of the apparatus

$$H = \lambda\left(|u\rangle\langle d| + |d\rangle\langle u|\right) \otimes |\phi_d\rangle\langle\phi_d|. \tag{1.11}$$

Strictly, the Hamiltonian for the apparatus should be constant in time. However, if the axial position of the particle is treated classically (which is a good approximation for finite times and suitable initial conditions), then the particle will effectively feel a time-varying potential. Thus, one can assume that the Hamiltonian (1.11) is turned on for the time $\pi/2\lambda$, as in the preceding subsection. The total state of the particle at the end of the feedback is thus

$$\begin{aligned} |\Psi(t_2)\rangle &= \exp\left[-\mathrm{i}(\pi/2)\left(|u\rangle\langle d| + |d\rangle\langle u|\right) \otimes |\phi_d\rangle\langle\phi_d|\right]|\Psi(t_1)\rangle & (1.12)\\ &= |u\rangle\left(c_u|\phi_u\rangle + c_d|\phi_d\rangle\right). & (1.13) \end{aligned}$$

As before, the spin system is now in the up state.

1.3.3 Environmental Decoherence

From the above subsections, it is apparent that the two descriptions of feedback, the first using measurement theory and the second describing the "apparatus" quantum-mechanically, are equivalent as far as the final state of the system is concerned. However, if one were to widen one's perspective to include the apparatus, then the two descriptions are not equivalent. In the first case, a measurement has been assumed to have taken place, and so the particle either has spin up and is in the beam with wavepacket ϕ_u, or has spin down and is located in the ϕ_d beam. After the feedback has taken effect, all particles have spin up, but are either located in one or the other beam. In the second case, the final state (1.13) is a superposition of the particle being in either of the two beams. If the beams are left to freely propagate, then dispersion will eventually cause them to overlap in 3-space. Alternatively, this could be caused deliberately with an applied force. In any case, if the two beams overlap, then there will be interference fringes if the state is a superposition, but none if it is a mixture. By this means, it could be distinguished whether or not a measurement did take place.

This analysis does not contradict the attitude towards measurement expressed in Sec. 1.2. There, a measurement was defined in terms of perception by a conscious observer. If an observer did know which way the particle went, then there would certainly be no interference fringes because orthogonal observer states would still be correlated to the respective beams. Note that in this resolution, it is not necessary to invoke the metaphysical abilities of consciousness: the equivalence is simply a consequence of the entanglement of the observer with the apparatus. This entanglement is the result of a physical process: in order for the observer to know which beam the particle is in, he has to observe it, for example by shining a torch at the apparatus. This fact implies that all that is necessary to destroy the interference fringes is for the particle to interact with some other quantum system which reliably distinguishes the two beams. An example of this was analyzed in Ref. [121], where the system causing the destruction of coherence was another two-level system in one beam. As a reaction to this paper, Zurek [157] published the now famous paper introducing the concept of a pointer basis. This paper and its successor [158] have spawned the enormous literature on environmental decoherence which has appeared over the past ten years (see Ref. [159] for a popular review).

The only point which I wish to take from this literature is that it is very easy for superpositions such as (1.13) to be destroyed for all practical purposes. For the case considered here, it may be possible to isolate the two beams from their environment for a sufficiently long time to see interference fringes which would distinguish the feedback model using measurement from that not using measurement. This is because the apparatus (the position of the particle) is very small. In most cases of quantum-limited feedback, the apparatus is enormous (in terms of numbers of particles) and, more importantly, is constantly interacting with an effectively infinite number of degrees of freedom in its environment. The continuum of electromagnetic field modes is one obvious component of that environment which is known to be a quantum system and which couples to all atomic matter. To build a feedback loop in which coherences were important, it would be necessary to use a non-material apparatus. An example of this would be an all-optical loop, because free light beams do not couple to each other. This possibility is considered in Ch. 7, and indeed can lead to behaviour which cannot be replicated by a material feedback loop. For most of this thesis, however, feedback will refer to feedback by a material apparatus. Because of environmental decoherence, this effectively means feedback by measurement, regardless of whether or not a conscious observer takes note of the measured results.

1.4 Structure of the Thesis

If this thesis had been commenced five years earlier, its title would have lacked "Trajectories and". As with all counterfactuals, the meaningfulness of this statement can always be disputed, but it does contain a grain of truth. At the beginning of 1987, the first theoretical work on quantum-limited feedback had just been published, by Yamamoto and co-workers [155, 73]. The more detailed theory of Shapiro *et al* [126] appeared during 1987. As noted above, these theories are not complete in that they do not fully take into account the quantum nature of the fluctuations in the photocurrent. To deal with these fluctuations exactly, it is necessary to generalize the input-output theory of Gardiner and Collett [53], which was also a relatively new theory in 1987. It is quite possible that this generalization could have been made any time after 1987. Since the explicit formulation of quantum trajectories as stochastic Schrödinger equations did not appear until 1992 [37], a fully quantum theory of feedback could have been derived quite independently of quantum trajectories. In reality, the exact theory presented in this thesis was derived first from quantum trajectories, and the quantum Langevin treatment only later.

The relevance of the above discussion to the structure of this thesis is that is possible to divide it into two streams: that part which could have been written five years earlier (the quantum Langevin treatment of feedback), and that part which relies on the concepts of quantum trajectories (which includes the measurement approach to quantum feedback). I will describe the former first. Skipping Ch. 2, the necessary background is found in Ch. 3. There I review classical stochastic calculus, with particular attention to the physical meaning of the mathematical distinction between the Itô and

Stratonovich forms of a Gaussian white noise stochastic differential. I suggest a non-rigorous way of deriving this distinction which is useful because it applies to more general forms of stochastic differential equations (SDEs). This is necessary because feedback requires one to deal with non-Markovian SDEs, and with point process noise as well as Gaussian noise. The next part of Ch. 3 introduces quantum SDEs, which are also called quantum Langevin equations (QLEs). These arise from a weak linear coupling between the electric field in free space, and a localized system dipole. Quantum stochastic calculus has the added feature that the increments are operators. I show how a Markovian master equation for the system state matrix can be derived from the QLE, and also include the effect of white noise in the bath. Next I consider a different system-bath coupling, appropriate to describe the radiation pressure force. This coupling is nonlinear in the bath amplitude, but is nevertheless shown to give sensible results when the general stochastic calculus defined earlier is used. The noise properties of the bath intensity are shown to be those of a point process, which can be interpreted as a stream of photons.

This background is all that is necessary to treat feedback in the quantum Langevin approach, which is done in Ch. 7. The basic idea of feedback is to use the measured photocurrent of the output beam from the system to control the system dynamics. First I treat intensity-dependent feedback, using the direct detection photocurrent. Identifying this photocurrent with the intensity operator of the output beam, this feedback can be described using a Hamiltonian of the same form as the photon-flux pressure one. The result is a modified QLE incorporating feedback. In the Markovian limit, a master equation can be derived. Quadrature-dependent feedback, which corresponds to using a homodyne detection photocurrent, is treated next. Unlike direct detection, this is possible even if the bath is contaminated by white noise. Homodyne-based feedback turns out to be a special case of cascaded systems theory, which describes the unidirectional linear coupling of two quantum systems. In fact, it is possible to reproduce the effect of both direct and homodyne feedback by using cascaded systems theory to describe a physical feedback mechanism involving only light beams (without detection). This all-optical feedback is examined in detail. It leads naturally to considering an all-optical feedback device in which both quadratures are fed back simultaneously. This cannot be reproduced by an electro-optical feedback loop because Heisenberg's uncertainty principle forbids the simultaneous measurement of conjugate variables. Referring to this as feedback requires broadening the definition of feedback to deal with such purely quantum systems.

The second strand of this thesis, based on quantum trajectories, has its roots in quantum measurement theory. This is the topic of Ch. 2, where I show why the concept of quantum measurements must be generalized from the projection postulate to the the theory of operations and effects in order to deal with realistic measurements. In particular, continuous monitoring of an open quantum system is shown to be described by instantaneous quantum jumps. In Ch. 4 I show that for a quantum optical system, these jumps may be interpreted in terms of photodetections. In doing this I use the quantum optical system-bath dipole coupling considered in Ch. 3. The point process which constitutes the measurement record can be used as the source of randomness in the stochastic Schrödinger equation (SSE) describing the evolution of the system conditioned on that record. By considering the addition of a local oscillator to the field before it is detected, the SSE for homodyne detection is derived also. More generally, a stochastic master equation (SME) is needed to deal with inefficient detection.

In Ch. 5 I discuss the interpretation of such quantum trajectories in the literature, and treat numerous examples both numerically and analytically. The importance of the measurement theory interpretation of quantum trajectories is emphasized in this chapter. Finally, in Ch. 6 I reach the topic of feedback using quantum trajectories. Here, as with the QLE approach, the general stochastic calculus is needed to deal with the noise properties of the photocurrent, and with the Markovian limit of what is necessarily a non-Markovian process (as the measurement must precede the feedback). The feedback master equations derived here are of course the same as those derived in Ch. 7. To deal with non-Markovian feedback, I show that the non-Markovian quantum trajectories can be solved analytically in some cases. However, the QLE treatment of these systems in Ch. 7 is much more straightforward. The practical application for feedback considered in Ch. 6 is noise reduction. The limits of noise reduction for linear feedback based on homodyne measurements are derived. To overcome these limits, intracavity QND measurements are also considered in Ch. 6. Extracavity QND measurements are considered in Ch. 7 in the context of all-optical feedback.

Chapter 8 cannot be divided into either stream, as it takes elements from both. The topic of this final chapter proper is using feedback to eliminate back-action in continuous quantum measurements. The basic idea is that any measurement consists of a quantum non-demolition measurement, which does not disturb the measured variable, followed by back-action, which may disturb that quantity. This back-action is described by a unitary transformation of the system, and can thus always be undone by controlling the system Hamiltonian by feedback. This idea was conceived using the measurement (quantum trajectory) approach to feedback, but the specific optical scheme which I propose is treated using the quantum Langevin approach. This underlines the usefulness of having equivalent formulations of quantum feedback: one method may be more easily applied, or more easily understood, in a particular situation. The concluding chapter of the thesis also ties together both threads of this thesis. It consists of a summary, a review of present and future applications of quantum trajectories and feedback, and a discussion of remaining questions.

Chapter 2

Quantum Measurement and Feedback Theory

In this chapter, I introduce the formal theory of quantum measurements. This involves a generalization of the projection postulate to the theory of operations and effects. This generalization is justified by considering indirect measurements; that is to say, projective measurements on an apparatus which has interacted with the system. Next, the general formalism is applied to continuous measurements. Such measurements are appropriate for open quantum systems which obey a master equation. Lastly, the action of feedback is investigated within the formal structure of measurement theory. By using the apparatus considered earlier, I show that any feedback of measurement results can be reproduced within quantum mechanics. Applying the feedback theory to continuous measurements yields a modified master equation in the nonselective case.

2.1 Quantum Measurement Theory

2.1.1 The Projection Postulate

In Sec. 1.2, it was stated that the only additional assumption necessary to derive quantum measurement theory from quantum mechanics was that measure in Hilbert space is observed as probability. To understand this assumption, consider a Hilbert space for a system incorporating an observer. Let the orthogonal subspaces of this Hilbert space containing different states of mind of the observer be assigned projection operators P_α. Let the state of the system (including observer) be the normalized state vector $|\Psi\rangle$. Then, the probability that the observer perceives the system to be in the subspace with projection operator P_α is

$$\Pr[\alpha] = \langle\Psi|P_\alpha|\Psi\rangle. \tag{2.1}$$

Then, as explained in Sec. 1.2, as far as the observer is concerned, the state of the system is

$$|\tilde{\Psi}_\alpha\rangle = P_\alpha|\Psi\rangle. \tag{2.2}$$

Here the tilde indicates that this ket is unnormalized. This allows the probability (2.1) to be rewritten

$$\Pr[\alpha] = \langle\tilde{\Psi}_\alpha|\tilde{\Psi}_\alpha\rangle. \tag{2.3}$$

In the standard (Copenhagen) interpretation of quantum physics, the state of the observer would not be included in the state vector. Apart from this difference, the theory stated above is essentially the projection postulate. In more conventional language, the system is restricted to an object about which the observer has some knowledge. Because this knowledge may be imperfect, it is necessary in general to describe the state of the system by a state matrix. Consider an observation of the system in the time interval $[t, t+T)$. If the observer has made previous measurements on the system, then its state $\rho(t)$ at time t is already a conditioned state. The probability for the system to be observed in the subspace of eigenstates of P_α is

$$\Pr[\alpha] = \mathrm{Tr}[\tilde{\rho}_\alpha(t+T)]. \tag{2.4}$$

Here, $\tilde{\rho}_\alpha(t+T)$ is an unnormalized state matrix given by

$$\tilde{\rho}_\alpha(t+T) = P_\alpha \rho(t) P_\alpha. \tag{2.5}$$

The new state of the system, conditioned on the result α, is

$$\rho_\alpha(t+T) = \tilde{\rho}_\alpha(t+T)/\Pr[\alpha]. \tag{2.6}$$

This fundamental statement of quantum measurement theory was first correctly given by Dirac [42].

2.1.2 Operations and Effects

The projection postulate theory of measurement given above, while fundamentally correct for complete systems (including the observer), is deficient for describing practical measurements on objects. For a start, while the above description included a finite time T for the measurement, the system was assumed not to evolve over that time. This is obviously unrealistic. To include the free evolution of the system, and other complications which will be justified in the following section, it is necessary to use the theory of operations and effects developed formally in Refs. [78, 86]. Readable accounts are found in Refs. [52, 17]. Using the same notation as above, the unnormalized state matrix given the result α is given by

$$\tilde{\rho}_\alpha(t+T) = \mathcal{J}[\Omega_\alpha(T)]\rho(t). \tag{2.7}$$

Here, $\Omega_\alpha(T)$ is an operator which is the argument of $\mathcal{J}$ which is a *superoperator* acting on $\rho(t)$. Superoperators transform one operator into another operator, and will always be indicated by the use of calligraphic font. The superoperator $\mathcal{J}[r]$ takes an arbitrary operator r as its argument and acts according to

$$\mathcal{J}[r]\rho \equiv r\rho r^\dagger. \tag{2.8}$$

In this case, the superoperator $\mathcal{J}$ can be called an *operation* because it takes density operators (bounded positive operators) to density operators. Other superoperators are not necessarily operations.

The set of operators $\{\Omega_\alpha(T)\}$ which define the measurement are arbitrary, apart from the condition

$$\sum_\alpha \Omega_\alpha^\dagger(T)\Omega_\alpha(T) = 1, \tag{2.9}$$

where the sum is over all possible measurement results α. This is known as the completeness condition [52]. It is simply a statement of conservation of probability. The probability for obtaining the result α is given by Eq. (2.4), which can be rewritten as

$$\Pr[\alpha] = \mathrm{Tr}[W_\alpha(T)\rho(t)], \tag{2.10}$$

where

$$W_\alpha(T) = \Omega_\alpha^\dagger(T)\Omega_\alpha(T). \tag{2.11}$$

The operator $W_\alpha(T)$ is known as the *effect* for the result α. The set of such effects is known as a decomposition of unity, as in Eq. (2.9). Note that different sets of operations $\mathcal{J}[\Omega_\alpha(T)]$ may have the same set of effects $W_\alpha(T)$, because the $\Omega_\alpha(T)$ are not necessarily Hermitian. Thus, to specify the measurement it is necessary to specify the operators $\Omega_\alpha(T)$, which I will thus refer to as the *measurement operators*.

The results of the preceding section follow if the measurement operators are projectors. A trivial example of effects and operations using measurement operators more general than projectors is the case of a freely evolving system observed at the end of the interval $[t, t+T)$. If the free Hamiltonian of the system is H, and if a unitary operator $U(T) = \exp[-\mathrm{i}HT]$ is defined, then the measurement operators become

$$\Omega_\alpha(T) = P_\alpha U(T). \tag{2.12}$$

In this example, the effects are still projectors

$$W_\alpha(T) = U^\dagger(T)P_\alpha U(T), \tag{2.13}$$

which means that the measurements are orthogonal. That is to say, it is possible to prepare the initial state of the system such that only one measurement result is possible. In general, this is not the case. The origin of nonorthogonal measurements will be investigated in the next section.

With the definitions given so far, if the system under observation is in a pure state at the start of the measurement, then it will still be in a pure state after the measurement, conditioned on the result. If the initial state is $|\psi(t)\rangle$, the unnormalized final state is

$$|\tilde{\psi}_\alpha(t+T)\rangle = \Omega_\alpha(T)|\psi(t)\rangle. \tag{2.14}$$

Of course, if the measurement is made but one ignores the result, the final state will not be pure, but a mixture of the possible outcomes weighted by their probabilities

$$\begin{aligned} \rho(t+T) &= \sum_\alpha \Pr[\alpha]\rho_\alpha(t+T) && (2.15)\\ &= \sum_\alpha \mathcal{J}[\Omega_\alpha(T)]\rho(t). && (2.16) \end{aligned}$$

Note that being given $\rho(t)$ and $\rho(t+T)$ does not allow one to determine the measurement operators $\Omega_\alpha(T)$ uniquely. This is because any alternative set of measurement operators, defined as a unitary combination of the original set, will yield the same nonselective evolution [80]. To see this, define

$$\Omega'_\alpha(T) = \sum_\beta U_{\alpha,\beta}\Omega_\beta(T), \tag{2.17}$$

where U is a $c-$number matrix satisfying

$$\sum_\alpha U_{\alpha,\beta}U^*_{\alpha,\gamma} = \delta_{\beta,\gamma}. \tag{2.18}$$

Then

$$\begin{aligned} \sum_\alpha \Omega'_\alpha(T)\rho(t)\Omega'^\dagger_\alpha(T) &= \sum_{\alpha,\beta,\gamma} U_{\alpha,\beta}\Omega_\beta(T)\rho(t)\Omega_\gamma^\dagger(T)U^*_{\alpha,\gamma} && (2.19)\\ &= \sum_{\beta,\gamma}\delta_{\beta,\gamma}\Omega_\beta(T)\rho(t)\Omega_\gamma^\dagger(T) = \sum_\beta \Omega_\beta(T)\rho(t)\Omega_\beta^\dagger(T). && (2.20) \end{aligned}$$

Measurements which preserve pure states, as in Eq. (2.14), I will call efficient measurements. In practice, many measurements are not efficient, and the conditioned states of the system will not remain pure. Such measurements can be defined by generalizing the operation for α to

$$\mathcal{O}_\alpha(T) = \sum_\beta \mathcal{J}[\Omega_{\alpha\beta}(T)], \tag{2.21}$$

where β is another subscript, which could be thought of as indexing measurement results which are not recorded, and so which must be averaged over as in Eq. (2.16). These unrealized results are results which could be distinguished without altering the nonselective evolution of the system, which now is given by

$$\rho(t+T) = \sum_{\alpha\beta} \mathcal{J}[\Omega_{\alpha\beta}(T)]\rho(t). \tag{2.22}$$

For this case of inefficient measurements, the effects are given by

$$W_\alpha(T) = \sum_\beta \Omega^\dagger_{\alpha\beta}(T)\Omega_{\alpha\beta}(T). \tag{2.23}$$

2.2 Indirect Measurements

As pointed out by Heisenberg [75], the placing of the boundary (the 'Heisenberg cut') between quantum system and classical apparatus is essentially arbitrary. However, if one includes too little in the quantum system, and simply uses the projection postulate to describe measurements, then one is likely to be misled. This is the source of error in many misunderstandings by quantum theoreticians and philosophers, such as in the original quantum Zeno paradox of Misra and Sudarshan [104]. The more of the world which is treated using quantum mechanics, the more accurate the description of measurement is expected to be. The simplest application of this principle is to treat the apparatus as a quantum system, and to apply the projection postulate to the apparatus rather than to the system. The measurement on the apparatus is effectively a measurement of the system, but must be described in terms of the operations and effects introduced in the preceding section. The analysis presented below is based on that in Sec. 3.4 of Ref. [17].

Consider an apparatus prepared in the state $|\phi\rangle$, and a system in the initial state $|\psi\rangle$. Let the apparatus and system interact for a time T, via the Hamiltonian

$$H = H_{\rm s} \otimes 1_{\rm a} + 1_{\rm s} \otimes H_{\rm a} + V, \tag{2.24}$$

where V is the interaction term. Then the combined state of the system plus apparatus at time $t+T$ is

$$|\Psi(t+T)\rangle = U(T)\left[|\phi\rangle|\psi(t)\rangle\right], \tag{2.25}$$

where $U(T) = \exp[-{\rm i}HT]$. In general, this is an entangled state, as discussed in the introduction. Now let a projective measurement of the apparatus be made at time $t+T$. The unnormalized state vector for the combined system is

$$|\tilde{\Psi}_\alpha(t+T)\rangle = P_\alpha|\Psi(t+T)\rangle, \tag{2.26}$$

where $P_\alpha = 1_{\rm s} \otimes |\phi_\alpha\rangle\langle\phi_\alpha|$. Such a measurement projects the apparatus into a pure state, and hence disentangles the system from the apparatus. The corresponding unnormalized disentangled system state is then

$$|\tilde{\psi}_\alpha(t+T)\rangle = \Omega_\alpha(T)|\psi(t)\rangle, \tag{2.27}$$

where the measurement operator in the system Hilbert space is defined by

$$\Omega_\alpha(T) = \langle\phi_\alpha|U(T)|\phi\rangle. \tag{2.28}$$

This simply shows why the general measurement operators need to be introduced to describe measurements in practice.

Inefficient measurements, requiring non purity-preserving operations, arise from indirect measurements in two ways. The first way is as described in Sec. 2.1.2, with a measurement on the apparatus state being made (in a virtual sense) but then ignored. The second way is if the initial apparatus state is mixed rather than pure. Let this initial state be represented by the density operator μ, and the initial state of the system by $\rho(t)$. Then the conditioned state of the system following the measurement is the reduced state matrix

$$\begin{aligned}\tilde\rho_\alpha(t+T) &= \mathrm{Tr_a}\left[P_\alpha U(T)\{\mu\otimes\rho(t)\}U^\dagger(T)P_\alpha\right] &(2.29)\\ &\equiv \mathcal{O}_\alpha(T)\rho(t), &(2.30)\end{aligned}$$

where the trace is over the apparatus Hilbert space. While it may not be obvious that the operations $\mathcal{O}_\alpha(T)$ so defined are as arbitrary as Eq. (2.21) implies, it can be shown (Theorem 5.1 of Ref. [78]) that they are. The requirement for an efficient measurement is thus that the state of the apparatus both before the interaction, and after its measurement, must be pure, indicating maximal knowledge by the observer. In practice, both sources of inefficiency are important.

2.3 Continuous Measurement Theory

2.3.1 Master Equations

One case of the above measurement theory which is of great importance is continuous observation, with an infinitesimal measurement interval $T = dt$. The form of such a measurement theory is restricted by the requirement that the nonselective evolution generated by the measurements should give a valid evolution equation for the state of the system. Explicitly, this nonselective evolution is

$$\rho(t+dt) = \sum_{\alpha\beta}\Omega_{\alpha\beta}(dt)\rho(t)\Omega_{\alpha\beta}(dt). \tag{2.31}$$

This is obviously Markovian [50], with the increment in the system state from time t to time $t+dt$ depending only on the state of the system at time t. As stated in the introduction, a Markovian evolution equation for the state matrix of a system is called a master equation. Physically, it can be derived from the interaction of the system with its environment, if the environment can be treated as a bath. That is to say, it is necessary for the environment to irreversibly carry away information from the system. In the ideal model of quantum measurements considered in Sec. 2.2, the environment must act as a continuous stream of independently prepared apparatuses which are wheeled up to the system, interact for an infinitesimal time dt, and are wheeled away again to be measured. Although this sounds unrealistic, there are some system-environment interactions which behave exactly like this to a good approximation. This will be explored in Ch. 3.

The most general form of master equation is the so-called Lindblad form [90, 52]

$$\dot\rho = -\mathrm{i}[H,\rho] + \sum_\mu \mathcal{D}[c_\mu]\rho. \tag{2.32}$$

Here H is an Hermitian operator and $\mathcal{D}$ is a superoperator taking an operator argument (enclosed in square brackets). It is defined for an arbitrary operator r by

$$\mathcal{D}[r] \equiv \mathcal{J}[r] - \mathcal{A}[r], \tag{2.33}$$

where $\mathcal{J}$ is as defined in Eq. (2.8), and $\mathcal{A}$ is a superoperator producing an anticommutator from the definition

$$\mathcal{A}[r]\rho \equiv \tfrac{1}{2}\{r^\dagger r, \rho\}, \tag{2.34}$$

where $\{a, b\} \equiv ab + ba$. The operators c_μ are completely arbitrary.

In Eq. (2.32), the vector of operators c_μ is not unique; a unitary transformation in the complex vector space indexed by μ will leave the master equation unchanged [54]. That is to say, it is invariant under the transformation

$$c_\mu \rightarrow U_{\mu\nu} c_\nu \quad , \quad U_{\mu\nu} U^*_{\lambda\nu} = \delta_{\mu\lambda} \tag{2.35}$$

where the Einstein summation convention is being used. In the context of the following subsection, this will be understood as a unitary re-arrangement of measurement operators as in Sec. 2.1.2. For simplicity, consider the case where there is only one source of irreversibility so that

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \mathcal{D}[c]\rho \equiv \mathcal{L}\rho. \tag{2.36}$$

Even here, this representation is not unique, for this master equation is invariant under the transformation

$$c \rightarrow c + \gamma \; ; \quad H \rightarrow H - \mathrm{i}\tfrac{1}{2}(\gamma^* c - \gamma c^\dagger), \tag{2.37}$$

where γ is an arbitrary complex number. This transformation will be very important in this thesis, from Ch. 4 onwards.

2.3.2 Continuous Observation

The general form of the master equation (2.36) and the general form of quantum measurement can be put in one-to-one correspondence once a particular representation of the master equation has been chosen. That is to say (for the case of one output channel), when a physically meaningful value for the parameter γ has been chosen. First consider the case of efficient measurements. Then by inspection, only two measurement operators $\Omega_\alpha(dt)$ are needed,

$$\Omega_1(dt) = \sqrt{dt}\, c, \tag{2.38}$$

$$\Omega_0(dt) = 1 - \left(\mathrm{i}H + \tfrac{1}{2}c^\dagger c\right) dt. \tag{2.39}$$

It is easy to verify that the nonselective evolution under this measurement is

$$\rho(t + dt) = \sum_{\alpha=0,1} \mathcal{J}[\Omega_\alpha(dt)]\rho(t) = (1 + \mathcal{L}dt)\rho(t), \tag{2.40}$$

where $\mathcal{L}$ is as given in Eq. (2.36). The action of the transformation (2.37) on these measurement operators can be recognized to be a unitary re-arrangement of the two operators, as discussed in Sec. 2.1.2, with the unitary matrix

$$\begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}|\gamma|^2 dt & \gamma\sqrt{dt} \\ -\gamma^*\sqrt{dt} & 1 - \frac{1}{2}|\gamma|^2 dt \end{pmatrix}. \tag{2.41}$$

From the measurement operators $\Omega_0(dt), \Omega_1(dt)$, it is evident that continuous measurements on a Markovian quantum system necessarily yields a measurement record which is a point process [34]. For almost all infinitesimal time intervals, the measurement result is $\alpha = 0$, which is thus regarded as a null result. In this case, the system changes infinitesimally, but not unitarily, via the operator $\Omega_0(dt)$. At randomly determined (but not necessarily Poisson distributed) times, there is a result $\alpha = 1$, which I will call a *detection*. When this occurs, the system undergoes a finite evolution induced by the operator $\Omega_1(dt)$. This change can validly be called a *quantum jump*, although it must be remembered that it represents a sudden change in the observer's knowledge, not an objective physical event as in Bohr's original conception [13]. Real measurements which correspond approximately to this ideal measurement theory are made routinely in experimental quantum optics. If c is the lowering operator for the quantum system, multiplied by the square root of the damping rate, then this theory describes the system evolution in terms of photodetections. This will be shown in Ch. 4. Because at present the primary application of this measurement theory is quantum optics, the terms photodetection and photocurrent *et cetera* will often be used instead of the more general terminology.

In the context of quantum optics, inefficiency in measurements arises because photodetectors sometimes miss detections. In this case, it is necessary to use measurement operations which do not preserve purity. If the proportion of detections which are actually registered is η, then the measurement operation for a detection becomes

$$\mathcal{O}_1(dt) = \eta dt \mathcal{J}[c]. \tag{2.42}$$

Note that this still preserves purity. However, the smooth evolution between jumps is now described by the operation

$$\mathcal{O}_0(dt)\rho = \rho + (-\mathrm{i}[H,\rho] - \eta\mathcal{A}[c]\rho + (1-\eta)\mathcal{D}[c]\rho)\, dt \tag{2.43}$$

For $\eta < 1$, this will not preserve purity in general. The modification of the smooth evolution operator is necessary so that

$$\mathcal{O}_0(dt) + \mathcal{O}_1(dt) = 1 + \mathcal{L}dt. \tag{2.44}$$

It would be possible to write $\mathcal{O}_0(dt)$ in the form of Eq. (2.21), but this would serve little purpose.

2.4 Feedback

2.4.1 General Feedback Theory

Feedback can be defined as the use of a measurement result to influence the dynamics of the system at any future time. To calculate the effect of feedback on a system, all that is necessary is the general measurement theory given in Sec. 2.1, combined with a specification of how the system evolution depends on those measurement results. An example of the latter would be to adjust the system Hamiltonian. This is possible with open systems if some dominant degrees of freedom of the environment can be treated classically to a good approximation. Examples of this will be considered in Chs. 6–8. The feedback-controlled Hamiltonian could be written symbolically as

$$H_{\mathrm{fb}}(t) = H(\{\alpha(t') : t' + T < t\}), \tag{2.45}$$

where this indicates an Hermitian operator function of the set of all prior measurement results $\alpha(t')$, where t' is the start of the measurement interval. Each result $\alpha(t')$ is a random variable which can be assumed to be a real number, so that the state of the system conditioned on these results is also a random variable. The dynamics generated by such feedback could be quite complicated, because the feedback could operate continually, while further measurements are being done. There are two ways to deal with such feedback theoretically. The first is to simulate a particular stochastic history of the system with measurement and feedback. The probabilities for each measurement result are determined by Eq. (2.10), and the choice of one particular result could be generated by a pseudo-random number generator. The overall effect of the feedback (ignoring the particular measurement record) could be calculated from a large ensemble of trajectories. The alternative approach would be to try to solve for the ensemble average evolution exactly. This would obviously be very difficult in general.

One special case in which the ensemble average evolution can be found is that of *instantaneous feedback*. That is to say, as soon as the measurement result is recorded, it is used to cause an immediate finite change in the system state. Of course, it is not possible to physically cause a finite change in zero time; instantaneous feedback is the limit of very fast feedback in which the system dynamics is changed by a very large amount for a very short time. Assuming that the system dynamics can be treated alone, the evolution of the system over this short time must be governed by a master equation. This more general Liouvillian evolution could be derived from the Hamiltonian case defined above (2.45) by including a bath in the system for the purposes of feedback, and then eliminating it in the usual way. Thus, the effect of the feedback will be to cause finite evolution by the superoperator $\exp(\mathcal{K})$, where $\mathcal{K}$ is a Liouville superoperator of the form of Eq. (2.32). The superoperator $\exp(\mathcal{K})$ is an operation, and operations form a (non-Abelian) group under multiplication. Thus, if each measurement result α has instantaneous feedback generated by the superoperator $\mathcal{K}_\alpha$, then the operation for each result is simply transformed by

$$\mathcal{O}_\alpha(T) \to \mathcal{O}_\alpha(T^+) = \exp(\mathcal{K}_\alpha)\mathcal{O}_\alpha(T). \tag{2.46}$$

The ensemble average description of the feedback is then given by

$$\rho(t+T^+) = \sum_\alpha \exp(\mathcal{K}_\alpha)\mathcal{O}_\alpha(T)\rho(t). \tag{2.47}$$

2.4.2 Feedback within Quantum Mechanics

As emphasized in the introduction, it should always be possible to give a description of a physical feedback apparatus within quantum mechanics, not using quantum measurement theory. Here, I will show that this can be done using the apparatus introduced in Sec. 2.2 to derive operations and effects. Consider the completely general case of Eq. (2.29), where the initial apparatus state μ is possibly mixed, and the initial system state is $\rho(t)$. The entangled state at time $t+T$ is

$$R(t+T) = U(T)\{\mu(t) \otimes \rho(t)\}U^\dagger(T). \tag{2.48}$$

Here I have introduced a time argument for the initial apparatus state in order to identify this apparatus as the one used for the measurement in the interval $[t, t+T)$. Assume that the evolution of the apparatus after it has interacted with the system commutes with the projection operators $P_\alpha(t)$. That is to say, the free evolution of the apparatus does not cause transitions between the measured subspaces. Here, the time argument for P_α merely associates the projection operator with the Hilbert space of the apparatus used for the measurement in the interval $[t, t+T)$. Now define an operator for this apparatus

$$A(t) = \sum_{\alpha(t)} \alpha(t) P_\alpha(t) \tag{2.49}$$

which has the measurement results α as eigenvalues and P_α as corresponding eigenstates. Then it is simple to define the Hamiltonian corresponding to Eq. (2.45)

$$H_{\rm fb}(t) = H\left(\{A(t') : t' + T < t\}\right). \tag{2.50}$$

The functional form of this Hamiltonian is the same as that in Eq. (2.45), except that where Eq. (2.45) generates a system Hamiltonian depending on the random variables $\alpha(t)$, Eq. (2.50 generates a Hamiltonian in the system and apparatus space, with the $\alpha(t)$ replaced by $A(t)$. Physically, it could be achieved by wheeling the used apparatuses back to the system so that they can interact again.

Because of the idempotency and orthogonality of projection operators, the Hamiltonian (2.50) can be decomposed into outer products of Hermitian system operators with projection operators in the apparatus Hilbert spaces. To determine the system evolution alone, one traces over the apparatus spaces. In doing this, the projection operators P_α in the feedback Hamiltonian project the apparatus states into the states corresponding to the measured result α. The effect of this projection on the system is exactly the operation $\mathcal{O}_\alpha(T)$, because the observable $A(t)$ is assumed to be constant after the measurement interval. Furthermore, the weightings for this component of the system state matrix evolution are the same as the probabilities of the measurement results, because of the dual rôle of $\mathcal{O}_\alpha(T)$. Thus, the evolution generated by the Hamiltonian (2.50) is precisely the same as that generated by an ensemble of trajectories with feedback generated by (2.45). In this analysis, however, no measurement theory has been used. The measurement results remain unrealized; the weightings produced by the operations do not have a probability interpretation. It is possible to realize these measurements at any stage without altering the argument given here. Another apparatus measuring the observable A will not affect the feedback, because all operators $A(t)$ commute with the feedback Hamiltonian. In practice, this is just what does occur. The feedback apparatus could be treated as a quantum system (as done here) but nevertheless it will interact with other systems (such as a cathode ray tube) which enable the actual result to be observed. The point made here is that the observation step is not necessary to formulate quantum-limited feedback.

The above argument is rather abstract and possibly hard to follow, because it is completely general. It is therefore helpful to consider the special case of instantaneous feedback as defined above. Consider again the entangled state (2.48). At time $t+T$, let the Hamiltonian for the system and apparatus become

$$H_{\rm fb} = \lambda \sum_\alpha Z_\alpha \otimes P_\alpha \tag{2.51}$$

for a time λ^{-1} which is considered very short. Here, the Z_α are Hermitian system operators. Then the combined state of the system plus apparatus after the feedback is

$$R(t+T^+) = \exp\bigl(-{\rm i}\textstyle\sum_\alpha Z_\alpha \otimes P_\alpha\bigr) U(T)\{\mu(t)\otimes\rho(t)\}U^\dagger(T)\exp\bigl({\rm i}\textstyle\sum_\alpha Z_\alpha \otimes P\alpha\bigr). \tag{2.52}$$

Using the orthonormality of projection operators, this can be rewritten

$$R(t+T^+) = \sum_\alpha \exp\left(-{\rm i}Z_\alpha\right) P_\alpha U(T)\{\mu(t)\otimes\rho(t)\}U^\dagger(T) P_\alpha \exp\left({\rm i}Z_\alpha\right). \tag{2.53}$$

Taking the trace over the apparatus states yields

$$\rho(t+T^+) = \sum_\alpha \exp\left(-{\rm i}Z_\alpha\right)[\mathcal{O}_\alpha(T)\rho(t)]\exp\left({\rm i}Z_\alpha\right). \tag{2.54}$$

Defining feedback generators $\mathcal{K}_\alpha\rho \equiv -{\rm i}[Z_\alpha, \rho]$, this can be rewritten

$$\rho(t+T^+) = \sum_\alpha \exp\left(\mathcal{K}_\alpha\right)\mathcal{O}_\alpha(T)\rho(t), \tag{2.55}$$

which is the same as that deduced above (2.47). The Stern-Gerlach device analyzed in Sec. 1.3 may be recognized to be feedback of this kind, although with a finite time for the feedback. As mentioned earlier, non-Hamiltonian generators $\mathcal{K}_\alpha$ can be produced by temporarily widening the definition of the system. By understanding the case of instantaneous feedback, the reader should be convinced of the general equivalence between the two treatments of feedback. The ensemble of states generated using measurement theory with feedback is identical to the state generated within quantum mechanics, with a quantum apparatus.

2.4.3 Continuous Markovian Feedback

To incorporate feedback into the theory for continuous measurements of Sec. 2.3 is quite straightforward. For simplicity, consider only instantaneous feedback and efficient detection. Thus, the mechanism must cause an immediate change in the system based only on the result of the measurement in the preceding infinitesimal time interval. Because the null result $\alpha = 0$ occurs almost all of the time, feeding back this information is pointless. The feedback must act immediately after a detection, and cause a finite amount of evolution. Let this finite evolution be effected by the superoperator $e^{\mathcal{K}}$, where $\mathcal{K}$ is a Liouville superoperator as before. Then the unnormalized density operator following a detection at time t is

$$\tilde{\rho}_1(t+dt) = e^{\mathcal{K}} c\rho(t)c^\dagger dt. \tag{2.56}$$

The superoperator acts on the product of all operators to its right. Note that the feedback preserves the trace of this state matrix, as is required by conservation of probability. The nonselective evolution of the system is still given by

$$\rho(t+dt) = \tilde{\rho}_1(t+dt) + \tilde{\rho}_0(t+dt). \tag{2.57}$$

Since $\tilde{\rho}_0(t+dt)$ is unchanged by feedback, one has simply

$$\dot{\rho} = -\mathrm{i}[H,\rho] + e^{\mathcal{K}}\mathcal{J}[c]\rho - \mathcal{A}[c]\rho. \tag{2.58}$$

This is the most general form of feedback master equation for perfect detection via a single loss source. If the detection is not perfect, or if there are other loss sources, then the Hamiltonian evolution term must be replaced by a more general Liouville term. It can be shown that Eq. (2.58) does conform to the required Lindblad form (2.32). As an example, assume that $\mathcal{K}$ acts as

$$\mathcal{K}\rho = -\mathrm{i}[Z,\rho] + \mathcal{D}[b]\rho. \tag{2.59}$$

Then the master equation (2.58) can then be written

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \sum_{m=0}^{\infty}\int_0^1 ds_m \int_0^{s_m} ds_{m-1}\ldots\int_0^{s_2} ds_1 \mathcal{D}[h_m(s_m,s_{m-1}\ldots,s_1)c]\rho, \tag{2.60}$$

where

$$h_m(s_m,s_{m-1}\ldots,s_1) = e^{-(\mathrm{i}Z+\frac{1}{2}b^\dagger b)(1-s_m)}be^{-(\mathrm{i}Z+\frac{1}{2}b^\dagger b)(s_m-s_{m-1})}b\ldots e^{-(\mathrm{i}Z+\frac{1}{2}b^\dagger b)s_1}. \tag{2.61}$$

In the special case where $\mathcal{K}\rho = -\mathrm{i}[Z,\rho]$, this simplifies greatly to

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \mathcal{D}[e^{-\mathrm{i}Z}c]\rho. \tag{2.62}$$

Using the formalism of the preceding Sec. 2.4.2, this last case of Hamiltonian feedback can be readily derived within quantum mechanics. However, I will leave this physical description of the feedback mechanism until after the physical origin of the master equation has been explored in the following chapter.

Chapter 3

Stochastic Differential Equations

This chapter has two main parts: the first, comprising Secs. 3.1 – 3.3, deals with classical stochastic differential equations, and the second, comprising Secs. 3.4 – 3.7, with quantum SDEs. Sec. 3.1 reviews the most familiar sort of SDEs, involving Gaussian white noise. This introduction to the Stratonovich and Itô differential calculi emphasizes physical significance rather than mathematical rigour. In Sec. 3.2, I introduce a more general form of stochastic differential calculus to deal with non-Gaussian, non-white noise, which occurs in the quantum theory of continuous observations. The Stratonovich – Itô distinction is replaced by a more general implicit – explicit distinction. In Sec. 3.3, I show that the latter distinction is more useful when considering feedback, even if Gaussian white noise is involved. Sec. 3.4 introduces quantum SDEs, in which the $q-$number noise source is the external quantized electromagnetic field. A dipole coupling of the system to the bath is considered, and a quantum Langevin equation (QLE) derived for an arbitrary Gaussian white-noise bath. The quantum optical master equation is derived in Sec. 3.5, both from the QLE, and directly in the interaction picture. In Sec. 3.6, a nonlinear bath-system coupling, appropriate for radiation pressure on a mirror, is treated using the general calculus defined in Sec. 3.2. This will be used again in Ch. 7 when considering quantum-mechanical feedback. Lastly, the quantum theory of cascaded open systems, which will also be needed later, is developed briefly in Sec. 3.7.

3.1 Classical Stochastic Differential Equations

3.1.1 Gaussian White Noise

Although the description "stochastic differential equations" (SDE) sounds rather general, it is often taken to refer only to differential equations with a Gaussian white noise term [50]. In this chapter, I wish to consider more general sorts of stochastic equations. This is necessary to consider quantum trajectories and feedback fully. However, it is useful first to review Gaussian stochastic differential equations. This review is not intended to be mathematically rigorous, but rather to emphasize the physical assumptions behind the formalism. In particular, the concept of stochastic integration will not be introduced at all; Ref. [50] contains the necessary formal treatment of this topic. Consider the one dimensional case for simplicity. The stochastic differential equation is then of the form

$$\dot{x} = \alpha(x) + \beta(x)\xi(t). \tag{3.1}$$

Here, the time argument of x has been omitted, α and β are arbitrary functions, and $\xi(t)$ is a rapidly varying stochastic continuous function of time. The idealized limit of such a noisy function is Gaussian white noise, which is characterized by

$$\mathrm{E}[\xi(t)\xi(t')] = \delta(t-t'), \tag{3.2}$$

$$\mathrm{E}[\xi(t)] = 0, \tag{3.3}$$

where E denotes an ensemble average, or expectation value. Note that the delta-function correlation expressed by the right side of Eq. (3.2) means that the correlation time for the noise is zero. Because of this singularity, one has to be very careful in finding the solutions of Eq. (3.1).

Physically, an equation like (3.1) could be obtained by deriving it for a physical (non-white) noise source $\xi(t)$, and then taking the idealized limit. In that case, Eq. (3.1) is known as a Stratonovich SDE [50]. The Stratonovich SDE for some function f of x is found by using the standard rules of differential calculus [50], *viz.*

$$\dot{f}(x) = f'(x)[\alpha(x) + \beta(x)\xi(t)], \tag{3.4}$$

where the prime denotes differentiation with respect to x. As stated above, the differences with standard calculus arise when actually solving Eq. (3.1). Let $x(t)$ be known. If one were to assume that the infinitesimally evolved variable x is given by

$$x(t+dt) = x(t) + [\alpha(x) + \beta(x)\xi(t)]dt \tag{3.5}$$

and further that the stochastic term $\xi(t)$ is independent of the state of the system at the same time, then one would derive the expected increment in x to be

$$\mathrm{E}[dx(t)] = \alpha[x(t)]dt. \tag{3.6}$$

The second assumption here seems perfectly reasonable since the noise is not correlated with any of the noise which has interacted with the system in the past, and so would be expected to be uncorrelated with the system. Applying the same arguments to $f(x)$ yields

$$\mathrm{E}[df(t)] = f'\big(x(t)\big)\alpha\big(x(t)\big)dt. \tag{3.7}$$

Now consider $f(x) = x^2$. The expected increment in the variance of x is

$$\mathrm{E}[x(t+dt)^2] - \mathrm{E}[x(t+dt)]^2 = \mathrm{E}[df(t)] - 2x(t)\mathrm{E}[dx(t)] = 0. \tag{3.8}$$

That is to say, the stochastic term has not introduced any noise into the variable x.

Obviously this result is completely contrary to what one would wish from a stochastic equation. The lesson is that it is invalid to simultaneously make the three assumptions that

1. The chain rule of standard calculus applies [Eq. (3.4)].
2. The infinitesimal increment of a quantity is equal to its rate of change by dt [Eq. (3.5)].
3. The noise and the system at the same time are independent.

With a Stratonovich SDE the first assumption is true, and the usual explanation is that the second is also true but that the third assumption is false. For reasons which will become apparent in Sec. 3.2, I prefer to characterize a Stratonovich SDE by saying that the second assumption is false (or true only implicitly) and that the third is still true. An alternative choice of which postulates to relax is that of the Itô stochastic calculus [50]. With an Itô SDE, the first assumption is false, the second is true in an explicit manner, and the third is also true. The Itô form has the advantage that it simply allows the increment in a quantity to be calculated, and also allows ensemble averages to be taken easily. It has the disadvantage that one cannot use the usual chain rule.

3.1.2 Itô Stochastic Differential Calculus

Because different rules of calculus apply to the Itô and Stratonovich forms of an SDE, the equation will appear differently in general. The Itô form of the Stratonovich equation (3.1) is

$$dx = [\alpha(x) + \tfrac{1}{2}\beta(x)\beta'(x)]dt + \beta(x)dW(t). \tag{3.9}$$

Here, the infinitesimal Wiener increment has been introduced, defined by

$$dW(t) = \xi(t)dt. \tag{3.10}$$

I have also introduced a convention of indicating Itô equations by an explicit representation of an infinitesimal increment [as in the left side of Eq. (3.9)], while Stratonovich equations will be indicated by an implicit equation with a fluxion on the left side [as in Eq. (3.1)]. If an Itô equation is given as

$$dx = a(x)dt + b(x)dW(t), \tag{3.11}$$

then the corresponding Stratonovich equation is

$$\dot{x} = a(x) - \tfrac{1}{2}b(x)b'(x) + b(x)\xi(t). \tag{3.12}$$

These equations will be derived within a broader context in Sec. 3.2.

In the Itô form, the noise is independent of the system state, so that the expected increment in x from Eq. (3.11) is simply

$$\mathrm{E}[dx] = a(x)dt. \tag{3.13}$$

However, the nonsense result (3.8) is avoided because the chain rule does not apply to calculating $df(x)$. The actual increment in $f(x)$ is simple to calculate by using a Taylor expansion for $f(x+dx)$. The difference with the usual chain rule is that second order infinitesimal terms cannot be necessarily ignored. This arises because the noise is so singular that second order noise infinitesimals are as large as first order deterministic infinitesimals. Specifically, the infinitesimal Wiener increment $dW(t)$ can be assumed to be defined by the following Itô rules

$$\mathrm{E}[dW(t)^2] = dt, \tag{3.14}$$

$$\mathrm{E}[dW(t)] = 0. \tag{3.15}$$

These are a consequence of Eqs. (3.2,3.3). Furthermore, it is possible to omit the expectation value in Eq. (3.14) because in any finite time, a time average effects an ensemble average of what is primarily a deterministic rather than stochastic quantity. This can be formulated more rigorously in terms of a mean square limit [50]. Expanding the Taylor series to second order gives the modified chain rule

$$df(x) = f'(x)dx + \tfrac{1}{2}f''(x)(dx)^2. \tag{3.16}$$

Specifically, with dx given by Eq. (3.11), and using the rule $dW(t)^2 = dt$,

$$df(x) = \left[f'(x)a(x) + \tfrac{1}{2}f''(x)b(x)^2\right]dt + f'(x)b(x)dW(t). \tag{3.17}$$

With this definition, and with $f(x) = x^2$, one finds that the expected variance of x an infinitesimal time after the known initial condition $x(t)$ is

$$\mathrm{E}[x(t+dt)^2] - \mathrm{E}[x(t+dt)]^2 = b\big(x(t)\big)^2 dt. \tag{3.18}$$

That is to say, the effect of the noise is to increase the variance of x. Thus, the correct use of the stochastic calculus evades the absurd result of Eq. (3.8). Because a non-zero $b(x)$ increases the variance in x, $b(x)^2$ is known as the diffusion coefficient, while $a(x)$ is called the drift coefficient. These names were originally applied to the two terms of a Fokker-Plank equation. This is an equation which expresses the drift and diffusion of a variable within a deterministic framework. To do this, it is necessary to use a probability distribution for x, rather than dealing with x itself as a random variable. This distinction was discussed in Sec. 1.1.1, in the Introduction of this thesis. The Fokker-Planck equation (FPE) for the probability distribution p of a variable x obeying the Itô SDE (3.11) is

$$\dot{p}(x) = \left[-\partial_x a(x) + \tfrac{1}{2}\partial_x{}^2 b(x)^2\right] p(x). \tag{3.19}$$

The Fokker-Planck equation has found much application in quantum optics, and this thesis is no exception. Usually, the distinction between a SDE and its corresponding FPE is not important, with the SDE commonly being used as a calculation tool. However, in this thesis, it is necessary to maintain this distinction, because the noise terms will often have a physical interpretation as measurement results. In this context, I find it necessary to consider stochastic equations for probability distributions. This mixture of SDEs and FPEs may seem unusual to those who use them interchangeably, but arises naturally in this thesis.

3.2 Implicit and Explicit Stochastic Equations

3.2.1 Non-Gaussian Non-White Noise

The preceding theory of stochastic differential equations is inadequate for this thesis for two reasons. Firstly, the sort of noise which arises in quantum trajectories is more general than Gaussian white noise. For example, as discussed in Sec. 2.3.2, continuous observation of a Markovian system naturally leads to a measurement record which is a point process. Secondly, when feedback is considered, the simple distinction between Itô and Stratonovich equations given above fails even for Gaussian white noise. That is because noise which is fed back is necessarily correlated with the system at the time it is fed back, and cannot be decorrelated by invoking Itô calculus. Thus, what is needed is a more general theory of stochastic calculus which can deal with non-Gaussian non-white noise, possibly including feedback. The case of feedback is considered in Sec. 3.3, and will be seen to fit naturally into the structure developed in this section. A different sort of generalization of the stochastic calculus of Sec. 3.1, involving operator-valued noise terms, will be discussed later in this chapter (Sec. 3.4 – 3.7). Although the noise considered in this and the following section may be non-white (white noise has a flat noise spectrum), it must still have an infinite bandwidth. If the correlation function for the noise were a smooth function of time, then there would no need to use any sort of stochastic calculus; the regular rules of calculus would apply.

The sort of general equation which must be considered is of the form (again only one-dimensional for now)

$$dx = k(x)dM(t). \tag{3.20}$$

Here, deterministic evolution is being ignored, so $dM(t)$ is some stochastic increment. If $dM(t) = dW(t)$ then Eq. (3.20) is an Itô SDE. More generally, $dM(t)$ will have well-defined moments which may depend on the state of the system x. A stochastic calculus will be necessary if second or higher order moments of $dM(t)$ are not of second or higher order in dt. For Gaussian white noise, only the second order moments fit this description, with $dW(t)^2 = dt$. An example in which all moments must be considered is a point process increment $dM(t) = dN(t)$. This can be defined by

$$\begin{aligned} \mathrm{E}[dN(t)] &= \wp(x)dt, & (3.21) \\ dN(t)^2 &= dN(t). & (3.22) \end{aligned}$$

Equation (3.21) indicates that the mean of $dN(t)$ is of order dt and may depend on the state of the system x. Equation (3.22) simply states that $dN(t)$ equals either zero or one, which is why it is called a point process. Obviously all moments of $dN(t)$ are of the same order as dt, so the chain rule for $f(x)$ will completely fail. In this case of a point process, it is simpler to explicitly do the calculation with the two possible values of $dN(t)$, and to weight the two results with probabilities $\wp(x)dt$ for $dN(t) = 1$ and $1 - \wp(x)dt$ for $dN(t) = 0$. In general, it would not be possible to do this, and one would have to expand a function of x to as many orders in $dM(t)$ as the stochastic rules for $dM(t)$ would require.

Equation (3.20) explicitly gives the increment in the quantity x. For this reason, I will call an equation of this form an *explicit* SDE. As noted, the usual chain rule for $f(x)$ does not apply. On the other hand, one could imagine a SDE which, like the Stratonovich equation for Gaussian white noise, arises from a physical process in which the singularity of the noise is an idealization. Such an equation would be written, using my convention, as

$$\dot{x} = \chi(x)\mu(t), \tag{3.23}$$

where $\mu(t)$ is a noisy function of time which is idealized by

$$\mu(t) = dM(t)/dt. \tag{3.24}$$

Equation (3.23) is an *implicit* equation in that it gives the increment in x only implicitly. It has the advantage that $f(x)$ would obey an implicit equation as given by the usual chain rule,

$$\dot{f}(x) = f'(x)\chi(x)\mu(t). \tag{3.25}$$

Notice that the third distinction between Itô and Stratonovich calculus, based on the independence of the noise term and the system at the same time, has not entered this discussion. That is because even in the explicit equation (3.20), the noise may depend on the system state. The independence condition is simply a peculiarity of Gaussian white noise. The implicit – explicit distinction is more general than the Stratonovich – Itô distinction. As I will show below, the relationship between the Stratonovich and Itô SDEs can be easily derived (albeit heuristically) within this more general framework without using the concept of stochastic integration.

So far I have not specified what qualities are necessary for a valid stochastic increment. Say the stochastic increment is specified by a probability distribution

$$\Pr[dM(t) = dM] = \Upsilon_M\big(dM, x(t), dt\big). \tag{3.26}$$

This equation contains the Markovian assumption that $dM(t)$ depends on previous increments only through the effect on x. The explicit rules for dW (3.14,3.15) and dN (3.21,3.22) are simple ways of expressing the respective distributions. One obvious consistency condition is that the distribution should scale correctly with dt, by which I mean

$$\Pr[dM(t) + dM(t+dt) = dM] = \Upsilon_M\big(dM, x(t), 2dt\big). \tag{3.27}$$

It is easy to verify that dW and dN scale this way. Presumably, if this law is obeyed then the probability distribution for x will obey a differential Chapman-Kolmogorov equation [50], which is simply the general time-continuous version of a Markovian process. The fact that this equation consists of three terms, representing drift, diffusion, and jumps, suggests that it would be possible to treat all stochastic increments as diffusion using dW or jumps using dN [1]. While this may be true, it does not rule out the use of other stochastic increments which may be cumbersome to express in these terms. For example, the Cauchy process [50, 137], has an increment $dM = dC$ with

$$\Upsilon_C(dC, x, dt) = \frac{dt}{\pi} \frac{1}{dt^2 + dC^2}. \tag{3.28}$$

This satisfies Eq. (3.27) and gives a differential Chapman-Kolmogorov equation with a jump term only. However, it is a very singular increment; even the second power of dC is unbounded. The only stochastic increments which will actually be used in this thesis are dN and dW.

3.2.2 Making Implicit Equations Explicit

The general problem to be solved in this subsection is to find the explicit form of an implicit SDE with arbitrary noise. The implicit form, which arises as the limit of an equation with a physical noise source, is given as

$$\dot{x}(t) = \chi(x)\mu(t), \tag{3.29}$$

where the symbols are as defined above. For implicit equations, the usual chain rule (3.25) applies, and can be rewritten

$$\dot{f} = f'(x)\chi(x)\mu(t) = \phi\big(f(x)\big)\mu(t), \tag{3.30}$$

where $\phi(f)$ is defined here implicitly. Now, in order to solve Eq. (3.29), it is necessary to find an explicit expression for the increment in x. The approach I have adopted is to solve Eq. (3.29) under the rules of regular calculus, expanding the Taylor series to all orders in dM. This can be written formally as

$$\begin{aligned} x(t+dt) &= \exp\left(dt\partial_s\right) x(s)|_{s=t} && (3.31)\\ &= \exp\left[\chi(x)dM(t)\partial_x\right] x|_{x=x(t)}. && (3.32)\end{aligned}$$

Here I have used the relation

$$\left[\frac{d}{ds}x(s) = \chi\big(x(s)\big)\frac{dM(t)}{dt}\right]_{s=t}, \tag{3.33}$$

which is the explicit meaning of the implicit Eq. (3.29). That $\mu(t)$ is assumed to be constant, while $x(s)$ is evolved, is an expression of the fact that the noise $\mu(t)$ cannot in reality be delta-correlated. If the noise $\mu(t)$ is the limit of a physical process [which is the limit for which Eq. (3.29) is intended to apply], then it must have some finite correlation time over which it remains relatively constant. The noise can be considered delta-correlated if that time can be considered to be infinitesimal compared to the characteristic evolution time of the system x.

The explicit SDE is thus defined to be

$$dx(t) = \left(\exp\left[\chi(x)dM(t)\partial_x\right] - 1\right) x(t), \tag{3.34}$$

which means

$$dx(t) = \left(\exp\left[\chi(x)dM(t)\partial_x\right] - 1\right) x|_{x=x(t)}. \tag{3.35}$$

[1] In fact, van Kampen [137] indicates that the classical master equation is fundamentally built on jump processes, and that diffusion is only one approximation.

This expression will converge for all $\chi(x)$ for $dM = dN$ or $dM = dW$. For $dM = dC$, it will only converge if $\chi(x)$ is constant, which is not a failure of the theory, but rather a statement of the extreme singularity of Cauchy noise. Assuming convergence, Eq. (3.35) is compatible with the requirement on the implicit form [Eq. (3.29)], independent of the nature of the stochasticity. This can be seen from calculating the increment in $f(x)$ using the explicit form:

$$\begin{aligned} df(t) &= f\big(x(t)+dx(t)\big) - f\big(x(t)\big) \\ &= f\big(\exp\left[\chi(x)dM(t)\partial_x\right] x|_{x=x(t)}\big) - f\big(x(t)\big) \\ &= \exp\left[\chi(x)dM(t)\partial_x\right] f(x)|_{x=x(t)} - f\big(x(t)\big) \\ &= \left(\exp\left[\phi(f)dM(t)\partial_f\right] - 1\right) f|_{f=f(x(t))}. \end{aligned} \tag{3.36}$$

The final expression (3.36) here is precisely what would have been obtained by turning the implicit equation (3.29) into an explicit equation. This completes the proof that Eq. (3.34) is the correct explicit form of the implicit Eq. (3.29).

For deterministic processes, there is no distinction between the explicit and implicit forms, as only the first order expansion of the exponential remains with dt infinitesimal. There is also no distinction if $\chi(x)$ is a constant, which was seen to be neccesary for a Cauchy process. For Gaussian white noise, the formula (3.34) is the rule given in Sec. 3.1.2 for converting from Stratonovich to Itô. That is, if the Stratonovich SDE is Eq. (3.29) with $dM(t) = dW(t)$, then the Itô SDE is

$$dx(t) = \chi(x)dW(t) + \tfrac{1}{2}\chi(x)\chi'\big(x(t)\big)dt. \tag{3.37}$$

Here, the Itô rule $dW(t)^2 = dt$ has been used. This rule implies that it is only necessary to expand the exponential to second order. This fact makes the inverse transformation (Itô to Stratonovich) easy. For the jump process, the rule $dN(t)^2 = dN(t)$ means that the exponential must be expanded to all orders. This gives

$$dx(t) = dN(t)\left(\exp\left[\chi(x)\partial_x\right] - 1\right) x(t). \tag{3.38}$$

In this case, the inverse transformation would not appear to be easy to find in general.

The multidimensional generalization of the above formulas is obvious. If the implicit form is (using the Einstein summation convention)

$$\dot{x}_i(t) = \chi_{ij}\big(\mathbf{x}(t)\big)\mu_j(t), \tag{3.39}$$

then the explicit form is

$$dx_i(t) = \left(\exp\left[\chi_{kj}(\mathbf{x})dM_j(t)\partial_k\right] - 1\right) x_i(t). \tag{3.40}$$

This is quite complicated in general. When considering quantum-limited feedback, the system is represented by the state matrix ρ which is in general specified by a double infinity of real numbers. Fortunately, however, its equations of motion are linear. Thus, if one has the implicit equation

$$\dot{\rho}(t) = \mu(t)\mathcal{K}\rho(t), \tag{3.41}$$

where $\mathcal{K}$ is a Liouville superoperator, then the explicit SDE is simply

$$d\rho(t) = \left(\exp\left[\mathcal{K}dM(t)\right] - 1\right)\rho(t). \tag{3.42}$$

3.3 Stochastic Equations with Feedback

3.3.1 Non-Markovian SDEs

In Sec. (3.2), it was established that it is necessary to generalize the Stratonovich – Itô distinction in order to deal with stochastic terms more general than Gaussian white noise. In this section, I argue that if one wishes to consider feedback, then the implicit – explicit distinction is more useful than the Stratonovich – Itô distinction, even if one is dealing with Gaussian white noise. In this context, feedback means that the noise, after having interacted with the system at some time, then interacts with it again at some later time. As an example, consider Gaussian white noise, and let the initial interaction of the noise with the system be via the implicit (Stratonovich) equation

$$\dot{x}(t) = \beta\big(x(t)\big)\xi(t). \tag{3.43}$$

This equation will be referred to as a measurement interaction. Now let the noise interact again with the system at a time τ later, via the implicit feedback equation

$$[\dot{x}(t)]_{\rm fb} = \gamma\big(x(t)\big)\xi(t-\tau). \tag{3.44}$$

Turning this into an explicit equation by the rule (3.37) gives the complete evolution

$$dx(t) = \beta\big(x(t)\big)dW(t) + \gamma\big(x(t)\big)dW(t-\tau) + \tfrac{1}{2}\beta\big(x(t)\big)\beta'\big(x(t)\big)dt + \tfrac{1}{2}\gamma\big(x(t)\big)\gamma'\big(x(t)\big)dt. \tag{3.45}$$

While it may be correct to call Eqs. (3.43,3.44) Stratonovich equations, it is not correct to call Eq. (3.45) an Itô equation. This is because, even though it is an explicit equation with Gaussian white noise, the noise in the feedback term is not independent of the state of the system at the time it acts. Thus it is not possible simply to neglect the noise terms in calculating the ensemble average evolution of the system. Rather, there is the non-zero correlation

$$\mathrm{E}\left[\gamma\big(x(t)\big)dW(t-\tau)\right] \tag{3.46}$$

which is central to the action of the feedback. Although Eq. (3.45) is not an Itô equation, it is nevertheless a valid explicit equation. If one were to solve this feedback system by stochastic numerical simulations, then the explicit equation (3.45) is the algorithm for calculating infinitesimal increments for the system. In general, this numerical method may be the only recourse.

3.3.2 The Markovian Limit

In order to make the feedback problem described in the preceding section more tractable, it would be desirable to derive a true Itô equation, with all stochastic increments being independent of the state. This would only be possible in the Markovian ($\tau \to 0$) limit. In taking this limit, it is necessary to remember that the feedback (3.44) must act after the measurement (3.43). To do this, it is useful to define a new variable

$$y = \exp\left[\beta(x)dW(t)\partial_x\right]x = x + \beta(x)dW(t) + \tfrac{1}{2}\beta(x)\beta'(x)dt \tag{3.47}$$

equal to x after the measurement but before the feedback. The feedback then acts on the new variable y to give the complete evolution in the time dt:

$$x(t+dt) = \exp\left[\gamma(y)dW(t)\partial_y\right]y = y + \gamma(y)dW(t) + \tfrac{1}{2}\gamma(y)\gamma'(y)dt. \tag{3.48}$$

Substituting in the expression (3.47), and using a Taylor expansion for $\gamma(y)$ in $y - x$ yields the explicit increment in x including measurement and feedback

$$dx = \gamma'(x)b(x)dt + \tfrac{1}{2}\gamma(x)\gamma'(x)dt + \tfrac{1}{2}\beta(x)\beta'(x)dt + [b(x) + \gamma(x)]dW(t). \tag{3.49}$$

This equation is a true Itô equation, with the noise $dW(t)$ being independent of the state $x(t)$. The first deterministic increment here represents the feedback, while the others are simply the usual Itô corrections.

Equation (3.49) can be derived directly by exponentiating differential evolution operators as follows

$$dx = (\exp[\beta(x)dW(t)\partial_x]\exp[\gamma(x)dW(t)\partial_x] - 1)\, x. \tag{3.50}$$

At first sight this equation appears confusing, as it seems to place the feedback *before* the measurement. However, on reflection, the terms are correctly placed. The measurement differential operator has the effect of replacing all of the xs to its right by ys, where y is as defined in Eq. (3.47). The action of the feedback is then just as described above in Eq. (3.48). Furthermore, the general equation with a possible time delay τ can be written

$$dx = (\exp[\beta(x)dW(t)\partial_x]\exp[\gamma(x)dW(t-\tau)\partial_x] - 1)\, x. \tag{3.51}$$

For τ finite, this reduces to the non-Markovian Eq. (3.45) above. With $\tau = 0$, it gives Eq. (3.49), which correctly takes into account the correlation (3.46).

For other sorts of noise, the preceding argument would be modified as follows. Let the initial ("measurement") interaction be given by

$$\dot{x}(t) = \psi\big(x(t)\big)\mu(t), \tag{3.52}$$

and let the noise $\mu(t) = dM/dt$ be fed back by a physical process giving the implicit SDE

$$\dot{x} = \chi\big(x(t)\big)\mu(t-\tau). \tag{3.53}$$

Then the explicit equation describing the measurement plus feedback is

$$dx(t) = (\exp[\psi(x)dM(t)\partial_x]\exp[\chi(x)dM(t-\tau)\partial_x] - 1)\, x(t). \tag{3.54}$$

In the limit $\tau \to 0$, this represents a Markovian SDE. It can only be called an Itô SDE for the case $dM = dW$, but in any case a Markovian equation has many advantages, as will be seen in Ch. 6. As another illustration, consider the Markovian limit with $dM = dN$. Then,

$$dx(t) = dN(t)\,(\exp[\psi(x)\partial_x]\exp[\chi(x)\partial_x] - 1)\, x(t). \tag{3.55}$$

3.4 Quantum Stochastic Differential Equations

3.4.1 Quantizing the Electromagnetic Field

The remaining four sections of this chapter deal with quantum stochastic differential equations. These differ from the classical differential equations considered so far in that the noise terms are operators rather than c numbers. The origin of such quantum noise terms is best understood in the quantum optical case. There, the noise operators arise from the quantization of the continuum of electromagnetic field modes in the environment [2]. Thus, it is useful to review the quantization of the electromagnetic field (see for example Refs. [119, 52]). Let the fundamental field be the vector potential $\mathbf{A}(\mathbf{r}, t)$ in the Coulomb gauge. The free Lagrangian density for this field is

$$\mathcal{L} = \frac{\varepsilon_0}{2}\left(\mathbf{E}^2 - c^2\mathbf{B}^2\right), \tag{3.56}$$

where ε_0 is the permittivity of free space and c is the speed of light. The transverse electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields are defined by

$$\mathbf{E} = -\dot{\mathbf{A}}\ ; \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{3.57}$$

From Eq. (3.56), the canonical field to $\mathbf{A}$ is $-\varepsilon_0\mathbf{E}$. In quantizing the field, these obey the canonical commutation relations

$$[A_j(\mathbf{r}, t), E_k(\mathbf{r}', t)] = -\mathrm{i}\frac{\hbar}{\varepsilon_0}\delta^{\perp}_{jk}(\mathbf{r} - \mathbf{r}'). \tag{3.58}$$

Here $\delta^{\perp}_{jk}$ denotes a three-dimensional transverse delta-function

$$\delta^{\perp}_{jk}(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\left(\delta_{jk} - \frac{k_j k_k}{\mathbf{k}^2}\right) \tag{3.59}$$

Note that the Heisenberg picture operators in the canonical commutation relations are at equal times. In the Schrödinger picture, the same relations hold, but the time argument is omitted. The Euler-Lagrange (which is also the Heisenberg) equation of motion from Eq. (3.56) is the wave equation

$$\ddot{\mathbf{A}} = c^2\nabla^2\mathbf{A}. \tag{3.60}$$

Now consider the case of a beam of polarized light. That is to say, consider only one component A of $\mathbf{A}$ and let its spatial variation be confined to one direction, say z. This simplifies the analysis, and is also appropriate for determining the inputs and outputs of a quantum optical cavity. In reality, the transverse spatial extent of the beam would be confined to some area Λ which is determined by the area of the optical components involved [52]. However, as long as the x and y extensions are much greater than a wavelength, the beam can be approximated by plane waves. The appropriate wave equation is

$$\ddot{A} = c^2\partial_z^2 A, \tag{3.61}$$

of which I am interested only in the forward propagating solutions

$$A(z, t + t') = A(z - ct', t). \tag{3.62}$$

If the field is reflected off a cavity mirror (say at $z = 0$) then the direction of z will change at the point of reflection. This is why only one direction of propagation need be considered. The field for $z < 0$ is incoming and that for $z > 0$ is outgoing. The canonical commutation relation is now

$$[A(z, t), E(z', t)] = -\mathrm{i}\frac{\hbar}{\varepsilon_0\Lambda}\delta(z - z'). \tag{3.63}$$

Solutions for A and E satisfying the wave equation (3.61) can be constructed using the annihilation and creation operators for the modes of frequency ω, which satisfy

$$[a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega'). \tag{3.64}$$

[2] Strictly, the quantization of the electromagnetic field is not essential. All of the results of quantum electrodynamics can be derived from pair-wise and self interactions of Dirac particles [8]. Nevertheless, independent quantization of the electromagnetic field is, like second quantization of particles, a convenient tool.

They are

$$A(z,t) = \sqrt{\frac{\hbar}{\varepsilon_0 \Lambda 2\pi c}} \int_0^\infty d\omega \frac{1}{\sqrt{2\omega}} \left\{ a(\omega) \exp[-\mathrm{i}\omega(t - z/c)] + a^\dagger(\omega) \exp[\mathrm{i}\omega(t - z/c)] \right\}, \quad (3.65)$$

$$E(z,t) = \sqrt{\frac{\hbar}{\varepsilon_0 \Lambda 2\pi c}} \int_0^\infty d\omega \sqrt{\frac{\omega}{2}} \left\{ \mathrm{i}a(\omega) \exp[-\mathrm{i}\omega(t - z/c)] - \mathrm{i}a^\dagger(\omega) \exp[\mathrm{i}\omega(t - z/c)] \right\}. \quad (3.66)$$

Localized Photons

This expression for the fields in terms of annihilation and creation operators for a continuum of modes defines the sense in which they are composed of photons of definite frequency. However, this sense is quite unlike the naïve picture of a beam of light made up of (possibly different frequencies of) photons, hurtling through space at the speed of light. Each mode is spread over all space, so there is no way in which a photon, as an excitation of such a mode, can move at all. To define an annihilation operator $b(z,t)$ for a localized photon of a particular frequency, it would be necessary to sum many different mode operators. Such operators can be defined, with slight variations in the details of the definition [53, 54]. The various definitions are effectively equivalent in application to quantum optical problems. The authors of Refs. [53, 54] construct the localized annihilation operator from the mode annihilation operators $a(\omega)$. In this thesis, I am introducing a different definition for $b(z,t)$, constructed from the original fields in space-time, $A(z,t)$ and $E(z,t)$, without the intervening use of Fourier space. The purpose of defining $b(z,t)$ is the same as in Refs. [53, 54], to derive quantum stochastic differential equations for quantum optical systems, and the end result is completely equivalent. However, it is valuable pedagogically to be familiar with different approaches to the problem, which is why I have chosen to present a new definition of $b(z,t)$.

As established above, A and $-E$ are canonically conjugate variables at each point in space-time. Motivated by the analogy with position and momentum, a local annihilation operator for an oscillator of angular frequency ω_0 can be defined as

$$b(z,t) = \exp[\mathrm{i}\omega_0(t - z/c)] \sqrt{\frac{\Lambda \varepsilon_0 c}{\hbar}} \left[\sqrt{\frac{\omega_0}{2}} A(z,t) - \frac{\mathrm{i}}{\sqrt{2\omega_0}} E(z,t) \right]. \quad (3.67)$$

Here, the normalization constant has been chosen so that

$$[b(z,t), b^\dagger(z',t)] = c\delta(z - z'), \quad (3.68)$$

and the rotating exponential so that a "localized photon" of frequency ω_0 has a slowly varying annihilation operator $b(z,t)$ [3]. It also has the property (3.62), obeying

$$b(z, t + t') = b(z - ct', t) \quad (3.69)$$

in free space.

In terms of the mode operators, $b(z,t)$ is given by

$$b(z,t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \left\{ \frac{\omega_0 + \omega}{2\sqrt{\omega_0 \omega}} a(\omega) \exp[\mathrm{i}(\omega_0 - \omega)(t - z/c)] + \frac{\omega_0 - \omega}{2\sqrt{\omega_0 \omega}} a^\dagger(\omega) \exp[\mathrm{i}(\omega_0 + \omega)(t - z/c)] \right\}. \quad (3.70)$$

Note that $b(z,t)$ includes both annihilation and creation mode operators, unlike previous definitions [53, 54] which have only included annihilation operators $a(\omega)$. However, if one is only concerned

[3] It has been pointed out to me by one examiner that this commutation relation is not exact, because the fields $A(z)$ and $E(z)$ do not commute with themselves at different positions. However, it is easily shown that the additional terms are negligible if one is concerned only with frequencies near to ω_0.

with light at frequency ω_0, which is the assumption behind the definition (3.67), then the coefficient in the integrand for creation operators at frequency ω_0 is zero, while the coefficient for annihilation operators near the same frequency is one. This is the sense in which $b(z,t)$ pertains to localized photons of frequency ω_0. If only frequencies near ω_0 are significantly excited, then the time-flux of energy can be easily seen to be

$$W(z,t) \simeq \hbar\omega_0 b^\dagger(z,t)b(z,t). \tag{3.71}$$

Thus, the operator $b(z,t)$ conforms to one's naïve expectations.

3.4.2 Linear System-Bath Coupling

Consider a localized system at $z=0$ interacting with the bath of electromagnetic field modes through a dipole coupling. That is to say, the interaction Hamiltonian is of the form

$$V(t) = D(t)E(0,t), \tag{3.72}$$

where $D(t)$ is the Heisenberg operator for the dipole moment of the system. Let this dipole be driven by the system Hamiltonian to oscillate at frequency ω_0. Thus the dipole can be assumed to be of the form

$$D(t) \propto c(t)e^{-\mathrm{i}\omega_0 t} + c^\dagger(t)e^{\mathrm{i}\omega_0 t}, \tag{3.73}$$

where $c(t)$ is a slowly varying (compared to ω_0) system operator which acts to lower the system energy by $\hbar\omega_0$. From Eq. (3.67), the interaction Hamiltonian can then be written

$$V(t) = \mathrm{i}\hbar[c(t)e^{-\mathrm{i}\omega_0 t} + c^\dagger(t)e^{\mathrm{i}\omega_0 t}][b(0,t)e^{-\mathrm{i}\omega_0 t} - b^\dagger(0,t)e^{\mathrm{i}\omega_0 t}], \tag{3.74}$$

where time is being measured inversely in units of the dipole energy damping rate. Now since ω_0 is assumed to be the dominant frequency, only resonant terms in this Hamiltonian need be kept. This is known as the rotating wave approximation. From Eq. (3.70), $b(0,t)$ has slowly varying components, and also components varying at frequency $2\omega_0$. However, as noted above, the coefficient in the integrand of these latter components is zero. Thus, they may be ignored, and $b(0,t)$ can be treated as if it had only slowly varying components. In the rotating wave approximation, the interaction Hamiltonian is therefore

$$V(t) = \mathrm{i}\hbar[b^\dagger(0,t)c(t) - c^\dagger(t)b(0,t)], \tag{3.75}$$

where this has been given an Hermitian operator ordering.

The Heisenberg equation of motion for an arbitrary system operator s, ignoring the free dynamics, is thus

$$\dot{s}(t) = -[b^\dagger(0,t)c(t) - c^\dagger(t)b(0,t), s(t)]. \tag{3.76}$$

This equation is a quantum analogue of a Stratonovich equation, where the white noise terms are now operators satisfying

$$[b(0,t), b^\dagger(0,t')] = \delta(t-t'), \tag{3.77}$$

where Eqs. (3.68) and (3.69) have been used. It is the singularity of the commutation relations (3.77) which make it necessary to use stochastic calculus to deal with Eq. (3.76). The Stratonovich form [as indicated by the use of a fluxion rather than an infinitesimal increment in Eq. (3.76)] is appropriate because in reality, the system would be coupled to the electric field over a finite region of space. If the spatial extent of the system is δz, then the bandwidth of the coupling would be of order $\delta\omega \sim c/\delta z$. This finite (but large compared to ω_0) bandwidth means that the noise is not strictly white. This also allows different system frequencies to be damped independently to the same physical bath, the electromagnetic field [52]. To find a useful version of Eq. (3.76), it is necessary to use the Itô quantum stochastic calculus.

In the Itô calculus, the noise is independent of the state of the system at the same time. This will be true if the bath operators $b(0,t)$ in Eq. (3.76) are assumed to represent the field immediately before it interacts with the system:

$$b_0(t) \equiv b(0^-,t). \tag{3.78}$$

This can be thought of as an *input* bath operator, and is the quantum analog of $\xi(t)$ [53]. The analog of the infinitesimal Wiener increment (3.10) is then

$$dB_0(t) = b_0(t)dt, \tag{3.79}$$

which satisfies

$$[dB_0(t), dB_0^\dagger(t)] = dt. \tag{3.80}$$

The evolution of an arbitrary operator is then given explicitly by

$$s(t+dt) = U^\dagger(t,t+dt)s(t)U(t,t+dt), \tag{3.81}$$

where

$$U(t,t+dt) = \exp\left[dB_0^\dagger(t)c(t) - c^\dagger(t)dB_0(t)\right] \tag{3.82}$$

must be expanded to second order because of the commutation relations (3.80).

In order to do the expansion of $U(t,t+dt)$, it is necessary to know the moments of $dB_0(t)$. If $dB_0(t)$ is to be thought of as a bath, it should be specifiable simply by its first and second order moments. In this subsection, I wish to consider only the simplest case, a bath in the vacuum state. Then it is completely specified by

$$dB_0(t)dB_0^\dagger(t) = dt, \tag{3.83}$$

with all other first and second order moments vanishing. This non-vanishing second order contribution could be thought of as vacuum noise. Using this relation, the explicit quantum Langevin equation is

$$ds = \left(c^\dagger sc - \tfrac{1}{2}sc^\dagger c - \tfrac{1}{2}c^\dagger cs\right)dt - [dB_0^\dagger c - c^\dagger dB_0, s]. \tag{3.84}$$

The final, stochastic term in this equation is essential to preserve canonical commutation relations [53]. That these will be preserved can be verified by showing that Eq. (3.84) has the property that

$$d(s_1 s_2) = s_1(ds_2) + (ds_1)s_2 + (ds_1)(ds_2), \tag{3.85}$$

for arbitrary system operators s_1, s_2, where each of the increments is calculated using Eq. (3.84). The second order infinitesimal is needed because of the Itô calculus.

In the approach of Gardiner and Collett [53], Eq. (3.84) (which can be called a quantum Langevin equation) could be either a Stratonovich or an Itô equation. In their treatment, the Itô – Stratonovich distinction only enters when considering the statistics of the input field, not its commutation relations. Here, I prefer to consider Eq. (3.76) as the Stratonovich form, so that the decay of the dipole exhibited by the deterministic term in the Itô equation (3.84) appears to be due to vacuum fluctuations of the quantized electromagnetic field [4]. This interpretation is more consistent with

[4] This is of course only one point of view. The dipole damping rate, and its alteration in cavity QED experiments, can be explained by radiation reaction, as in classical electromagnetism [102, 32, 39].

the general treatment of classical implicit and explicit equations given above. The difference arises because in my treatment the bath operators also evolve during the time interval, and obey the same Stratonovich equation (3.76) as system operators. By contrast, the bath operator of Gardiner and Collett is treated as a noise input, and does not obey the same Stratonovich equation as the system. Yet another alternative, but equivalent, approach to treating the bath operators in quantum optics is given by Carmichael [19].

Under the Hamiltonian (3.75), the bath field also evolves in time via the unitary operator (3.82). However, the overall evolution of the bath is dominated by the free bath dynamics, which causes the field to be translated forward in space as in Eq. (3.69). Thus, if one defines an *output* bath operator at time t by

$$b_1(t) \equiv b(0^+, t), \tag{3.86}$$

then this will be related to the input bath operator by

$$b_1(t) = U^\dagger(t, t + dt) b_0(t) U(t, t + dt). \tag{3.87}$$

This argument can be proven more rigorously using the frequency components of the field [53]. Expanding the unitary operator to lowest order in dt gives

$$b_1(t) = b_0(t) + c(t), \tag{3.88}$$

which is independent of dt, as is necessary. If the system is an optical cavity, then this operator represents the field immediately after it has been reflected at the cavity mirror. Just as $b_0(t)$ commutes with an arbitrary system operator $s(t')$ at an earlier time $t' < t$, it can simply be shown [53] that $b_1(t)$ commutes with the system operators at a later time $t' > t$. This fact is crucial to the quantum mechanical approach to feedback to be developed in Ch. 8. As a consequence of this, the output field obeys the same commutation relations as the input field

$$[b_1(t), b_1^\dagger(t')] = \delta(t - t'), \tag{3.89}$$

as required because it is a free field also. Unlike $b_0(t)$ for earlier times, $b_1(t)$ for later times is not independent of the state of the system. This is why a measurement of the output field effects a measurement of the system, as will be explored in the following Chapter.

3.4.3 Non-Vacuum Bath Input

In the preceding subsection, the bath was assumed to be in the vacuum state. That is to say, in the eigenstate of zero eigenvalue of the annihilation operators $b_0(t)$ for all t. An almost trivial generalization of this case is if the bath is in a non-zero eigenstate of $b_0(t)$ with eigenvalues $\beta(t)$. That is to say, the input is in a time-varying coherent state. Such an input simply changes $b_0(t)$ to $\nu(t) + \beta(t)$, where $\nu(t)$ represents the operator part of $b_0(t)$, with zero mean. This adds the following term to the system Hamiltonian:

$$i\hbar[c(t)\beta^*(t) - c^\dagger(t)\beta(t)]. \tag{3.90}$$

This alteration could represent any pseudo-classical input (having a positive Glauber-Sudarshan P representation [63, 130]) by allowing $\beta(t)$ to be stochastic. However, it cannot represent a nonclassical input such as produced by a degenerate parametric oscillator (DPO) with finite bandwidth. That is because the output of such a DPO cannot be considered to be independent of its source, and so is not a bath. It is possible to consider such an input using the formalism presented so far, but in general only by explicitly including the source. This will be considered in Sec. 3.7. The nonclassical correlations in the output of the DPO would cause the source and driven system to become entangled. This widens the scope of what must be included as the system, and narrows the scope of the bath to the input to the DPO. At some stage, the input should be what would be expected from the background radiation in the laboratory. That is to say, a vacuum state or a thermal state.

A thermal state is a pseudo-classical state, and so can be represented by a suitable stochastic amplitude $\beta(t)$. In fact, because thermal fluctuations are present at all frequencies, $\beta(t)$ can be approximated by white noise in the frequency regime of interest, around ω_0. That is to say, $\beta(t)$ would satisfy

$$\mathrm{E}[\beta^*(t)\beta(t')] = N\delta(t-t')\ ;\ \ \mathrm{E}[\beta(t)] = 0, \tag{3.91}$$

where

$$N = [\exp(\hbar\omega_0/k_BT) - 1]^{-1}. \tag{3.92}$$

Consider the case where $c = \sqrt{\kappa}a\exp[\mathrm{i}\omega_0 t]$, where a is the annihilation operator for a cavity mode of frequency ω_0, and the damping rate κ has been specifically included. Then the equation of motion for a is, from Eq. (3.84),

$$\dot{a}(t) = -\left(\mathrm{i}\omega_0 + \frac{\kappa}{2}\right)a(t) - \sqrt{\kappa}[\beta(t) + \nu(t)], \tag{3.93}$$

Note that, apart from the vacuum noise operator $\nu(t)$, this equation is equivalent to the first Langevin equation (1.1) in the Introduction, pertaining to a classical optical system with amplitude $\alpha(t)$. Indeed, the classical equation is identical to the pseudo-classical one which can be derived from the Glauber-Sudarshan P function [which obeys the Fokker-Planck equation (1.2)]. The vacuum operator in Eq. (3.93) is necessary only to preserve the commutation relations between a and $a^\dagger$.

Equation (3.93) is a Stratonovich equation with respect to $\beta(t)$. To calculate the evolution of moments of the intracavity field it would be necessary to use the Itô form. This seems clumsy, because a Stratonovich to Itô conversion has already been made in converting the original equation quantum Stratonovich equation (3.76) to the quantum Itô equation (3.84). It would be more elegant to take into account the approximately white thermal noise of $b_0(t)$ at the same time as its exactly white vacuum noise. This can be done by specifying the higher order moments of $dB_0(t)$ to be

$$dB_0(t)dB_0^\dagger(t) = (N+1)dt, \tag{3.94}$$

rather than (3.83). From the commutation relations (3.80), $dB_0^\dagger(t)dB(t) = N dt$, while for a thermal state the phase-dependent moments $dB_0(t)^2$ vanish. However, there is no physical reason why $dB_0(t)^2$ must always vanish. Therefore in general, this moment can be defined to be

$$dB_0(t)^2 = M dt, \tag{3.95}$$

where M is a complex number limited only by the requirement [52]

$$|M|^2 \leq N(N+1). \tag{3.96}$$

As will be shown in Sec. 3.5, this condition follows from the positivity of the density operator for the bath.

Using these rules in expanding the unitary operator in Eq. (3.81) gives the general QLE for a white noise bath

$$\begin{aligned} ds &= \tfrac{1}{2}\left\{(N+1)(2c^\dagger sc - sc^\dagger c - c^\dagger cs) + N(2csc^\dagger - scc^\dagger - cc^\dagger s)\right. \\ &\quad \left. + M[c^\dagger,[c^\dagger,s]] + M^*[c,[c,s]]\right\}dt - [dB_0^\dagger c - c^\dagger dB_0 - \mathrm{i}H\,dt, s], \end{aligned} \tag{3.97}$$

where the free Hamiltonian for the system, $\hbar H$, has been included. If $|M|^2 \leq N^2$, then this equation can be derived as the Itô form of the Stratonovich equation derived from the stochastic Hamiltonian (3.90), with

$$\mathrm{E}[\beta(t)\beta(t')] = M\delta(t-t') \tag{3.98}$$

in addition to Eq. (3.91). However, if

$$N(N+1) \geq |M|^2 > N^2, \tag{3.99}$$

then Eq. (3.98) is impossible, and there is no pseudo-classical derivation of Eq. (3.97). Instead, the condition (3.99) indicates that the input light is what is called squeezed white noise [53]. It can be considered to be the broad-band limit of the squeezed light such as produced by a DPO. The entanglement which would ordinarily be produced between the source and driven cavities is avoided because the broad-band assumption indicates that the state of the source of squeezing changes so rapidly that nonclassical correlations persist only over time scales much shorter than those of interest for the driven system. This allows the output from the DPO to be treated as a bath as in Eq. (3.95).

3.5 The Master Equation

Although the stochastic terms in the QLE (3.97) are necessary to preserve commutation relations, they can be omitted in calculating the rate of change of expectation values, because they are quantum Itô terms, independent of all system operators at equal times. Thus,

$$\begin{aligned}\langle \dot{s} \rangle \;&=\; \tfrac{1}{2}\langle (N+1)(2c^\dagger sc - sc^\dagger c - c^\dagger cs) + N(2csc^\dagger - scc^\dagger - cc^\dagger s)\rangle \\ &\quad + \tfrac{1}{2}\langle M[c^\dagger,[c^\dagger,s]] + M^*[c,[c,s]]\rangle - \langle [\beta^* c - \beta c^\dagger - \mathrm{i}H, s]\rangle,\end{aligned} \tag{3.100}$$

where a coherent bath amplitude has been included for completeness. Note that a fluxion $\dot{s}$ rather than an increment ds has been used, because there is no implicit – explicit distinction for deterministic equations such as this one. Now this equation is Markovian, depending only on the average of system operators at the same time. Therefore, it should be able to be derived from a Markovian evolution equation for the system in the Schrödinger picture. That is to say, there should exist a master equation for the system density operator such that

$$\langle \dot{s}(t)\rangle = \mathrm{Tr}[s\dot{\rho}(t)]. \tag{3.101}$$

Here, the placement of the time argument indicates the picture (Heisenberg or Schrödinger).

By inspection of Eq. (3.100), the corresponding master equation is

$$\dot{\rho} = (N+1)\mathcal{D}[c]\rho + N\mathcal{D}[c^\dagger]\rho + \frac{M}{2}[c^\dagger,[c^\dagger,\rho]] + \frac{M^*}{2}[c,[c,\rho]] - \mathrm{i}[H + \mathrm{i}\beta^* c - \mathrm{i}\beta c^\dagger, \rho]. \tag{3.102}$$

This is the general master equation for a white noise optical bath. It can be shown [52] that it is of the Lindblad form [Eq. (2.32)], providing $|M|^2 \leq N(N+1)$, as before. For the cavity case defined above where $H = \omega_0 a^\dagger a$, and with $M = 0$ and $\beta = 0$, the master equation is given by Eq. (1.3) in the Introduction.

It must be emphasized that the use of the quantum Langevin equation in deriving the master equation is not necessary. It is possible to derive the master equation directly from the system-bath interaction (3.74). Furthermore, such a derivation will be found useful in later chapters, and so is worth pursuing. For convenience, I will work in the interaction picture, rather than the Schrödinger picture, so that the free evolution causing the oscillation of the dipole can be ignored. The interaction between the system and the bath is now given by

$$V(t) = \mathrm{i}\hbar[b^\dagger(t)c - c^\dagger b(t)], \tag{3.103}$$

where c is a slowly-varying interaction picture operator, and $b(t)$ is also a slowly-varying interaction picture bath operator. The time-dependence is maintained for the bath operator, because the free Hamiltonian of the bath causes propagation at the speed of light, so a new part of the bath interacts with the system at each new point in time. These parts are labeled by the time of interaction t. Thus, $b(t)$ is an operator in the Hilbert space for a particular part of the bath. Each part has its own state matrix $\mu(t)$. For the incoming field to be a bath requires that its total state matrix be the direct product of the state matrices of the parts [54]. That is to say, the temporally separate parts of the bath must be unentangled.

Let the system at time t be known to be $\rho(t)$. Thus the initial state of the system and (relevant part of the) bath at time t is

$$R(t) = \mu(t) \otimes \rho(t). \tag{3.104}$$

The infinitesimally evolved state is

$$R(t+dt) = U(t,t+dt)[\mu(t) \otimes \rho(t)]U^\dagger(t,t+dt), \tag{3.105}$$

where

$$U(t,t+dt) = \exp\left[dB^\dagger(t)c - c^\dagger dB(t)\right]. \tag{3.106}$$

Now the infinitesimal quantum Wiener increment $dB(t) = b(t)dt$ has all of the properties defined in Sec. 3.3. Thus, it is necessary to expand $U(t,t+dt)$ to second order. The result for $R(t+dt)$ is

$$\begin{aligned}
&\mu(t) \otimes \rho(t) + \left[dB^\dagger(t)c - c^\dagger dB(t), \mu(t) \otimes \rho(t)\right] \\
&+dB^\dagger(t)\mu(t)dB(t) \otimes c\rho(t)c^\dagger - \tfrac{1}{2}dB(t)dB^\dagger(t)\mu(t) \otimes c^\dagger c\rho - \tfrac{1}{2}\mu(t)dB(t)dB^\dagger(t) \otimes \rho c^\dagger c \\
&+dB(t)\mu(t)dB^\dagger(t) \otimes c^\dagger\rho(t)c - \tfrac{1}{2}dB^\dagger(t)dB(t)\mu(t) \otimes cc^\dagger\rho - \tfrac{1}{2}\mu(t)dB^\dagger(t)dB(t) \otimes \rho cc^\dagger \\
&-dB(t)\mu(t)dB(t) \otimes c^\dagger\rho(t)c^\dagger + \tfrac{1}{2}dB(t)dB(t)\mu(t) \otimes c^\dagger c^\dagger\rho + \tfrac{1}{2}\mu(t)dB(t)dB(t) \otimes \rho c^\dagger c^\dagger \\
&-dB^\dagger(t)\mu(t)dB^\dagger(t) \otimes c\rho(t)c + \tfrac{1}{2}dB^\dagger(t)dB^\dagger(t)\mu(t) \otimes cc\rho + \tfrac{1}{2}\mu(t)dB^\dagger(t)dB^\dagger(t) \otimes \rho cc.
\end{aligned} \tag{3.107}$$

The infinitesimally evolved reduced state matrix for the system is given by

$$\rho(t+dt) = \mathrm{Tr}_\mu[R(t+dt)]. \tag{3.108}$$

Taking the trace over the bath state in Eq. (3.107) yields the general master equation (3.102) providing the following assignments are made:

$$\mathrm{Tr}[dB(t)\mu(t)] = \beta dt, \tag{3.109}$$

$$\mathrm{Tr}[dB^\dagger(t)dB(t)\mu(t)] = N dt, \tag{3.110}$$

$$\mathrm{Tr}[dB(t)dB(t)\mu(t)] = M dt. \tag{3.111}$$

The simplest state matrix which has these properties is what is known as a Gaussian state matrix [52]. To exhibit this state, it is useful to define an annihilation operator in the Hilbert space of $\mu(t)$

$$a = \sqrt{dt}\, b(t), \tag{3.112}$$

which gives the usual commutation relations for a single-mode field, $[a, a^\dagger] = 1$. The Wigner function for $\mu(t)$ is then defined by

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda \mathrm{Tr}\big[\mu(t) \exp[\lambda(a^\dagger - a^*) - \lambda^*(a - \alpha)]\big], \tag{3.113}$$

which is normalized so that

$$\int d^2\alpha W(\alpha, \alpha^*) = 1. \tag{3.114}$$

The Gaussian state matrix which generates the moments (3.109,3.110,3.111) is then

$$W(\alpha, \alpha^*) = \left(\pi\sqrt{\left(N + \tfrac{1}{2}\right)^2 - |M|^2}\right)^{-1} \exp\left(-\frac{\left(N + \frac{1}{2}\right)|\bar{\alpha}|^2 - \frac{1}{2}M^*(\bar{\alpha})^2 - \frac{1}{2}M(\bar{\alpha}^*)^2}{\left(N + \frac{1}{2}\right)^2 - |M|^2}\right), \tag{3.115}$$

where

$$\bar{\alpha} = \alpha - \sqrt{dt}\,\beta. \tag{3.116}$$

In the special case where $M = 0$ and $\beta = 0$, this represents a thermal equilibrium state of temperature T given in Eq. (3.92), which can be more easily expressed as

$$\mu(t) = \mathrm{Tr}[\exp(-\hbar\omega_0 a^\dagger a/k_B T)]^{-1} \exp(-\hbar\omega_0 a^\dagger a/k_B T). \tag{3.117}$$

The more general state is not so easily expressed directly. There are in general other, non-Gaussian, states which will have the correct mean amplitude, and correct second order moments N and M. These will give the same master equation, but the state of the reflected field will be different. This could conceivably give different statistics when it comes to considering detection on the output field. However, over any finite interval of time, an ensemble of non-Gaussian statistics will reproduce Gaussian statistics. This follows from the central limit theorem, providing that the white noise is the limit of a non-white process which is a continuous function of time, which is the case physically. This is essentially the same reason that the classical white noise satisfying Eqs. (3.2,3.3) was assumed to be Gaussian white noise, and why $b_0(t)$ was assumed to be quantum Gaussian white noise in Sec. 3.4. This argument can be rigorously defined using mean square limits [50]. Thus there is no harm in assuming the Gaussian form (3.115) for the localized input field states.

3.6 Photon Flux Pressure

The quantum stochastic calculus used so far assumed a coupling linear in the bath amplitude. To my knowledge, applying the quantum stochastic calculus to a nonlinear coupling has not been attempted before. However, such a nonlinear coupling arises naturally in physics from the light pressure force. This example will be used to illustrate the use of the nonlinear coupling which will be needed for feedback. The classical expression for the pressure due to an axially reflected light beam at position z and time t is [140]

$$P(t) = 2\,\overline{U(z,t)}, \tag{3.118}$$

where U is the energy density of the field, and the overline represents an average over an optical cycle. For a freely propagating field, the energy density is related to the electric field by $U = \varepsilon_0|E|^2$. Assume that the energy is concentrated at an optical frequency ω_0. Using the expression (3.67), while remembering that b and $b^\dagger$ rotate oppositely at frequency $\omega_0 = ck$, gives

$$\overline{U(z,t)} = \frac{\hbar k}{\Lambda} b^\dagger(z,t) b(z,t), \tag{3.119}$$

where Λ is the transverse area of the beam. Now the pressure (3.118) is related to the potential energy V by $P\Lambda = \nabla V$. Thus the interaction Hamiltonian is

$$V(z,t) = 2\hbar k b^\dagger(z,t)b(z,t)z, \tag{3.120}$$

where z is the coordinate of the mirror. As promised, this expression is bilinear in the field amplitude.

Now, if the mirror does not move significantly, the argument z for the field operators can be assumed constant (say $z = 0$ as before). Then the evolution of some arbitrary mirror operator is given by

$$\dot{s} = \mathrm{i}[2kzb^\dagger(0,t)b(0,t), s]. \tag{3.121}$$

This is an implicit equation, and must be treated using the formalism developed in Sec. 3.2, combined with the stochastic calculus introduced in Sec. 3.4. Define an input photon flux operator

$$I_0(t) = b^\dagger(0^-,t)b(0^-,t). \tag{3.122}$$

This expression will not be well-defined if the input bath state is in a Gaussian state with nonzero N, as the photon flux would be N/dt which diverges as $dt \to 0$. That is because the white noise approximation assumes an infinite bandwidth of modes, each with a nonzero occupation number. This adds up to give an infinite photon flux. On the other hand, a zero temperature bath will give a zero photon flux, and have no effect on the mirror at all.

It is possible for Eq. (3.122) to be well-defined and non-zero, if for example the input operator is in a coherent state. This corresponds to replacing b_0 by $\beta + \nu$, where ν represents a vacuum state. Then by using the commutation relations (3.80) the operator

$$dN_0(t) = I_0(t)dt \tag{3.123}$$

satisfies

$$\begin{aligned} \langle dN_0(t)\rangle &= |\beta|^2 dt, & (3.124)\\ dN_0(t)^2 &= dN_0(t). & (3.125) \end{aligned}$$

These are identical to those of Eqs. (3.21,3.22) for a classical point process. However, being operators, one also has to take into account the commutators of $dN_0(t)$ with the field amplitudes $dB_0(t)$ [5]

$$[dB_0(t), dN_0(t)] = dB_0(t) = [dN_0(t), dB_0^\dagger(t)]^\dagger. \tag{3.126}$$

The reason that $dN_0^2 = dN_0$, expressing the fact that there is either zero photons or one photon, can be seen from the fact that $dN_0(t) = a^\dagger a$, where a is as defined in the preceding section (3.112). For a coherent input of amplitude β, the local state matrix $\mu(t)$, expanded to second order in $\sqrt{dt}$ in eigenstates of $a^\dagger a$ gives

$$\mu(t) = (1 - |\beta|^2 dt)|0\rangle\langle 0| + \sqrt{dt}(\beta|1\rangle\langle 0| + \beta^*|0\rangle\langle 1|) + |\beta|^2 dt|1\rangle\langle 1|. \tag{3.127}$$

This shows that the possibility of more than one photon arriving at once can be ignored. In fact, only the photon number populations are important for this result, so the relations (3.124,3.125) will also hold for an input state with no phase information, such as that produced by a laser.

The implicit equation (3.121) can be rewritten

$$\dot{s} = -I_0(t)\mathcal{K}s, \tag{3.128}$$

where $\mathcal{K}s = -\mathrm{i}[2kz, s]$. The explicit counterpart is then

$$\begin{aligned} ds(t) &= \left(\exp[-\mathcal{K}dN_0(t)] - 1\right) s(t) && (3.129)\\ &= dN_0(t)\left[U^\dagger s(t) U - s(t)\right], && (3.130) \end{aligned}$$

where $U = \exp[-2\mathrm{i}kz]$. Just as in Eq. (3.84), the stochastic term is necessary to preserve the commutation relations. This can be seen by calculating the increment in the product of two system operators, using the Itô (explicit calculus) rule

$$d(s_1 s_2) = (ds_1)s_2 + s_1(ds_2) + (ds_1)(ds_2). \tag{3.131}$$

The result is

$$d(s_1 s_2) = dN_0(t)\left[U^\dagger s_1 s_2 U - s_1 s_2\right], \tag{3.132}$$

as necessary for (3.130) to be a valid quantum Langevin equation.

When turning (3.130) into a master equation for the mirror by the relation (3.101), the noise term $dN_0(t)$ is replaced by its expectation value giving

$$\dot{\rho} = |\beta|^2\left(U\rho U^\dagger - \rho\right) = |\beta|^2\mathcal{D}[\exp(-2\mathrm{i}kz)]\rho. \tag{3.133}$$

This is precisely what could have been predicted straight away from Eq. (3.118), with a photon flux of $|\beta|^2$ and a photon momentum of $\hbar k$. Each photon gives a kick to the momentum of the mirror of magnitude $2\hbar k$ as it is reflected. The effect on the output field is to cause a phase shift

$$b_1(t) \equiv b(0^+, t) = e^{-\mathcal{K}dN_0(t)} b_0(t) = e^{-2\mathrm{i}kz} b_0(t), \tag{3.134}$$

due to the shifting of the mirror away from the position $z = 0$. Of course, the photon flux operator is unchanged by the feedback, as

$$I_1(t) = b_1^\dagger(t) b_1(t) = b_0^\dagger(t) e^{2\mathrm{i}kz} e^{-2\mathrm{i}kz} b_0(t) = b_0^\dagger(t) b_0(t) = I_0(t). \tag{3.135}$$

3.7 Cascaded Open Systems

The theory of quantum SDEs presented so far has been limited to a single system with a bath input. Often in quantum optics experiments, the output of one system is used as an input to another open system. This has been called "cascaded systems theory" by Carmichael [20]. Cascaded systems have more than one application in this thesis, so it is useful to develop the general formalism here. The present treatment is restricted to a dipole coupling at both systems. This is done so that white noise may be included, which has not been done in previous treatments [51, 20]. Cascaded systems are different from coupled systems, because the interaction only goes one way. That is to say, the first system influences the second, but not *vice versa*. One mechanism for achieving the required unidirectionality is the Faraday isolator which utilizes Faraday rotation and polarization-sensitive beam splitters. This is most practical for the case of cavities, where the output is a beam of light. A quantum theoretical treatment which incorporates this spatial symmetry breaking at the level of the Hamiltonian was given recently by Gardiner [51]. If the propagation time between the source system and the driven system is ignored, then a master equation for both systems may be derived. This result was obtained simultaneously by Carmichael [20], who used quantum trajectories to illustrate the nature of the process.

Begin by considering the output field b_1 from the first cavity, as in Eq. (3.88), which is given by

$$b_1 = b_0 + \sqrt{\gamma_1} c_1, \tag{3.136}$$

where b_0 has white noise statistics as in Sec. 3.4.3, and c_1 is the anihilation operator for the source system. Here, I have included the damping rate γ_1 for this cavity, because that of the driven system need not be the same. Now let this field be the input into the second cavity with annihilation operator $c_2(t)$. If the damping rate for this cavity is γ_2, then the Hamiltonian coupling is

$$V_2(t) = \mathrm{i}\hbar\sqrt{\gamma_2}[b^\dagger(c\tau, t)c_2(t) - c_2^\dagger(t)b(c\tau, t)], \tag{3.137}$$

where $c\tau$ is the path length between the two cavities. Proceeding as in Sec. 6.4, the unitary evolution generated by this Hamiltonian is

$$U_2(t, t+dt) = \exp\left(\sqrt{\gamma_2}\left[dB_1^\dagger(t-\tau)c_2(t) - c_2^\dagger(t)dB_1(t-\tau)\right]\right), \tag{3.138}$$

where $dB_1(t) = b_1(t)dt$. An arbitrary operator in the source or driven cavity obeys the equation

$$s(t+dt) = U_1^\dagger(t, t+dt)U_2^\dagger(t, t+dt)s(t)U_2(t, t+dt)U_1(t, t+dt), \tag{3.139}$$

where

$$U_1(t, t+dt) = \exp\left(\sqrt{\gamma_1}\left[dB_0^\dagger(t)c_1(t) - c_1^\dagger(t)dB_0(t)\right]\right) \tag{3.140}$$

is the evolution due to the first cavity damping. Note that $U_2(t, t+dt)$ commutes with $U_1(t, t+dt)$ because of the finite τ, so the ordering in the above equation is not significant. Expanding the terms as above gives

$$\begin{aligned} ds &= \frac{\gamma_1}{2}\Big[(N+1)(2c_1^\dagger s c_1 - s c_1^\dagger c_1 - c_1^\dagger c_1 s) + N(2c_1 s c_1^\dagger - s c_1 c_1^\dagger - c_1 c_1^\dagger s) \\ &\quad + M[c_1^\dagger, [c_1^\dagger, s]] + M^*[c_1, [c_1, s]]\Big]\, dt \\ &\quad + \frac{\gamma_2}{2}\Big[(N+1)(2c_2^\dagger s c_2 - s c_2^\dagger c_2 - c_2^\dagger c_2 s) + N(2c_2 s c_2^\dagger - s c_2 c_2^\dagger - c_2 c_2^\dagger s) + \\ &\quad + M[c_2^\dagger, [c_2^\dagger, s]] + M^*[c_2, [c_2, s]]\Big]\, dt \\ &\quad - \sqrt{\gamma_1}[dB_0^\dagger c_1 - dB_0 c_1^\dagger, s] - \sqrt{\gamma_2}[dB_1^\dagger c_2 - dB_1 c_2^\dagger, s]. \end{aligned} \tag{3.141}$$

Here, the implicit time argument of s, c_1 and c_2 is t, while that of $dB_1 = (dB_0 + \sqrt{\gamma_1} c_1 dt)$ is $t - \tau$.

Equation (3.141) is an explicit equation, in the sense of Sec. 3.2, in that it gives an explicit algorithm for calculating an infinitesimal increment in some operator, given all of the other operators at the start of the time interval. However, it is not an Itô equation because the noise terms are not independent of the other operators. Although the noise input $dB_0(t)$ is independent, $dB_0(t-\tau)$ in $dB_1(t-\tau)$ is not independent of an arbitrary system operator $s(t)$. Thus it is not possible to turn this QLE into a master equation by setting the noise terms equal to zero in the average. In order to derive a master equation, it is necessary to take the formal limit $\tau \to 0$. This is a formal limit because the physics of the problem is independent of τ because the second system does not interact with the first (although this possibility will be considered in Ch. 7). However, if τ is set to zero, then the operators U_1 and U_2 above no longer commute, and so the order is important. The order in Eq. (3.139) is logical as it indicates that the unitary evolution at the first system occurs before that at the second. However, if $\tau = 0$ then the bath operator at the second system should be the same as that at the first. That is to say, $dB_1(t)$ should be replaced by $dB_0(t)$ in Eq. (3.138) to give

$$U_2(t, t+dt) = \exp\left(\sqrt{\gamma_2}\left[dB_0^\dagger(t)c_2(t) - dB_0(t)c_2^\dagger(t)\right]\right). \tag{3.142}$$

In Eq. (3.139) it is evident that the operator for the coupling at the second system actually acts first on the system operator $s(t)$, and the operator for the first coupling acts second. This perhaps confusing ordering can be reversed. Taking into account the commutation relations gives

$$s(t+dt) = U^\dagger(t,t+dt)s(t)U(t,t+dt), \tag{3.143}$$

where

$$U(t,t+dt) = U_1(t,t+dt)U_2(t,t+dt), \tag{3.144}$$

and where now the evolution for the driven cavity is generated by

$$U_2(t,t+dt) = \exp\left(\sqrt{\gamma_2}\left[dB_1^\dagger(t)c_2(t) - c_2(t)dB_1(t)\right]\right). \tag{3.145}$$

In this expression, the driving of the second system by the first is more apparent. From Eq. (3.143) one can easily derive the total increment in the system operator s to be

$$\begin{aligned} ds &= (N+1)\left[\frac{\gamma_1}{2}(2c_1^\dagger s c_1 - sc_1^\dagger c_1 - c_1^\dagger c_1 s) + \frac{\gamma_2}{2}(2c_2^\dagger s c_2 - sc_2^\dagger c_2 - c_2^\dagger c_2 s)\right. \\ &\qquad\left. + \sqrt{\gamma_1\gamma_2}(c_2^\dagger s c_1 - sc_2^\dagger c_1 + c_1^\dagger s c_2 - c_1^\dagger c_2 s)\right] dt \\ &\quad + N\left[\frac{\gamma_1}{2}(2c_1 s c_1^\dagger - sc_1 c_1^\dagger - c_1 c_1^\dagger s) + \frac{\gamma_2}{2}(2c_2 s c_2^\dagger - sc_2 c_2^\dagger - c_2 c_2^\dagger s)\right. \\ &\qquad\left. + \sqrt{\gamma_1\gamma_2}(c_2 s c_1^\dagger - sc_2 c_1^\dagger + c_1 s c_2^\dagger - c_1 c_2^\dagger s)\right] dt \\ &\quad + M\left[\frac{\gamma_1}{2}[c_1^\dagger,[c_1^\dagger,s]] + \frac{\gamma_2}{2}[c_2^\dagger,[c_2^\dagger,s]] + \sqrt{\gamma_1\gamma_2}[c_1^\dagger,[c_2^\dagger,s]]\right] dt \\ &\quad + M^*\left[\frac{\gamma_1}{2}[c_1,[c_1,s]] + \frac{\gamma_2}{2}[c_2,[c_2,s]] + \sqrt{\gamma_1\gamma_2}[c_1,[c_2,s]]\right] dt \\ &\quad - \sqrt{\gamma_1}[dB_0^\dagger c_1 - dB_0 c_1^\dagger, s] - \sqrt{\gamma_2}[dB_0^\dagger c_2 - dB_0 c_2^\dagger, s]. \end{aligned} \tag{3.146}$$

In this Itô equation, all operators have the same time argument. It agrees with the results of Refs. [51, 20] for the case $N = M = 0$, but it should be noted that the details of the derivation in both of these papers differ from those presented here.

It is now a simple matter to convert this stochastic Heisenberg equation into a master equation for the state matrix W of both systems

$$\begin{aligned} \dot{W} &= (N+1)\left[\gamma_1\mathcal{D}[c_1]W + \gamma_2\mathcal{D}[c_2]W + \sqrt{\gamma_1\gamma_2}\left([c_1 W, c_2^\dagger] + [c_2, W c_1^\dagger]\right)\right] \\ &\quad + N\left[\gamma_1\mathcal{D}[c_1^\dagger]W + \gamma_2\mathcal{D}[c_2^\dagger]W + \sqrt{\gamma_1\gamma_2}\left([c_1^\dagger W, c_2] + [c_2^\dagger, W c_1]\right)\right] \\ &\quad + M\left[\frac{\gamma_1}{2}[c_1^\dagger,[c_1^\dagger,W]] + \frac{\gamma_2}{2}[c_2^\dagger,[c_2^\dagger,W]] + \sqrt{\gamma_1\gamma_2}[c_2^\dagger,[c_1^\dagger,W]]\right] \\ &\quad + M^*\left[\frac{\gamma_1}{2}[c_1,[c_1,W]] + \frac{\gamma_2}{2}[c_2,[c_2,W]] + \sqrt{\gamma_1\gamma_2}[c_2,[c_1,W]]\right] \\ &\quad + \sqrt{\gamma_1}[\beta^*(t)c_1 - \beta(t)c_1^\dagger, W] + \sqrt{\gamma_2}[\beta^*(t)c_2 - \beta(t)c_2^\dagger, W] - \mathrm{i}[H,W] \end{aligned} \tag{3.147}$$

Here I have allowed for a coherent amplitude in the bath, and also included the intrinsic evolution for the two systems, generated by the Hamiltonian $\hbar H$. This is the general equation for two open quantum systems, linked unidirectionally by a bath of harmonic oscillators with an optical frequency coherent amplitude contaminated by white noise. It has the necessary property that, if H is the sum of Hamiltonians operating in the Hilbert subspaces of the two systems, then the source system (c_1) is unaffected by the driven system (c_2). That is to say, the density operator ρ for the source system (obtained by tracing over the driven system) obeys the original master equation (3.102). This is evident from the fact that the only terms in Eq. (3.147) containing operators from both systems involve an exterior commutator with a driven system operator, which gives zero when traced over the driven subspace. It is not possible in general to derive a master equation for the second system alone. Its evolution is literally driven by the source system.

Chapter 4

Quantum Trajectories

As defined in the Introduction, a quantum trajectory is the evolution of a system conditioned on the results of measurements. Specifically, in this and later chapters I am concerned with quantum trajectories for continuously monitored systems. This has already been covered formally in Sec. 2.3. In this chapter, I examine one physical basis for such continuous observation, the quantum optical dipole interaction introduced in the preceding chapter. Three types of measurement are considered: direct, homodyne and heterodyne detection. The latter two can be generalized to cover the case of a non-vacuum white noise input. I show the relation between the detection schemes and various distribution functions for the case of a cavity field decaying into the vacuum.

4.1 Photon Detection Theory

4.1.1 The Bath as an Apparatus

Recall the Schrödinger picture derivation of the quantum optical master equation in Sec. 3.5. Let the input bath be in the vacuum state. That is to say, the input bath state is $\mu(t) = |0\rangle\langle 0|$ for all time t, where $|0\rangle$ is the lowest eigenstate for $a^\dagger a$. Here, $a = \sqrt{dt}\, b_0(t) = dB_0(t)/\sqrt{dt}$, as defined in Sec. 3.5, so that $a^\dagger a$ has integer eigenvalues representing the number of photons arriving in the interval of time $[t, t+dt)$. Let the state of the system at time t be $\rho(t)$, independent of $\mu(t)$. The entangled state after the interaction of duration dt is, from Eq. (3.107)

$$\begin{aligned} R(t+dt) \;&=\; |0\rangle\langle 0| \otimes \rho(t) + \sqrt{dt}\left[|1\rangle\langle 0| \otimes c\rho(t) + |0\rangle\langle 1| \otimes \rho(t)c^\dagger\right] - \mathrm{i}dt|0\rangle\langle 0| \otimes [H, \rho(t)] \\ &\quad + dt\left\{|1\rangle\langle 1| \otimes c\rho(t)c^\dagger - \tfrac{1}{2}|0\rangle\langle 0| \otimes \left[c^\dagger c\rho(t) + \rho(t)c^\dagger c\right]\right\}. \end{aligned} \tag{4.1}$$

Here, as in the remainder of this thesis, I am setting $\hbar = 1$ so that H is the system Hamiltonian. The free dynamics of the electromagnetic field will now remove the bath state from the system, so that they will remain entangled. If the outgoing bath is ignored, then its only effect is to cause the system to evolve to

$$\rho(t+dt) = \rho(t) + \mathcal{D}[c]\rho(t)dt - \mathrm{i}[H, \rho(t)]dt, \tag{4.2}$$

which causes damping of the system dipole amplitude $\langle c\rangle$.

There is no reason that the output bath state must be ignored. In practice, the output of the quantum system is often the sole purpose of an experiment. To obtain information about the system, the outgoing field must be measured. The obvious measurement to consider is photon counting. This I will model by projecting the field states onto eigenstates of $a^\dagger a$. It is possible to consider specific models for photon detectors, usually based on atomic systems [52]. However, these simply remove the measurement step, where possibilities become actualities, one step further along the von Neumann chain [139]. There is little which results from such models which cannot be achieved by including losses and convoluting the classical photocurrent with an empirically derived detector response function. Therefore, I will simply use projection measurement operators

$$P_0 = |0\rangle\langle 0| \; ; \quad P_1 = |1\rangle\langle 1|. \tag{4.3}$$

It is evident from Eq. (4.1) that for almost all time intervals no photons are detected in the output. The unnormalized system state matrix (as defined in Ch. 2) conditioned on a null count is

$$\tilde{\rho}_0(t+dt) = \rho(t) - \tfrac{1}{2}\{c^\dagger c, \rho(t)\}dt - \mathrm{i}[H, \rho(t)]dt. \tag{4.4}$$

If a photon is detected, the system jumps into the unnormalized conditioned state

$$\tilde{\rho}_1(t+dt) = c\rho(t)c^\dagger dt. \tag{4.5}$$

Clearly, the unconditioned master equation evolution (4.2) is retrieved by averaging over the two possible results

$$\rho(t+dt) = \tilde{\rho}_0(t+dt) + \tilde{\rho}_1(t+dt). \tag{4.6}$$

The stochastic evolution described in the preceding paragraph is identical to that derived from measurement theory in Sec. 2.3, from the master equation (4.2). Here, one sees the physical significance of the point process measurement results as photodetections. In fact the measurement operators found in Sec. 2.3 can be derived directly from the theory of indirect measurements given in Sec. 2.2. The initial apparatus state for the time interval $[t, t+dt)$ is the bath state $\mu(t) = P_0$. The unitary interaction operator is

$$U(dt) = \exp[(a^\dagger c - c^\dagger a)\sqrt{dt} - \mathrm{i}H\,dt], \tag{4.7}$$

as in Eq. (3.106), with free evolution added. This must be expanded to first order in dt. Then the measurement operators are given by

$$\begin{aligned} \Omega_1(dt) &= \langle 1|U(dt)|0\rangle = \sqrt{dt}\,c, & (4.8)\\ \Omega_0(dt) &= \langle 0|U(dt)|0\rangle = 1 - \left(\mathrm{i}H + \tfrac{1}{2}c^\dagger c\right)dt. & (4.9) \end{aligned}$$

Again, these are identical to those found by inspection in Sec. 2.3. It might be thought that the measurement operators should be altered in some fashion because of the delay in the output beam reaching the detector, or perhaps even the delay before the experimentalist reads the current meter. In fact, this is not necessary, as long as the measurement operators act on the state matrix representing the system at the time of interaction, rather than the time of measurement.

4.1.2 Quantum Trajectories

The stochastic evolution of a damped system with output subject to direct photodetection is completely determined by the measurement operators (4.8,4.9). However, it is useful and elegant to reformulate this evolution in the form of an explicitly stochastic evolution equation. This equation of motion specifies the quantum trajectory of the system. Since the measurement result is a point process, it can be represented by a random variable $dN_{\rm c}(t)$, as introduced in Sec. 3.2. It represents the increment (either zero or one) in the photon count in the interval $[t, t+dt)$, and is defined by

$$\begin{aligned} {\rm E}[dN_{\rm c}(t)] &= {\rm Tr}[c^\dagger c \rho_{\rm c}(t)]dt, & (4.10)\\ dN_{\rm c}(t)^2 &= dN_{\rm c}(t). & (4.11)\end{aligned}$$

Here the subscript c indicates that the quantity to which it is attached is conditioned on previous measurement results, arbitrarily far back in time. The conditioned state matrix obeys the explicit (in the sense of Ch. 3) stochastic master equation (SME)

$$d\rho_{\rm c}(t) = \left\{ dN_{\rm c}(t)\mathcal{G}[c] - dt\mathcal{H}\left[{\rm i}H + \tfrac{1}{2}c^\dagger c\right] \right\} \rho_{\rm c}(t). \tag{4.12}$$

Here, the nonlinear (in ρ) superoperators $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$\begin{aligned} \mathcal{G}[r]\rho &= \frac{r\rho r^\dagger}{{\rm Tr}[r\rho r^\dagger]} - \rho, & (4.13)\\ \mathcal{H}[r]\rho &= r\rho + \rho r^\dagger - {\rm Tr}[r\rho + \rho r^\dagger]\rho. & (4.14)\end{aligned}$$

The nonlinearity of the SME (4.12) is indicative of the fundamental nonlinearity of quantum measurements. The original master equation for $\rho(t) = {\rm E}[\rho_{\rm c}(t)]$ can be restored simply by replacing $dN_{\rm c}(t)$ in Eq. (4.12) by its ensemble average value (4.10).

Because of the assumed perfect detection, the stochastic equation for the state matrix is equivalent to the following stochastic equation for the state vector

$$d|\psi_{\rm c}(t)\rangle = \left[dN_{\rm c}(t)\left(\frac{c}{\sqrt{\langle c^\dagger c\rangle_{\rm c}(t)}} - 1 \right) + dt\left(\frac{\langle c^\dagger c\rangle_{\rm c}(t)}{2} - \frac{c^\dagger c}{2} - {\rm i}H \right) \right] |\psi_{\rm c}(t)\rangle. \tag{4.15}$$

This equation can be called a stochastic Schrödinger equation (SSE), although the evolution it generates is obviously nonlinear. In this case, the nonselective, or unconditioned state matrix is defined by

$$\rho(t) = {\rm E}\left[|\psi_{\rm c}(t)\rangle\langle\psi_{\rm c}(t)| \right]. \tag{4.16}$$

The unraveling of the master equation into a SSE of the form of (4.15) is the most commonly used quantum trajectory for numerical simulations [37, 43, 44, 136, 97]. This will be reviewed in the following chapter. From the point of view of measurement theory, the stochastic master equation is of more use because it is more transparently related to the unconditioned master equation and because it can be generalized to cope with inefficient detectors. If the efficiency is η, then the measurement operations introduced in Sec. 2.3 generate the SME evolution

$$d\rho_{\rm c}(t) = \left\{ dN_{\rm c}(t)\mathcal{G}[\sqrt{\eta}\,c] + dt\mathcal{H}[-{\rm i}H - \eta\tfrac{1}{2}c^\dagger c] + dt(1-\eta)\mathcal{D}[c] \right\} \rho_{\rm c}(t), \tag{4.17}$$

where now Eq. (4.10) becomes

$${\rm E}[dN_{\rm c}(t)] = \eta{\rm Tr}[c\rho(t)c^\dagger]dt. \tag{4.18}$$

In this case a SSE does not exist.

4.1.3 Output Field Correlation Functions

In experimental quantum optics, it is more usual to consider a photocurrent than a photocount. For this reason, it is useful to define the photocurrent by

$$I_{\rm c}(t) = dN_{\rm c}(t)/dt. \tag{4.19}$$

This is a very singular quantity, consisting of a series of Dirac delta functions at the times of photodetections. The reader will no doubt have noticed the use of the symbols dN and I to represent both classical and quantum quantities. In this section, $I_c(t)$ is a classical photocurrent, distinguished by the conditioned c subscript. In Sec. 3.6, $I_1(t) = dN_1/dt$ was a Heisenberg picture bath operator. The reason for this is that the two quantities are effectively identical statistically. To see this, consider the Heisenberg operator $dN_1(t) = b_1^\dagger(t)b_1(t)dt$ for the output field. For the linear coupling being considered in this chapter, this is found from Eq. (3.88) to be

$$dN_1(t) = [c^\dagger(t) + \nu^\dagger(t)][c(t) + \nu(t)]dt. \tag{4.20}$$

Here I am using $\nu(t)$ for $b_0(t)$ to emphasize that the input field is in the vacuum state and so satisfies $\langle \nu(t)\nu^\dagger(t')\rangle = \delta(t-t')$, with all other moments vanishing. It is then easy to show that

$$\begin{aligned} \langle dN_1(t)\rangle &= \mathrm{Tr}[c^\dagger(t)c(t)\rho]dt, & (4.21)\\ dN_1(t)^2 &= dN_1(t). & (4.22)\end{aligned}$$

These moments are identical to those of the photocount increment (4.10,4.11), with a change from Schrödinger to Heisenberg picture, and the dropping of conditioned subscripts. This should not be surprising, because $dN_1(t) = a^\dagger a$, the operator being measured by the projection measurements (4.3).

The identity between the statistics of the output photon flux operator $I_1(t)$ and the photocurrent $I_c(t)$ does not stop at the equal time moments (4.21,4.22). The most commonly calculated higher order moment in quantum optics is the autocorrelation function. This is defined as

$$F^{(2)}(t, t+\tau) = \langle I_1(t+\tau)I_1(t)\rangle. \tag{4.23}$$

From the expression (4.20), this is found to be

$$F^{(2)}(t, t+\tau) = \langle c^\dagger(t)c^\dagger(t+\tau)c(t+\tau)c(t)\rangle + \delta(\tau)\langle c^\dagger(t)c(t)\rangle. \tag{4.24}$$

Here the commutation relations $[b_1(t), b_1^\dagger(t')] = \delta(t-t')$ have been used deliberately to put the field operators in normal order. That is to say, with creation operators at the front and annihilation operators at the rear. Doing this eliminates the input field operators, because they act directly on the vacuum giving a null result. The autocorrelation function can be rewritten in the Schrödinger picture using the linearity of the master equation evolution (4.2). The result is

$$F^{(2)}(t, t+\tau) = \mathrm{Tr}\left[c^\dagger c e^{\mathcal{L}\tau} c\rho(t)c^\dagger\right] + \mathrm{Tr}\left[c^\dagger c\rho(t)\right]\delta(\tau), \tag{4.25}$$

where

$$\mathcal{L}\rho = \mathcal{D}[c]\rho - \mathrm{i}[H, \rho], \tag{4.26}$$

and superoperators act on the product of all operator to their right, as is the convention which I am using.

Now consider the same autocorrelation function, but calculated from the observed currents

$$F^{(2)}(t, t+\tau)(dt)^2 = \mathrm{E}\left[dN_c(t+\tau)dN_c(t)\right]. \tag{4.27}$$

First consider τ finite. Now $dN_c(t)$ is either zero or one. If it is zero, then the function is automatically zero. Hence,

$$F^{(2)}(t, t+\tau)(dt)^2 = \Pr\left[dN_c(t) = 1\right] \times \mathrm{E}\left[dN_c(t+\tau)|_{dN_c(t)=1}\right]. \tag{4.28}$$

This is equal to

$$F^{(2)}(t,t+\tau)(dt)^2 = \mathrm{Tr}[c^\dagger c\rho(t)]dt \times \mathrm{Tr}\left[c^\dagger c\, dt\mathrm{E}\left[\rho_c(t+\tau)|_{dN_c(t)=1}\right]\right]. \tag{4.29}$$

Now if $dN_c(t) = 1$, then $\rho_c(t+dt) = c\rho(t)c^\dagger/\mathrm{Tr}[c\rho(t)c^\dagger]$, and by linearity of the ensemble average evolution (4.26),

$$\mathrm{E}\left[\rho_c(t+\tau)|_{dN_c(t)=1}\right] = \exp(\mathcal{L}\tau)c\rho(t)c^\dagger/\mathrm{Tr}[c\rho(t)c^\dagger]. \tag{4.30}$$

Thus, the final expression for τ finite is

$$F^{(2)}(t,t+\tau) = \mathrm{Tr}\left[c^\dagger c e^{\mathcal{L}\tau}c\rho(t)c^\dagger\right]. \tag{4.31}$$

For c an annihilation operator for a cavity mode, this is equal to Glauber's second order coherence function, $G^{(2)}(t,t+\tau)$ [62].

If $\tau = 0$, then the expression (4.27) diverges, because $dN_c(t)^2 = dN_c(t)$. Naively,

$$F^{(2)}(t,t) = \mathrm{Tr}[c^\dagger c\rho(t)]/dt. \tag{4.32}$$

Properly interpreted, this is identical to the delta-function in Eq. (4.25). Thus, the expression for the autocorrelation function of the observed photocurrent is the same as that of the output field photon flux. In fact, any statistical comparison between the two will agree because they are merely different representations of the same physical quantity. Conceptually, the two representations are quite different. From the Heisenberg operator derivation, the shot noise term (the delta function) in the autocorrelation function arises from the commutation relations of the electromagnetic field. On the other hand, the quantum trajectory model produces shot noise because photodetections are discrete events. The latter explanation is far more appealing from an intuitive point of view. Any theorems to do with photocurrents which are usually derived using field operators can be derived from the quantum trajectory model. Some results may be more obvious using one method, others more obvious with the other, so it is good to be familiar with both.

4.1.4 Coherent Field Input

As noted in Sec. 3.6, a bath in a thermal or squeezed white noise state has a theoretically infinite photon flux. Thus, it is not possible to count the photons in such a beam; no matter how small the time interval one would still expect a finite number of counts. In practice, such noise is not white, and the bandwidth of the detector will keep the count rate finite. Nevertheless, in the limit where the white noise approximation is a good one, the photon flux due to the bath will be much greater than that due to the system. Hence, direct detection will yield negligible information about the system state, so the quantum trajectory will simply be the unconditioned master equation (4.2). However, the vacuum input considered so far can still be generalized by adding a coherent field, as explained in Sec. 3.6. If the amplitude of this field is $\beta(t)$, then $\mu(t) = |\beta(t)\sqrt{dt}\rangle\langle\beta(t)\sqrt{dt}|$. Expanded to first order in dt, this is given by Eq. (3.127). The measurement operators are

$$\begin{aligned}\Omega_1(dt) &= \langle 1|U(dt)|\beta(t)\sqrt{dt}\rangle = \sqrt{dt}\,[c+\beta(t)], &(4.33)\\ \Omega_0(dt) &= \langle 0|U(dt)|\beta(t)\sqrt{dt}\rangle = 1 - \left[\mathrm{i}H + \tfrac{1}{2}c^\dagger c + c^\dagger\beta(t)\right]dt. &(4.34)\end{aligned}$$

Note that both the smooth and the jump evolution are altered by the coherent field. With detectors of efficiency η, and the time argument in β suppressed, the SME is

$$\begin{aligned}d\rho_c(t) &= dN_c(t)\mathcal{G}[\sqrt{\eta}(c+\beta)]\rho_c(t) + dt\left\{-\mathrm{i}[H,\rho_c(t)] - \eta\mathcal{H}\left[\tfrac{1}{2}c^\dagger c + c^\dagger\beta\right]\rho_c(t)\right\}\\ &\quad + (1-\eta)dt\left\{\mathcal{D}[c]\rho_c(t) + [\beta^* c - c^\dagger\beta, \rho_c(t)]\right\}, \end{aligned}\tag{4.35}$$

where

$$\mathrm{E}[dN_{\rm c}(t)] = \eta \mathrm{Tr}\big[\mathcal{J}[c+\beta]\rho(t)\big]\,dt. \tag{4.36}$$

In the case of unit efficiency, an equivalent SSE can be defined

$$\begin{aligned} d|\psi_{\rm c}(t)\rangle &= \Bigg[dN_{\rm c}(t)\left(\frac{c+\beta}{\sqrt{\langle(c^\dagger+\beta^*)(c+\beta)\rangle_{\rm c}(t)}}-1\right) \\ &\quad + dt\left(\frac{\langle c^\dagger c\rangle_{\rm c}(t)}{2} - \frac{c^\dagger c}{2} + \frac{\langle c^\dagger\beta+\beta^* c\rangle_{\rm c}(t)}{2} - c^\dagger\beta - \mathrm{i}H\right)\Bigg]|\psi_{\rm c}(t)\rangle. \end{aligned} \tag{4.37}$$

This simple modification to the vacuum input SSE was (to my knowledge) first derived for this thesis. The ensemble average evolution is the master equation

$$\dot\rho = \mathcal{D}[c]\rho - \mathrm{i}[H + \mathrm{i}\beta^* c - \mathrm{i}c^\dagger\beta, \rho], \tag{4.38}$$

which is as expected from Eq. (3.102) with $N = M = 0$. To unravel this master equation for purposes of numerical calculation, one could choose the SSE as for the vacuum input (4.15), merely changing the Hamiltonian as indicated in the master equation (4.38). However, this would be a mistake if the trajectories were meant to represent the actual conditional evolution of the system, which is given by Eq. (4.37). This distinction will be pursued in the following chapter.

4.2 Homodyne Detection Theory

4.2.1 Adding a Local Oscillator

In Sec. 2.3 on continuous measurement theory it was noted that the unraveling of the master equation (4.2) as a quantum trajectory is not unique. Under the transformation (2.37) which leaves the master equation unchanged, the measurement operators (4.8,4.9) transform to

$$\Omega_1(dt) = \sqrt{dt}(c+\gamma), \tag{4.39}$$

$$\Omega_0(dt) = 1 - dt\left[\mathrm{i}H + \tfrac{1}{2}(c\gamma^* - c^\dagger\gamma) + \tfrac{1}{2}(c^\dagger+\gamma^*)(c+\gamma)\right]. \tag{4.40}$$

Physically, this transformation can be achieved by homodyne detection. In the simplest configuration, the output field of the cavity, $b_1 = \nu + c$, is sent through a beam splitter of transmittance η very close to one. The other input port of the beam splitter is a very strong coherent field. This has the same frequency as the system dipole, and is known as the local oscillator. The transmitted field is then represented by the operator

$$b_1' = \nu + c + \gamma, \tag{4.41}$$

where γ is a complex number representing a coherent amplitude, such that $|\gamma|^2/(1-\eta)$ is equal to the input photon flux of the local oscillator. A perfect measurement of $dN_1' = b_1'^\dagger b_1' dt$ by a photodetector leads to the above measurement operators.

Let the coherent field γ be real, so that the homodyne detection leads to a measurement of the x quadrature of the system dipole. This can be seen from the rate of photodetections at the (perfect) detector

$$\mathrm{E}[dN_{\rm c}(t)] = \mathrm{Tr}[(\gamma^2 + \gamma x + c^\dagger c)\rho_{\rm c}(t)]. \tag{4.42}$$

In this thesis, I am defining the two quadratures by

$$x = c + c^\dagger\ ;\ \ y = -\mathrm{i}(c - c^\dagger). \tag{4.43}$$

In the limit that γ is much larger than c, this rate consists of a large constant term plus a term proportional to x, plus a small term. From the measurement operators (4.39,4.40), the stochastic master equation for the conditioned state matrix is

$$d\rho_c(t) = \left\{dN_c(t)\mathcal{G}[c+\gamma] + dt\mathcal{H}\left[-iH - \gamma c - \tfrac{1}{2}c^\dagger c\right]\right\}\rho_c(t). \tag{4.44}$$

For this case of unit efficiency detectors, the SME can be equivalently written as the SSE

$$\begin{aligned} d|\psi_c(t)\rangle \;&=\; \Bigg[dN_c(t)\left(\frac{c+\gamma}{\sqrt{\langle(c^\dagger+\gamma)(c+\gamma)\rangle_c(t)}} - 1\right) \\ &\quad + dt\left(\frac{\langle c^\dagger c\rangle_c(t)}{2} - \frac{c^\dagger c}{2} + \frac{\langle c^\dagger\gamma + \gamma c\rangle_c(t)}{2} - \gamma c - iH\right)\Bigg]|\psi_c(t)\rangle. \end{aligned} \tag{4.45}$$

This shows how the master equation (4.2) can be unraveled in a completely different manner from the usual quantum trajectory (4.15). Note the minor difference from the coherently driven SSE (4.37), which makes the latter simulate a different master equation (4.38).

4.2.2 The Continuum Limit

The ideal limit of homodyne detection is when the local oscillator amplitude goes to infinity. In this limit, the rate of photodetections goes to infinity, but the effect of each on the system goes to zero, because the field being detected is almost entirely due to the local oscillator. Thus, it should be possible to approximate the photocurrent by a continuous function of time, and also to derive a smooth evolution equation for the system. This was done first by Carmichael [21]; the following is a more rigorous working of the derivation he sketched.

Let the system operators be of order unity, and let γ be an arbitrarily large parameter. Consider a time interval $[t, t+\delta t)$, where $\delta t \sim \gamma^{-3/2}$. This scaling is chosen so that in δt, the number of detections $\delta N \sim \gamma^2\delta t \sim \gamma^{1/2}$ is very large, but the change in the system $\sim \delta t \sim \gamma^{-3/2}$ is very small. The mean number of detections in this time will be

$$\begin{aligned} \mu \;&=\; \mathrm{Tr}\left[(\gamma^2 + \gamma x + c^\dagger c)[\rho_c(t) + O(\gamma^{-3/2})]\right]\delta t \\ &=\; [\gamma^2 + \gamma\langle x\rangle_c(t) + O(\gamma^{1/2})]\delta t. \end{aligned} \tag{4.46}$$

The error in μ (due to the change in the system over the interval) is larger than the contribution from $c^\dagger c$. The variance in δN will be dominated by the Poisson statistics of the local oscillator. Because the number of counts is very large, these statistics will be approximately Gaussian. Specifically, I have shown [149] that the statistics of δN are consistent with that of a Gaussian random variable of mean (4.46) and variance

$$\sigma^2 = [\gamma^2 + O(\gamma^{3/2})]\delta t. \tag{4.47}$$

The error in σ^2 is necessarily as large as expressed here for the statistics of δN to be consistent with Gaussian statistics. Thus, δN can be written as

$$\delta N = \gamma^2\delta t[1 + \langle x\rangle_c(t)/\gamma] + \gamma\delta W, \tag{4.48}$$

where the accuracy in each term is only as great as the highest order expression in $\gamma^{-1/2}$. Here δW is a Wiener increment satisfying $\mathrm{E}[(\delta W)^2] = \delta t$.

Now as far as the system is concerned, the time δt is still very small. Expanding Eq. (4.44) in powers of γ^{-1} gives

$$\begin{aligned}\delta\rho_{\rm c}(t) &= \delta N_{\rm c}(t)\left(\frac{\mathcal{H}[c]}{\gamma}+\frac{\langle c^\dagger c\rangle_{\rm c}(t)\mathcal{G}[c]-\langle x\rangle_{\rm c}(t)\mathcal{H}[c]}{\gamma^2}+O(\gamma^{-3})\right)\rho_{\rm c}(t)\\ &\quad+\delta t\mathcal{H}\left[-{\rm i}H-\gamma c-\tfrac{1}{2}c^\dagger c\right]\rho_{\rm c}(t),\end{aligned} \tag{4.49}$$

where $\mathcal{H}$ and $\mathcal{G}$ are as defined previously (4.14,4.13). Although Eq. (4.44) requires that $dN_{\rm c}(t)$ be a point process, it is possible to simply substitute the expression obtained above for δN as a Gaussian random variable into Eq. (4.49). This is because each jump is infinitesimal, so the effect of many jumps is approximately equal to the the effect of one jump scaled by the number of jumps. This can be justified more rigorously [149] by considering an expression for the system state given precisely δN detections, and then taking the large δN limit. The simple procedure I am adopting here gives the correct answer more rapidly. Keeping only the lowest order terms in $\gamma^{-1/2}$ and letting $\delta t\to dt$ yields the SME

$$d\rho_{\rm c}(t)=-{\rm i}[H,\rho_{\rm c}(t)]dt+\mathcal{D}[c]\rho_{\rm c}(t)dt+dW(t)\mathcal{H}[c]\rho_{\rm c}(t). \tag{4.50}$$

Here $dW(t)$ is an infinitesimal Wiener increment, as introduced in Sec. 3.1. Thus, the jump evolution of Eq. (4.44) has been replaced by diffusive evolution. As Eq. (4.50) is, by its derivation, an explicit equation with white noise, it can be called an Itô stochastic master equation. It is trivial to see that the ensemble average evolution reproduces the nonselective master equation by eliminating the noise term.

Just as the $\gamma\to\infty$ leads to continuous evolution for the state, it also changes the point process photocount into a continuous photocurrent with white noise. Removing the constant local-oscillator contribution gives

$$I_{\rm c}^{\rm hom}(t)\equiv\lim_{\gamma\to\infty}\frac{\delta N_{\rm c}(t)-\gamma^2\delta t}{\gamma\delta t}=\langle x\rangle_{\rm c}(t)+\xi(t), \tag{4.51}$$

where $\xi(t)=dW(t)/dt$. It is not difficult to see that if the detector efficiency is η, the homodyne photocurrent becomes

$$I_{\rm c}^{\rm hom}(t)=\eta\langle x\rangle_{\rm c}(t)+\sqrt{\eta}\xi(t), \tag{4.52}$$

and the SME (4.50) is modified to

$$\dot\rho=-{\rm i}[H,\rho_{\rm c}(t)]+\mathcal{D}[c]\rho_{\rm c}(t)+\sqrt{\eta}dW(t)\mathcal{H}[c]\rho_{\rm c}(t). \tag{4.53}$$

For the case $\eta=1$, the homodyne SME is equivalent to the SSE

$$d|\psi_{\rm c}(t)\rangle=\left\{-{\rm i}Hdt-\tfrac{1}{2}\left[c^\dagger c-2\langle x/2\rangle_{\rm c}(t)c+\langle x/2\rangle_{\rm c}^2(t)\right]dt+[c-\langle x/2\rangle_{\rm c}(t)]dW(t)\right\}|\psi_{\rm c}(t)\rangle. \tag{4.54}$$

If one ignores the normalization of the state vector, then one gets the simpler equation

$$d|\bar\psi_{\rm c}(t)\rangle=dt\left[-{\rm i}H-\tfrac{1}{2}c^\dagger c+I_{\rm c}^{\rm hom}(t)c\right]|\bar\psi_{\rm c}(t)\rangle. \tag{4.55}$$

Here I have used a bar rather than a tilde to denote the unnormalized state because its non-unit norm does not have any interpretation as a probability for a measurement result, unlike the unnormalized states introduced in Ch. 2. This SSE (which is the form which Carmichael originally derived) very elegantly shows how the state is conditioned on the measured photocurrent. It is possible to derive this equation in a much more direct manner. However that method tends, in my opinion, to obscure the relationship with quantum measurement theory which I wish to keep central in this thesis. For that reason, I present the alternative derivation in App. B, along with analogous rederivations of the direct detection SSE.

Finally, it is worth noting that these equations can all be derived from balanced homodyne detection, in which the beam splitter transmittance is one half, rather than close to one. In that case, one photodetector is used for each output beam, and the signal photocurrent is the difference between the two currents. This configuration has the advantage of needing smaller local oscillator powers to achieve the same ratio of system amplitude to local oscillator amplitude, because all of the local oscillator beam is detected. Also, if the local oscillator has classical fluctuations, then these cancel when the photocurrent difference is taken, whereas with simple homodyne detection, these fluctuations are indistinguishable from (and may even swamp) the signal fluctuations. Furthermore, the local oscillator intensity fluctuations, which modulate the signal homodyne detection, can be removed by demodulating the difference current using the information in the sum current, which depends only on the local oscillator intensity. Thus in practice, balanced homodyne detection has many advantages over simple homodyne detection. In theory, the ideal limit is the same for both, which is why I have considered only simple homodyne detection. The analysis for balanced homodyne detection is found in Ref. [149].

4.2.3 Output Field Correlation Functions

Just as for direct detection, the output autocorrelation functions for homodyne detection can be derived either from the quantum trajectories introduced here, or from Heisenberg picture field operators. In this case, the photon flux operator for the output field after the local oscillator has been added is, from Eq. (4.41)

$$I_1' = \gamma^2 + \gamma(c + c^\dagger + \nu + \nu^\dagger) + (c^\dagger + \nu^\dagger)(c + \nu). \tag{4.56}$$

In the limit that $\gamma \to \infty$, the last term can be ignored for the homodyne photocurrent operator

$$I_1^{\mathrm{hom}}(t) \equiv \lim_{\gamma\to\infty} \frac{I_1'(t) - \gamma^2}{\gamma} = x(t) + \xi(t). \tag{4.57}$$

Here, x is the quadrature operator defined before, and $\xi(t)$ is the vacuum input operator

$$\xi(t) = \nu(t) + \nu^\dagger(t), \tag{4.58}$$

which has statistics identical to the normalized Gaussian white noise for which the same symbol is used. Its operator nature is evident only from its commutation relations with its conjugate variable

$$[\xi(t), \upsilon(t')] = 2\mathrm{i}\delta(t - t'), \tag{4.59}$$

where

$$\upsilon(t) = -\mathrm{i}\nu(t) + \mathrm{i}\nu^\dagger(t). \tag{4.60}$$

The output quadrature operator (4.57) evidently has the same single-time statistics as the homodyne photocurrent (4.51), with a mean equal to the mean of x, and a white-noise variation. The two-time correlation function of the operator $I_1^{\mathrm{hom}}(t)$ is defined by

$$F_{\mathrm{hom}}^{(1)}(t, t+\tau) = \langle I_1^{\mathrm{hom}}(t+\tau) I_1^{\mathrm{hom}}(t)\rangle. \tag{4.61}$$

Using the commutation relations for the output field (3.89), this can be put into normal order as

$$F_{\mathrm{hom}}^{(1)}(t, t+\tau) = \langle : x(t+\tau)x(t) :\rangle + \delta(\tau), \tag{4.62}$$

where the annihilation of the vacuum has been used as before. Here, the colons denote time and normal ordering of the operators c, $c^\dagger$. The meaning of this can be seen in the Schrödinger picture

$$F_{\rm hom}^{(1)}(t,t+\tau) = {\rm Tr}\left[xe^{\mathcal{L}\tau}\left(c\rho(t)+\rho(t)c^\dagger\right)\right] + \delta(\tau), \tag{4.63}$$

where $\mathcal{L}$ is as before and $\rho(t)$ is the state of the system at time t which is assumed known.

In the quantum trajectory approach, the autocorrelation function is defined as

$$F_{\rm hom}^{(1)}(t,t+\tau) = {\rm E}[I_{\rm c}^{\rm hom}(t+\tau)I_{\rm c}^{\rm hom}(t)]. \tag{4.64}$$

From Eq. (4.51), and the fact that $\xi(t+\tau)$ is independent of the system at the past times t, this expression can be split into three terms

$$F_{\rm hom}^{(1)}(t,t+\tau) = {\rm E}[\langle x\rangle_{\rm c}(t+\tau)\xi(t)] + {\rm E}[\xi(t+\tau)\xi(t)] + {\rm E}[\langle x\rangle_{\rm c}(t+\tau)]\langle x\rangle(t), \tag{4.65}$$

where the factorization of the third term is due to the fact that $\rho(t)$ is known. The second term here is equal to the shot noise in Eq. (4.63). The first term is nonzero because the conditioned state of the system at time $t+\tau$ depends on the noise in the photocurrent at time t. That noise enters by the conditioning equation (4.50), so that

$$\rho_{\rm c}(t+dt) = \rho(t) + O(dt) + dW(t)\mathcal{H}[c]\rho(t) \tag{4.66}$$

The subsequent stochastic evolution of the system will be independent of the noise $\xi(t) = dW(t)/dt$ and hence may be averaged, giving

$${\rm E}[\langle x\rangle_{\rm c}(t+\tau)\xi(t)] = {\rm Tr}\big[xe^{\mathcal{L}\tau}{\rm E}[\{1+dW(t)\mathcal{H}[c]\}\rho_{\rm c}(t)dW(t)/dt]\big]. \tag{4.67}$$

Using the Itô rules for $dW(t)$ and expanding the superoperator $\mathcal{H}$ yields

$${\rm E}[\langle x\rangle_{\rm c}(t+\tau)\xi(t)] = {\rm Tr}\left[xe^{\mathcal{L}\tau}\left(c\rho(t)+\rho(t)c^\dagger\right)\right] - {\rm Tr}\left[xe^{\mathcal{L}\tau}\rho(t)\right]{\rm Tr}[x\rho(t)]. \tag{4.68}$$

The second term here cancels the third term in Eq. (4.65) to give the final expression

$$F_{\rm hom}^{(1)}(t,t+\tau) = {\rm Tr}\left[xe^{\mathcal{L}\tau}\left(c\rho(t)+\rho(t)c^\dagger\right)\right] + \delta(\tau), \tag{4.69}$$

in exact agreement with that calculated from the operator expressions (4.63). Once again, the operator quantity $I_1^{\rm hom}(t)$ has exactly the same statistics as the classical photocurrent $I_{\rm c}^{\rm hom}(t)$. The different conceptual basis is reflected in the origin of the delta function in the autocorrelation function. Quantum mechanically it is due to operator commutation relations, while from quantum trajectories it appears as local oscillator shot noise. These two complementary views were discussed by Carmichael [19] before quantum trajectories were introduced.

4.3 Heterodyne Detection Theory

4.3.1 Detuning the Local Oscillator

Homodyne detection has the advantage over direct detection in that it can detect phase-dependent properties of the system. By choosing the phase of the local oscillator, any given quadrature of the system can be measured. However, only one quadrature can be measured at a time. It would be possible to obtain information about two orthogonal quadratures simultaneously by splitting the system output beam into two by a beam splitter, and then homodyning each beam with the same local oscillator apart from a $\pi/2$ phase shift. If the balanced homodyne detection is used, then this measurement scheme is known as eight-port homodyne detection [94] for obvious reasons. An alternative way to achieve this double measurement is to detune the local oscillator from the system dipole frequency by an amount Δ much larger than any other system frequency. The photocurrent will then oscillate rapidly at frequency Δ, and the two Fourier components of this oscillation will correspond to two orthogonal quadratures of the output field. This is known as heterodyne detection, and is the subject of this section. It has not been treated by quantum trajectories prior to this thesis.

Begin with the homodyne SME (4.50), which assumed a constant local oscillator amplitude. Detuning the local oscillator to a frequency Δ above that of the system will affect only the final (stochastic) term in Eq. (4.50). Its effect will be simply to replace c by $c\exp(\mathrm{i}\Delta t)$ to get

$$\begin{aligned} d\rho_{\mathrm{c}}(t) &= -\mathrm{i}[H,\rho_{\mathrm{c}}(t)]dt + \mathcal{D}[c]\rho_{\mathrm{c}}(t)dt \\ &\quad + dW(t)\left\{e^{\mathrm{i}\Delta t}\left[c\rho_{\mathrm{c}}(t) - \langle c\rangle_{\mathrm{c}}(t)\rho_{\mathrm{c}}(t)\right] + e^{-\mathrm{i}\Delta t}\left[\rho_{\mathrm{c}}(t)c^\dagger - \langle c^\dagger\rangle_{\mathrm{c}}(t)\rho_{\mathrm{c}}(t)\right]\right\} \end{aligned} \tag{4.70}$$

Consider a time δt small on a characteristic time scale of the system, but large compared to Δ^{-1} so that there are many cycles due to the detuning. One might think that averaging the rotating exponentials over this time would eliminate the terms in which they appear. However this is not the case because these terms are stochastic, and since the noise is white by assumption, it will vary even faster than the the rotation at frequency Δ. Define two new Gaussian random variables

$$\delta W_x(t) = \int_t^{t+\delta t} \sqrt{2}\cos(\Delta s)dW(s), \tag{4.71}$$

$$\delta W_y(t) = \int_t^{t+\delta t} (-)\sqrt{2}\sin(\Delta s)dW(s). \tag{4.72}$$

It is easy to show that, to zeroth order in Δ^{-1}, these obey

$$\mathrm{E}[\delta W_q(t)\delta W_{q'}(t')] = \delta_{q,q'}(\delta t - |t-t'|)H(\delta t - |t-t'|), \tag{4.73}$$

where q and q' stand for x or y, and where H is the Heaviside function which is zero when its argument is negative and one when its argument is positive.

On the system's time scale δt is infinitesimal so the $\delta W_q(t)$ can be replaced by infinitesimal Wiener increments $dW_q(t)$ obeying

$$dW_q(t)dW_{q'}(t) = \delta_{q,q'}dt. \tag{4.74}$$

Taking the average over many detuning cycles therefore transforms Eq. (4.70) into

$$d\rho_{\mathrm{c}}(t) = -\mathrm{i}[H,\rho_{\mathrm{c}}(t)]dt + \mathcal{D}[c]\rho_{\mathrm{c}}(t)dt + \sqrt{1/2}\big(dW_x(t)\mathcal{H}[c] + dW_y(t)\mathcal{H}[-ic]\big)\rho_{\mathrm{c}}(t). \tag{4.75}$$

This is clearly equivalent to homodyne detection of the two quadratures simultaneously, each with efficiency $1/2$. Inefficient photodetection would simply change the $1/2$ into $\eta/2$. In order to record these two independent measurements, it is necessary to find the Fourier components of the photocurrent. These are defined by

$$I_{\mathrm{c}}^x(t) = (\delta t)^{-1}\int_t^{t+\delta t} 2\cos(\Delta s)I_{\mathrm{c}}^{\mathrm{hom}}(s)ds, \tag{4.76}$$

$$I_{\mathrm{c}}^y(t) = (\delta t)^{-1}\int_t^{t+\delta t} (-)2\sin(\Delta s)I_{\mathrm{c}}^{\mathrm{hom}}(s)ds. \tag{4.77}$$

To zeroth order in Δ^{-1}, these are

$$\begin{aligned} I_{\rm c}^{x}(t) &= \langle x\rangle_{\rm c}(t)+\sqrt{2}\xi_x(t), && (4.78)\\ I_{\rm c}^{y}(t) &= \langle y\rangle_{\rm c}(t)+\sqrt{2}\xi_y(t). && (4.79)\end{aligned}$$

Again, these are proportional to the homodyne photocurrents expected with an efficiency of 1/2.

The SME (4.75) is equivalent to the SSE

$$\begin{aligned} d|\psi_{\rm c}(t)\rangle &= \Big\{-{\rm i}H\,dt-\tfrac{1}{2}\left[c^\dagger c-2\langle c^\dagger\rangle_{\rm c}(t)c+\langle c^\dagger\rangle_{\rm c}(t)\langle c\rangle_{\rm c}(t)\right]dt \\ &\quad +\sqrt{1/2}dW_x(t)[c-\langle x/2\rangle_{\rm c}(t)]+\sqrt{1/2}dW_y(t)[-{\rm i}c-\langle y/2\rangle_{\rm c}(t)]\Big\}|\psi_{\rm c}(t)\rangle. \quad (4.80)\end{aligned}$$

This can be put into a simpler form analogous to Eq. (4.55) by ignoring normalization to get

$$d|\bar\psi_{\rm c}(t)\rangle = dt\left[-{\rm i}H-\tfrac{1}{2}c^\dagger c+I_{\rm c}^{\rm het}(t)^*c\right]|\bar\psi_{\rm c}(t)\rangle. \quad (4.81)$$

Here a complex heterodyne photocurrent has been defined as

$$I_{\rm c}^{\rm het}=\tfrac{1}{2}[I_{\rm c}^{x}+{\rm i}I_{\rm c}^{y}]. \quad (4.82)$$

It is more directly defined as

$$I_{\rm c}^{\rm het}(t)=\langle c\rangle_{\rm c}(t)+\zeta(t), \quad (4.83)$$

where $\zeta(t)=\sqrt{1/2}[\xi_x(t)+{\rm i}\xi_y(t)]$ is complex Gaussian white noise satisfying

$${\rm E}[\zeta^*(t)\zeta(t')]=\delta(t-t'), \quad (4.84)$$

with all other first and second order moments vanishing. Eq. (4.81), with the expression (4.83) in place of $I_{\rm c}^{\rm het}$, was first written down by Gisin and Percival [58]. However, as will be discussed in the following chapter, they did not derive it from any physical considerations, nor attribute to it any interpretation in terms of any detection scheme. Like the homodyne SSE (4.55), Eq. (4.81) can be derived more directly from the Langevin equation in the Schrödinger picture, as shown in App. B.

4.3.2 Output Field Correlation Functions

The equivalence between the quantum mechanical and quantum trajectory calculations of the output field correlation functions go through for heterodyne detection in much the same way as for homodyne detection. For variation, I will construct the Heisenberg operator for the 'heterodyne photocurrent' from two homodyne measurements, rather than from the Fourier components of the heterodyne signal. To make two homodyne measurements, it is necessary to use a 50/50 beam splitter to divide the cavity output before the local oscillator is added. This gives two output beams,

$$b_1^{\pm}=\sqrt{1/2}(\nu+c\pm\mu). \quad (4.85)$$

Here I have introduced another vacuum annihilation operator μ which enters at the free port of the beam splitter. Let the b_1^+ beam enter a homodyne apparatus to measure the x quadrature, and the b_1^- one to measure the y quadrature. Then the operators for the two photocurrents, normalized to the system signal, are

$$\begin{aligned} I_1^{x} &= x+[\nu+\nu^\dagger+\mu+\mu^\dagger], && (4.86)\\ I_1^{y} &= y-{\rm i}[\nu-\nu^\dagger-\mu+\mu^\dagger]. && (4.87)\end{aligned}$$

Defining the complex heterodyne photocurrent as before (4.82) gives

$$I_1^{\text{het}} = c + \nu + \mu^\dagger. \tag{4.88}$$

It is simple to see that $\zeta(t) = \nu(t) + \mu^\dagger(t)$ is effectively a complex Gaussian white noise term. Thus, the operator (4.88) has the same statistics as the photocurrent (4.83). Because it is a complex photocurrent, the autocorrelation function is now defined as

$$F_{\text{het}}^{(1)}(t, t+\tau) = \langle I_1^{\text{het}}(t+\tau)^\dagger I_1^{\text{het}}(t) \rangle, \tag{4.89}$$

which evaluates to

$$F_{\text{het}}^{(1)}(t, t+\tau) = \langle c^\dagger(t+\tau)c(t) \rangle + \delta(\tau). \tag{4.90}$$

For an optical system with c the annihilation operator, this is equal to Glauber's first order coherence function [62] for τ finite. Using the same method as in Sec. 4.2.3, the quantum trajectory approach yields

$$\mathrm{E}\left[I_{\mathrm{c}}^{\text{het}}(t+\tau)^* I_{\mathrm{c}}^{\text{het}}(t)\right] = \mathrm{Tr}[c^\dagger e^{\mathcal{L}\tau} c\rho(t)] + \delta(\tau). \tag{4.91}$$

Again, this agrees with the result (4.90).

4.4 Detection with White Noise

4.4.1 Quantum Trajectories with White Noise

So far, I have considered optical measurements with the input in a vacuum, or coherent state (the extension of the coherent input case from direct detection in Sec. 4.1.4 to homodyne and heterodyne detection is trivial). As explained in Sec. 4.1.4, the photon flux is undefined for an input bath in a more general state, such as with thermal or squeezed white noise. This indicates that direct detection is impossible, or in practice possible but useless. However, the output field quadrature operators are well-defined even with white noise, because they are only linear in the noise. Thus, field operators for homodyne and heterodyne detection photocurrents can be defined without difficulty, simply by replacing the vacuum operator ν by the more general input bath operator b_0. This indicates that it should be possible to treat homodyne and heterodyne detection in such situations. In this section, I develop the quantum trajectory theory for detection in the presence of white noise. This has not been done prior to this thesis.

The homodyne detection theory of Sec. 4.2 began with a finite local oscillator amplitude γ, so that the quantum trajectories were jump-like. Diffusive trajectories were obtained when the $\gamma \to \infty$ limit was taken. This approach would fail if the input field were contaminated by white noise, for the same reason that direct detection is impossible with white noise: the infinite photon flux due to the noise would swamp the signal. However, as noted in Sec. 4.1, a physical noise source would not be truly white, and in any case the bandwidth of the detector would give a cutoff to the flux. If the local oscillator is made sufficiently intense, then the signal due to this would overcome that due to the noise. Thus, in order to treat detection with white noise, it is necessary to begin with an infinitely large local oscillator. Then one can assume that the homodyne detection effects an ideal measurement of the instantaneous quadrature of the output field, without worrying about individual jumps.

Consider a section of the input bath $\mu(t)$ of temporal length dt interacting with the system ρ at time t. The state matrix for the entangled system and bath is, from Eq. (3.107)

$$R(t+dt) = R(t) + \sqrt{dt}\left[a^\dagger c - c^\dagger a, R(t)\right] + O(dt), \tag{4.92}$$

where

$$R(t) = \mu(t) \otimes \rho(t) \tag{4.93}$$

is the initial unentangled state. In order to consider a bath with arbitrary Gaussian statistics, it is useful to transform the bath state into a Wigner probability distribution. That is, define

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda \mathrm{Tr}_\mu \left[R \exp[\lambda(a^\dagger - a^*) - \lambda^*(a - \alpha)] \right]. \tag{4.94}$$

Note that the trace is only over the bath Hilbert space. $W(\alpha, \alpha^*)$ is still a density operator in the system Hilbert space.

From Eq. (3.115) The initial state is

$$W(\alpha, \alpha^*; t) = \left(\pi \sqrt{\left(N + \tfrac{1}{2}\right)^2 - |M|^2} \right)^{-1} \exp\left(- \frac{\left(N + \frac{1}{2}\right) |\alpha|^2 - \frac{1}{2} M^* (\alpha)^2 - \frac{1}{2} M (\alpha^*)^2}{\left(N + \frac{1}{2}\right)^2 - |M|^2} \right) \rho(t), \tag{4.95}$$

where I have assumed for simplicity that the bath has no coherent amplitude. Using the relations [52]

$$\begin{aligned} a\mu &\rightarrow \left(\alpha + \tfrac{1}{2}\partial_{\alpha^*}\right) W(\alpha, \alpha^*) && (4.96)\\ a^\dagger\mu &\rightarrow \left(\alpha^* - \tfrac{1}{2}\partial_{\alpha}\right) W(\alpha, \alpha^*) && (4.97)\\ \mu a &\rightarrow \left(\alpha - \tfrac{1}{2}\partial_{\alpha^*}\right) W(\alpha, \alpha^*) && (4.98)\\ \mu a^\dagger &\rightarrow \left(\alpha^* + \tfrac{1}{2}\partial_{\alpha}\right) W(\alpha, \alpha^*), && (4.99) \end{aligned}$$

the entangled state is

$$\begin{aligned} W(t + dt) &= W(t) + \sqrt{dt} \left[\left(\alpha^* - \tfrac{1}{2}\partial_\alpha\right) cW(t) - \left(\alpha + \tfrac{1}{2}\partial_{\alpha^*}\right) c^\dagger W(t) \right. \\ &- \left. \left(\alpha^* + \tfrac{1}{2}\partial_\alpha\right) W(t) c + \left(\alpha - \tfrac{1}{2}\partial_{\alpha^*}\right) W(t) c^\dagger \right] + O(dt). \end{aligned} \tag{4.100}$$

Now consider a measurement on the bath after it has left the system. As stated above, a photon counting measurement is not well defined. However, a measurement of one quadrature by homodyne measurement could be well defined. Photon counting can be avoided by modeling such a measurement as a projective measurement of $a + a^\dagger$. The unnormalized conditioned state following such a measurement is

$$\tilde{R}_{\mathrm{x}}(t + dt) = \langle \mathrm{x} | R(t + dt) | \mathrm{x} \rangle. \tag{4.101}$$

Here, I am using the Roman font x to denote eigenstates of $a + a^\dagger$, so

$$(a + a^\dagger)|\mathrm{x}\rangle = \mathrm{x}|\mathrm{x}\rangle. \tag{4.102}$$

In terms of the Wigner function (4.100), the conditioned state is

$$\tilde{W}_{\mathrm{c}}(\mathrm{x}; t + dt) = \int_{-\infty}^{\infty} d\mathrm{y} W(\mathrm{x}, \mathrm{y}; t + dt), \tag{4.103}$$

where here $\mathrm{x} = \alpha + \alpha^*$ and $\mathrm{y} = -\mathrm{i}(\alpha - \alpha^*)$. Because of the Gaussian initial bath state, the derivatives in Eq. (4.100) and the integral in Eq. (4.103) are elementary, but tedious, to evaluate. The result is

$$\begin{aligned} \tilde{W}_{\mathrm{c}}(\mathrm{x}; t + dt) &= G(\mathrm{x}, 0, L) \left\{ \rho(t) + \sqrt{dt}\, \mathrm{x}\, L^{-1} \left[(N + M^* + 1) c\rho(t) - (N + M) c^\dagger \rho(t) \right. \right. \\ &+ \left. \left. (N + M + 1)\rho(t) c^\dagger - (N + M^*)\rho(t) c \right] + O(dt) \right\}, \end{aligned} \tag{4.104}$$

where, for arbitrary arguments, G is a Gaussian distribution

$$G(x,\mu,\sigma^2) \equiv \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \tag{4.105}$$

so that $G(\mathrm{x},0,L)$ is the initial probability distribution for x, where

$$L \equiv 2N + 1 + M + M^*. \tag{4.106}$$

The probability distribution for the result x is

$$p(\mathrm{x};t+dt) = \mathrm{Tr}_\rho\left[\tilde{W}_{\mathrm{c}}(\mathrm{x};t+dt)\right]. \tag{4.107}$$

Evaluating the trace gives

$$\begin{aligned} p(\mathrm{x};t+dt) &= G(\mathrm{x},0,L)\left[1 + \sqrt{dt}\,\mathrm{x}\,L^{-1}\langle c + c^\dagger\rangle + O(dt)\right] & (4.108)\\ &= G(\mathrm{x},\sqrt{dt}\langle c + c^\dagger\rangle, L)[1 + O(dt)]. & (4.109)\end{aligned}$$

Thus, the quadrature of the outgoing field is the Gaussian random variable

$$\sqrt{dt}\,\mathrm{x} = \langle c + c^\dagger\rangle(t)dt + \sqrt{L}\,dW(t), \tag{4.110}$$

where $dW(t)^2 = dt$. The instantaneous photocurrent is

$$I_{\mathrm{c}}^{\mathrm{hom}}(t) = \langle c + c^\dagger\rangle_{\mathrm{c}}(t) + \sqrt{L}\xi(t), \tag{4.111}$$

where $\xi(t)$ represents Gaussian white noise as usual. Note that the coefficient L (4.106) for the white noise may take any real value [52].

The expression for the normalized conditioned system state matrix is

$$\begin{aligned} \rho_{\mathrm{c}}(t+dt) &= \tilde{W}_{\mathrm{c}}(\mathrm{x};t+dt)/p(\mathrm{x};t+dt) \\ &= \left\{1 + \sqrt{dt}\,\mathrm{x}\,L^{-1}\mathcal{H}[(N+M^*+1)c - (N+M)c^\dagger] + O(dt)\right\}\rho_{\mathrm{c}}(t), & (4.112)\end{aligned}$$

where $\mathcal{H}$ is as in Eq. (4.14). Substituting the expression for x in Eq. (4.112) gives

$$d\rho_{\mathrm{c}}(t) = \left(\frac{1}{\sqrt{L}}dW(t)\mathcal{H}\left[(N+M^*+1)c - (N+M)c^\dagger\right] + O(dt)\right)\rho_{\mathrm{c}}(t). \tag{4.113}$$

Up till now, I have consistently ignored terms of order dt in order to keep the problem manageable. Now, I use the argument that the nonselective evolution of the state matrix must be the master equation $\dot{\rho} = \mathcal{L}\rho$, where

$$\mathcal{L}\rho = (N+1)\mathcal{D}[c]\rho + N\mathcal{D}[c^\dagger]\rho - \frac{M}{2}[c^\dagger,[c^\dagger,\rho]] - \frac{M^*}{2}[c,[c,\rho]] - \mathrm{i}[H,\rho], \tag{4.114}$$

to claim that the term of order dt in Eq. (4.113) must be $\mathcal{L}dt$. This result should be obtained automatically if terms of order dt were included in the derivation I have presented. The final result for the stochastic master equation describing ideal homodyne detection in the presence of white noise is

$$d\rho_{\mathrm{c}}(t) = \left(dt\mathcal{L} + \frac{1}{\sqrt{L}}dW(t)\mathcal{H}\left[(N+M^*+1)c - (N+M)c^\dagger\right]\right)\rho_{\mathrm{c}}(t). \tag{4.115}$$

Note that for $N = M = 0$, this agrees with the result obtained in Sec. 4.2. Unlike the case of vacuum input, it is only possible to rewrite this SME as a SSE if $|M|^2 = N(N+1)$. This is the condition for the input bath to be in a pure state. For anything other than a vacuum state, such a pure state will be nonclassical, with $|M|^2 > N^2$. The simplest form of the SSE is again for unnormalized state vector $|\bar{\psi}_{\rm c}(t)\rangle$. It is

$$\begin{aligned} d|\bar{\psi}_{\rm c}(t)\rangle &= dt\Big\{ -{\rm i}H - \tfrac{1}{2}\left[(N+1)c^\dagger c + Ncc^\dagger - Mc^{\dagger 2} - M^* c^2\right] \\ &\quad + I_{\rm c}^{\rm hom}(t)\frac{1}{L}\left[(N+M^*+1)c - (N+M)c^\dagger\right]\Big\}|\bar{\psi}_{\rm c}(t)\rangle. \end{aligned} \tag{4.116}$$

An alternate, more elegant, derivation of this equation is presented in App. B. Of course, this unraveling is only possible for unit efficiency detectors. An efficiency of η multiplies L by $1/\eta$ in the SME (4.115) and the photocurrent (4.111).

Finally, I will briefly give the equivalent results for heterodyne detection. The two heterodyne photocurrents are

$$I_{\rm c}^x(t) = \langle x\rangle_{\rm c}(t) + \sqrt{2L_x}\,\xi_x(t), \tag{4.117}$$

$$I_{\rm c}^y(t) = \langle y\rangle_{\rm c}(t) + \sqrt{2L_y}\,\xi_y(t). \tag{4.118}$$

Here,

$$L_x \equiv 2N + 1 + M + M^* = L, \tag{4.119}$$

$$L_y \equiv 2N + 1 - M - M^*. \tag{4.120}$$

The conditioning SME is

$$\begin{aligned} d\rho_{\rm c}(t) &= dt\mathcal{L}\rho_{\rm c}(t) + \frac{1}{\sqrt{2L_x}}dW_x(t)\mathcal{H}\left[(N+M^*+1)c - (N+M)c^\dagger\right]\rho_{\rm c}(t) \\ &\quad + \frac{1}{\sqrt{2L_y}}dW_y(t)\mathcal{H}\left[(N-M^*+1)(-{\rm i}c) - (N-M)({\rm i}c^\dagger)\right]\rho_{\rm c}(t). \end{aligned} \tag{4.121}$$

4.4.2 Output Field Correlation Functions

From the conditioning equation and the expressions for the photocurrent, it is easy to find the two-time correlation function for the output field using the method of Sec. 4.2.3. The result for homodyne measurement is

$$F^{(1)}_{\rm hom}(t,t+\tau) = {\rm E}[I_{\rm c}^{\rm hom}(t+\tau)I_{\rm c}^{\rm hom}(t)] \tag{4.122}$$

$$\begin{aligned} &= {\rm Tr}\left[(c+c^\dagger)e^{\mathcal{L}\tau}\left\{(N+M^*+1)c\rho(t) - (N+M)c^\dagger\rho(t)\right.\right. \\ &\quad \left.\left. + (N+M+1)\rho(t)c^\dagger - (N+M^*)\rho(t)c\right\}\right] + L_x\delta(\tau). \end{aligned} \tag{4.123}$$

For heterodyne detection,

$$F^{(1)}_{\rm het}(t,t+\tau) = \tfrac{1}{4}{\rm E}[\{I_{\rm c}^x(t+\tau) - {\rm i}I_{\rm c}^y(t+\tau)\}\{I_{\rm c}^x(t) + {\rm i}I_{\rm c}^y(t)\}] \tag{4.124}$$

$$\begin{aligned} &= {\rm Tr}\left[c^\dagger e^{\mathcal{L}\tau}\left\{(N+1)c\rho(t) - Mc^\dagger\rho(t) + M\rho(t)c^\dagger - N\rho(t)c\right\}\right] \\ &\quad + (2N+1)\delta(\tau). \end{aligned} \tag{4.125}$$

Note that, unlike the case of a vacuum input, there is no simple relationship between these formulae and the Glauber coherence functions.

These correlation functions could be derived from the Heisenberg field operators. The relevant expressions are

$$F^{(1)}_{\rm hom}(t,t+\tau) = \langle [x(t+\tau) + b_0(t+\tau) + b_0^\dagger(t+\tau)][x(t) + b_0(t) + b_0^\dagger(t)]\rangle \tag{4.126}$$

and

$$F^{(1)}_{\text{het}}(t,t+\tau) = \langle [c^\dagger(t+\tau) + b_0^\dagger(t+\tau)][c(t)+b_0(t)]\rangle + (N+1)\delta(\tau), \tag{4.127}$$

where the final term in the expression for $F^{(1)}_{\text{het}}(t,t+\tau)$ comes from the independent noise operator $\mu(t)$ which now has the same statistics as the input field operator b_0 . I will not attempt to prove that these evaluate to Eqs. (4.123,4.125), because it is considerably more difficult than with a vacuum input $b_0 = \nu$. The reason for this is that it is impossible to chose an operator ordering such that the contributions due to the bath input vanish. The necessary method would have to be more akin to that used in obtaining Eqs. (4.123,4.125), where the stochastic equation analogous to the SME is the quantum Langevin equation (3.97).

4.5 Relation to Distribution Functions

Obviously the quantum trajectories for direct, homodyne, and heterodyne detection describe measurements of the intensity, one quadrature, and the complex amplitude of the dipole of the system respectively. Equally obviously, the measurements described are far removed from simple measurements of these quantities, such as by the projective measurements of Sec. 2.1.1. Nevertheless, there must be some elementary relation between the two types of measurements for at least some cases. For example, counting the number of photons in a cavity should give the same results (statistically) whether this is done by allowing the photons to escape gradually through an end mirror into a photodetector, or whether the measurement is a projection of the cavity mode into photon number eigenstates, provided the system Hamiltonian commutes with photon number. The purpose of this section is to establish the relationship between quantum trajectory and projective measurements, for the three schemes discussed. For simplicity, I will consider only a vacuum input. Also, I will consider only a freely decaying optical cavity with unit linewidth, so that $c = a$ with $[a, a^\dagger] = 1$. Other systems, such as atoms, are surprisingly difficult to deal with because of the different algebra of the raising and lowering operators. I begin with photon counting.

4.5.1 Photon Number Distribution

The most obvious difference between a projective measurement of photon number, and an external counting of escaped photons from a freely decaying cavity is that the final state of the cavity mode is the appropriate photon number eigenstate in the first case, and the vacuum in the second. The latter result is because the counting time must be infinite to allow all photons to escape. Although extra-cavity detection is not equivalent to projective detection, it should give the same statistics. That is to say, the probability for m photodetection events in an infinite time should be equal to the photon number population $p_m = \langle m|\rho(0)|m\rangle$ at the start of the interval. To determine this probability, it is useful to return to the methods of Sec. 4.1.1, before the explicit notation of quantum trajectories was introduced. Following Srinivas and Davies [128], I denote the jump superoperator $\mathcal{J}[a]$ by $\mathcal{J}$ and define a smooth evolution superoperator

$$\mathcal{S}(t)\rho = \exp\left[-\left(\tfrac{1}{2}a^\dagger a + \mathrm{i}H\right)t\right]\rho \exp\left[-\left(\tfrac{1}{2}a^\dagger a - \mathrm{i}H\right)t\right]. \tag{4.128}$$

Here, the Hamiltonian H has been retained, but will be set to zero for the case of a freely decaying cavity. Assuming perfect detection, the infinitesimally evolved unnormalized state matrices conditioned on one or zero detections in the interval $[t, t+dt)$ are

$$\tilde{\rho}_1(t+dt) \;=\; dt\mathcal{J}\rho(t), \tag{4.129}$$

$$\tilde{\rho}_0(t+dt) \quad = \quad \mathcal{S}(dt)\rho(t). \tag{4.130}$$

So far, this is nothing but new notation. However, it is now evident that the infinitesimal evolution can be easily extended to arbitrary lengths of times. Let the count interval be $[0,T)$, and let there be m detections at times $t_1, t_2, \ldots, t_m$. Then the unnormalized state matrix conditioned on this result is

$$\tilde{\rho}_{m:t_1,t_2,\ldots,t_m}(T) = \mathcal{S}(T-t_m)\mathcal{J}\,dt\mathcal{S}(t_m-t_{m-1})\mathcal{J}\,dt\ldots\mathcal{J}\,dt\mathcal{S}(t_1)\rho(0). \tag{4.131}$$

Here, I have used the semigroup property of $S(t)$ that $S(t_1)S(t_2) = S(t_1+t_2)$. Now, if one is interested only in the number of detections, then one can integrate over the times to get

$$\tilde{\rho}_m(T) = \int_0^T dt_m \int_0^{t_m} dt_{m-1}\ldots\int_0^{t_2} dt_1 \mathcal{S}(T-t_m)\mathcal{J}\mathcal{S}(t_m-t_{m-1})\mathcal{J}\ldots\mathcal{J}\mathcal{S}(t_1)\rho(0). \tag{4.132}$$

The superoperator preceding $\rho(0)$ on the right side of this equation is called $\mathcal{N}_m(T)$ by Srinivas and Davies. The total evolution of the system, ignoring all information gained by photodetection, is generated by the superoperator

$$\mathcal{T}(T) = \sum_{m=0}^{\infty}\mathcal{N}_m(T) = \exp[\mathcal{L}T], \tag{4.133}$$

where the last equality follows from the infinitesimal forms (4.129,4.130), with $\mathcal{L}dt = \mathcal{J}\,dt + \mathcal{S}(dt)$.

The probability for detecting m photons in time T is given by the trace of $\tilde{\rho}_m(T)$. With $H=0$, this can be evaluated in the photon number basis as

$$\begin{aligned}
\mathrm{Tr}[\tilde{\rho}_m(T)] \quad &= \quad \sum_{n=0}^{\infty}\int_0^T dt_m \int_0^{t_m} dt_{m-1}\ldots\int_0^{t_2} dt_1 \exp[-n(T-t_m)](n+1) \\
&\quad\quad \times\ \exp[-(n+1)(t_m-t_{m-1})](n+2)\ldots(n+m)\exp[-(n+m)t_1]p_{n+m} \\
&= \quad \sum_{n=0}^{\infty} e^{-nT}\int_0^T dt_m \int_0^{t_m} dt_{m-1}\ldots\int_0^{t_2} dt_1 \\
&\quad\quad \times\ \exp(-t_m)\exp(-t_{m-1})\ldots\exp(-t_1)\frac{(n+m)!}{n!}p_{n+m}.
\end{aligned} \tag{4.134}$$

This can further be rearranged to

$$\begin{aligned}
\mathrm{Tr}[\tilde{\rho}_m(T)] \quad &= \quad \sum_{n=0}^{\infty} e^{-nT}\frac{1}{m!}\left(\int_0^T e^{-t}dt\right)^m \frac{(n+m)!}{n!}p_{n+m} \\
&= \quad \sum_{n=0}^{\infty} e^{-nT}(1-e^{-T})^m\frac{(n+m)!}{n!m!}p_{n+m}.
\end{aligned} \tag{4.135}$$

In the limit $T\to\infty$, the only term in this sum which will contribute is that with $n=0$. Hence,

$$\mathrm{Tr}[\tilde{\rho}_m(\infty)] = p_m, \tag{4.136}$$

as expected. Counting the photons escaping from the cavity is thus equivalent to a measurement with the measurement operators

$$\Omega_N(\infty) = |0\rangle\langle N|, \tag{4.137}$$

where $a^\dagger a|N\rangle = N|N\rangle$. For the case of detection with efficiency η, it is easiest to calculate the probabilities by assuming perfect efficiency, and distributing the counts as registered or unregistered according to a Bernoulli distribution with probabilities η and $1-\eta$. The result is

$$\mathrm{Tr}[\tilde{\rho}_m(\infty)] = \sum_{n=0}^{\infty}\eta^m(1-\eta)^n\frac{(n+m)!}{n!m!}p_{m+n}. \tag{4.138}$$

Note that this is also what is obtained from Eq. (4.135), for finite counting times, where the effective efficiency is

$$\eta = 1 - e^{-T}. \tag{4.139}$$

Heisenberg Picture

The result (4.136) can also be obtained without using quantum trajectories, from the output field operators. The output photon flux is

$$I(t) = [a^\dagger(t) + \nu^\dagger(t)][a(t) + \nu(t)]. \tag{4.140}$$

Under the free decay evolution of Eq. (3.84),

$$a(t) = a(0)e^{-t/2} - \int_0^t e^{(s-t)/2}\nu(s)ds, \tag{4.141}$$

so

$$I(t) = \left[a^\dagger(0)e^{-t/2} - \int_0^t e^{(s-t)/2}\nu^\dagger(s)ds + \nu^\dagger(t)\right]\left[a(0)e^{-t/2} - \int_0^t e^{(s-t)/2}\nu(s)ds + \nu(t)\right]. \tag{4.142}$$

The operator for the total photocount is then

$$N = \int_0^\infty I(t)dt. \tag{4.143}$$

Ignoring bath operators in normal order (since they act directly on the vacuum and are thereby eliminated) gives simply

$$N = a^\dagger(0)a(0) = a^\dagger a. \tag{4.144}$$

This confirms that the integral of the photocurrent does indeed measure the operator $a^\dagger a$ for the initial cavity state.

4.5.2 Wigner Distribution

Now consider homodyne detection. This is rather more difficult than direct detection because an expression for the unnormalized state matrix (whose trace gives the wanted probability) can only be written down for γ finite so that the photodetections are discrete. Another difficulty is that it would not be advisable simply to integrate the photocurrent from a constant local oscillator from zero to infinity. Unlike direct detection, when all light has escaped the cavity the homodyne measurement continues to give a nonzero current. Thus, for long times the additional current is merely adding noise to the record. This can be circumvented by properly mode-matching the local oscillator to the system. That is to say, by matching the decay rate, as well as the frequency, of the local oscillator amplitude to that of the signal. In fact, since the system cavity is freely decaying, the local oscillator can be assumed to reside in the same mode as the system, and will then obviously decay at the same rate. The local oscillator amplitude is added to the system at the start of the interval by replacing $\rho(0)$ by

$$\rho'(0) = \exp[-\gamma(a - a^\dagger)]\rho(0)\exp[\gamma(a - a^\dagger)]. \tag{4.145}$$

Counting all of the photons which escape from this cavity will effect a measurement of $a^\dagger a$ on $\rho'(0)$. Equivalently, it will measure the operator

$$N' = \exp[\gamma(a - a^\dagger)](a^\dagger a)\exp[-\gamma(a - a^\dagger)] = (a^\dagger + \gamma)(a + \gamma) \tag{4.146}$$

on the original state $\rho(0)$. In the limit $\gamma \to \infty$, and with the constant count due to the local oscillator subtracted, the measured quantity is

$$\lim_{\gamma\to\infty} \frac{N' - \gamma^2}{\gamma} = x. \tag{4.147}$$

As expected, a homodyne measurement measures the quadrature x.

Because of the first difficulty mentioned above, it is not easy to derive this result from the diffusive quantum trajectories for homodyne measurement of Sec. 4.2. Nevertheless, it can be shown to be consistent with such ideal trajectories. Consider the case where the initial state of the cavity is coherent, with $\rho(0) = |\alpha\rangle\langle\alpha|$, where $a|\alpha\rangle = \alpha|\alpha\rangle$. It is evident from Eq. (4.50) that a coherent state will not be affected by the stochastic term, because it is an eigenstate of a. Furthermore, with $H = 0$, the irreversible evolution simply causes the coherent state to decay. Thus, this special solution to the SME (4.50) will be the deterministic

$$\rho(t) = |\alpha e^{-t/2}\rangle\langle\alpha e^{-t/2}|. \tag{4.148}$$

From Eq. (4.51), the homodyne photocurrent will be

$$I_{\rm c}^{\rm hom}(t) = e^{-t/2}\left[(\alpha + \alpha^*)\, e^{-t/2} + \xi(t)\right]. \tag{4.149}$$

The first decaying exponential factor here is due to the mode-matching of the local oscillator referred to before. The integrated photocurrent is thus

$$X = \int_0^\infty I_{\rm c}^{\rm hom}(t)dt = \alpha + \alpha^* + \sigma, \tag{4.150}$$

where

$$\sigma = \int_0^\infty e^{-t/2} dW(t) \tag{4.151}$$

is a Gaussian random variable of unit variance. That is to say, the probability distribution for X will be

$$p(X) = \frac{1}{\sqrt{2\pi}} \exp\left[(X - \alpha - \alpha^*)^2/2\right]. \tag{4.152}$$

Now consider a mixture of initial coherent states with a Glauber-Sudarshan P function representation $P(\alpha, \alpha^*)$. The probability distribution for X will then be

$$p(X) = \frac{1}{\sqrt{2\pi}} \int d^2\alpha \exp\left[-(X - \alpha - \alpha^*)^2/2\right] P(\alpha, \alpha^*). \tag{4.153}$$

This is equal to

$$p(X) = \int_{-\infty}^{\infty} dY \frac{1}{2\pi} \int d^2\alpha \exp\left[-(X - \alpha - \alpha^*)^2/2 - (Y + {\rm i}\alpha - {\rm i}\alpha^*)^2/2\right] P(\alpha, \alpha^*). \tag{4.154}$$

In this form, $p(X)$ will be recognized to be the marginal distribution for $x = a + a^\dagger$, obtained by integrating over Y in the Wigner function distribution,

$$p(X) = \int_{-\infty}^{\infty} dY W(X, Y), \tag{4.155}$$

where now $X = \alpha + \alpha^*$ and $Y = -{\rm i}(\alpha - \alpha^*)$. It is well known that the Wigner function marginal distribution for X is the true distribution for x. Thus, the prediction of quantum trajectories is consistent with that obtained above. This argument could perhaps be generalized for arbitrary initial states by an appeal to generalized functions for $P(\alpha, \alpha^*)$ [85].

The measurement operator for homodyne detection of a decaying cavity for an infinitely long time with a fully mode-matched local oscillator is thus

$$\Omega_X(\infty) = |0\rangle\langle X|, \tag{4.156}$$

where $(a + a^\dagger)|X\rangle = X|X\rangle$. In the case of detectors of efficiency η, the Gaussian noise in the integrated photocurrent is increased with respect to the signal. This simply broadens the distribution, so that the measurement operator must be generalized to a measurement operation

$$\mathcal{O}_X(\infty) = \int dx \sqrt{\frac{\eta}{2\pi(1-\eta)}} \exp\left[-\frac{\eta(x-X)^2}{2(1-\eta)}\right] \mathcal{J}\left[|0\rangle\langle x|\right]. \tag{4.157}$$

It can be shown that a finite integration time yields the same effect as this inefficient measurement operation, with η given by Eq. (4.139).

Heisenberg Picture

Finally, I wish to show that the above results can be quickly obtained using the Heisenberg operators for the output field. Recall that the output quadrature is

$$I_1^{\rm hom}(t) = e^{-t/2}[x(t) + \xi(t)], \tag{4.158}$$

where a decaying local oscillator has again been included. From the expression for $x(t)$ from Eq. (4.141), this is equal to

$$I_1^{\rm hom}(t) = e^{-t/2}\left[x(0)e^{-t/2} - \int_0^t e^{(s-t)/2}\xi(s)ds + \xi(t)\right], \tag{4.159}$$

and so

$$X = \int_0^\infty I_1^{\rm hom}(t)dt = x(0) + \int_0^\infty dt e^{-t/2} dW(t) - \int_0^\infty dt e^{-t} \int_0^t e^{s/2} dW(s) \tag{4.160}$$

Now, it is elementary to show that

$$\int_0^\infty dt e^{-t} \int_0^t e^{s/2} f(s)ds = \int_0^\infty e^{s/2} f(s)ds \int_s^\infty e^{-t}dt = \int_0^\infty e^{-t/2} f(t)dt \tag{4.161}$$

for an arbitrary well-behaved function f. Thus, the final expression for the operator of the integrated photocurrent is

$$X = x, \tag{4.162}$$

in agreement with Eq. (4.147).

4.5.3 Husimi Distribution

Heterodyne detection is different from direct and homodyne detection in that it does not measure an Hermitian operator. That is because it measures both quadratures simultaneously. Thus, the first method of the preceding section is not applicable. The second method is applicable, as the heterodyne SME (4.75) also causes coherent states to decay deterministically. The heterodyne photocurrent (4.83), with decaying local oscillator, is

$$I_c^{\rm het}(t) = e^{-t/2}[\alpha e^{-t/2} + \zeta(t)], \tag{4.163}$$

and the integrated photocurrent is

$$A = \int_0^\infty I_{\rm c}^{\rm het}(t)dt = \alpha + \omega, \tag{4.164}$$

where

$$\omega = \int_0^\infty e^{-t/2}\zeta(t)dt \tag{4.165}$$

is a complex Gaussian random variable with $\mathrm{E}[\omega^*\omega] = 1$.

For a general initial state with Glauber-Sudarshan function $P(\alpha, \alpha^*)$, the probability distribution for A is

$$p(A) = \frac{1}{\pi}\int d^2\alpha \exp\left[-|A-\alpha|^2/2\right] P(\alpha, \alpha^*). \tag{4.166}$$

Again, this will be recognized to be a familiar distribution, the Husimi or Q function [52]

$$p(A) = Q(A, A^*) = \frac{1}{\pi}\langle A|\rho(0)|A\rangle, \tag{4.167}$$

where $a|A\rangle = A|A\rangle$. The Q function is a well-defined quasiprobability function satisfying

$$\int d^2\alpha Q(\alpha, \alpha^*) = 1. \tag{4.168}$$

Another way of expressing the result (4.167) is that the measurement operator for a completed heterodyne measurement is

$$\Omega_A(\infty) = \frac{1}{\sqrt{\pi}}|0\rangle\langle A|. \tag{4.169}$$

The $1/\sqrt{\pi}$ factor is necessary because the coherent states are overcomplete. This is an example of a nonorthogonal measurement, as defined in Sec. 2.1.2. For inefficient measurements, a measurement operation must be used as in Eq. (4.157).

Heisenberg Picture

Once again, the above results are confirmed quickly from the Heisenberg operators for the output field. The non-Hermitian operator for the heterodyne photocurrent (4.88) becomes with local oscillator damping

$$I_1^{\rm het}(t) = e^{-t/2}[a(t) + \zeta(t)] \tag{4.170}$$

where $\zeta = \nu + \mu^\dagger$. Proceeding as in Sec. 4.5.2, and integrating over an infinite interval gives the operator

$$A = a + d^\dagger, \tag{4.171}$$

where

$$d^\dagger = \int_0^\infty e^{-t/2}\mu^\dagger(t)dt, \tag{4.172}$$

where μ is the independent vacuum annihilation operator. It might be thought that the non-Hermiticity of the integrated complex heterodyne photocurrent A would imply an ambiguity in the calculation of its moments. However, this is not the case because

$$[a, a^\dagger] = -[d^\dagger, d] = 1, \tag{4.173}$$

so that A commutes with its Hermitian conjugate, and ordering is therefore unimportant. If a normal ordering with respect to d is chosen, then terms involving d will not contribute. Normal ordering with respect to d is antinormal ordering with respect to a, so that the statistics of A are the antinormally ordered statistics of a. These statistics are precisely those generated by the Q function as defined above [52].

Chapter 5

Interpretation of Quantum Trajectories

The aim of this chapter is to discuss the nature of quantum trajectories, and to illustrate this by various examples. The first section reviews the considerable body of recently published work on quantum trajectories, categorizing them according to the authors' attitude towards the reality of the trajectories. Three categories are recognized: non-real, for which quantum trajectories are treated as numerical tools only; subjectively real, as in this thesis; and objectively real, in which quantum trajectories are treated as a solution to the quantum measurement 'problem'. The second section illustrates quantum trajectories on the Bloch sphere of a two-level atom. The third discusses the approximate quantum jumps of an atom produced by spectral detection in the limit of large Rabi frequency, with reference to one of the objectively real models discussed in the first section. The fourth section treats the quantum trajectories of a freely decaying cavity, and the fifth those of a cavity containing an active medium, acting as an ideal laser.

5.1 Literature Review

5.1.1 Motivations for Quantum Trajectories

As is evident from the preceding chapters, I consider quantum trajectories (for continuously monitored systems) to be a particular branch of quantum measurement theory. The state of the system conditioned on the photocurrent (the continuous measurement readout) will obey a stochastic evolution equation. In general, this will be a stochastic master equation (SME), but for evolution which preserves purity, a stochastic Schrödinger equation (SSE) can also be defined. By this meaning, quantum trajectories are characterized by a number of features, including

1. The nature of the trajectory depends on the detection scheme used to monitor the system output.

2. The conditioned system state is not necessarily pure, because of detector inefficiency or a noisy input.

Both of these features are to be found in the work of Carmichael [21] who introduced the term quantum trajectory in this context. In this section, I will also use quantum trajectory to refer to any stochastic unraveling of a master equation, not necessarily with a measurement theory interpretation. With this looser definition, the two features listed above may not be appropriate.

The present surge of interest in quantum trajectories (as stochastic unravelings of a master equation) began in January 1992 (when, incidentally, this thesis began), with the first publication of the stochastic Schrödinger equation method of treating resonance fluorescence, by Dalibard, Castin and Mølmer [37]. The same conclusions had been drawn by Carmichael [21], from work dating some years earlier [23, 24, 2], but this was not published until 1993. Since January 1992, there has been a large output of work on quantum trajectories [43, 44, 48, 54, 58, 59, 60, 61, 135, 136, 21, 22, 25, 106, 97, 98, 149, 150, 151, 152, 153, 144, 145, 146, 64, 65]. I do not intend to review each of these works (some of which were authored or co-authored by me), but rather to classify them according to the motivation the authors have for using quantum trajectories. These motivations are based on the reality which the authors attribute to their trajectories, and as such reconnects with quantum metaphysics as discussed in the Introduction. I distinguish three interpretations of the reality of quantum trajectories: not real, subjectively real, and objectively real. Interestingly, it is only with quantum trajectories ascribed the intermediate level of reality that the two characteristics listed above are appropriate.

5.1.2 Non-Real Quantum Trajectories

The "non-real" interpretation of quantum trajectories treats the trajectories simply as numerical tools for solving the master equation. This is essentially the attitude taken by Dalibard, Castin and Mølmer [37], although they do mention that their "Monte Carlo Wavefunction" (MCWF) approach also provides "new physical pictures". Note the emphasis on wavefunctions in their terminology. The value of quantum trajectories for pure states as a computational tool is that it only takes $2N-2$ real numbers to store a state vector in an $N-$dimensional Hilbert space, compared to N^2-1 real numbers for a state matrix. Of course, this gain is offset by the fact that many trajectories are required to obtain a reliable ensemble average. However, for a large Hilbert space, quantum trajectories may offer a real advantage for numerical computation. One practical application is computing the centre-of-mass motion of an atom undergoing resonance fluorescence [106, 44, 97, 98]. The trajectories used in these simulations involve quantum jumps, which could be interpreted as photodetections of the spontaneously emitted field. There is one collapse operator for each possible direction of spontaneous emission, so that a jump kicks the momentum of the atom as well as lowering the internal atomic state.

The numerical applications of quantum trajectories are also the main emphasis of a pair of papers by Gardiner, Parkins, and Zoller [54, 43], the latter also authored by Dum. The first treated the general case of simulating the quantum jumps for a squeezed and thermal reservoir. In the case of a vacuum input, the quantum jumps could be interpreted in terms of photodetections, as they point out [54]. The second paper gives specific quantum optical examples, and gives a simulation procedure for calculating spectra and correlation functions. These algorithms are not based on any interpretation of the quantum jumps as photocounts, unlike the autocorrelation functions derived in Ch. 4. Only quantum jumps are considered; there is no obvious computational advantage for diffusive quantum trajectories. Goetsch and Graham [64] have done quantum trajectory simulations of various nonlinear optical systems, again emphasizing the numerical advantages compared to other techniques such as stochastic simulations of the positive P equations. However, they use diffusive as well as point process trajectories, and recognize that these can be interpreted using measurement theory as in Ref. [151]. A later work by Goetsch and Graham [65], in which they give a novel derivation for the diffusive SSEs, is discussed in App. B.

5.1.3 Subjectively Real Quantum Trajectories

In this thesis, quantum trajectories are attributed some degree of reality, in that they represent the evolution of a system conditioned on continuous observation. This can be called "subjective reality", because the nature of the trajectories depend on the measurement scheme used. As noted above, with this interpretation it is sensible, even necessary, to consider different types of measurements, and also to consider trajectories which do not preserve pure states. It is this interpretation, as well as the computational convenience, of quantum trajectories which is the motivation for the work by Carmichael and co-workers [2, 136, 20, 21, 22, 25]. Most of this work was numerical, but analytic results were obtained for some particular systems and particular measurement schemes [20, 22]. Both quantum jump and quantum diffusion (homodyne detection) trajectories were used. Quantum jump (direct detection) simulations of atoms were also used by Gagen and Milburn [48] to illustrate the quantum Zeno effect. In the category of quantum trajectories as applied quantum measurement theory should also go a paper by Gagen, Wiseman, and Milburn [49]. This illustrated the quantum Zeno effect by a highly idealized model for position measurements which involved a diffusive quantum trajectory akin to the homodyne detection model. Both of these papers were numerical rather than analytical. A similar model of ideal position measurements was considered by Belavkin and Staszewski [11].

There are a number of reasons for wishing to consider the measurement interpretation of quantum trajectories. Firstly, the simulated photocurrents can be used to calculate such quantities as the waiting time distribution for photodetections, which are quite difficult to calculate by traditional methods [23]. Another measurable quantity which can be calculated by quantum trajectories is the spectrum, by putting the output light into a filter cavity [136]. The algorithm to do this is essentially the same as that of Ref. [43], but is given a physical interpretation by Tian and Carmichael. A second motivation is that subjectively real quantum trajectories give valuable physical insight into irreversible quantum processes, different from that offered by the master equation. For example, coherences which are lost in the nonselective (master equation) picture are seen to be merely distributed among different quantum trajectories [2, 21, 22]. It is this second motivation which is the subject of much of this chapter. Thirdly, the measured results can be used to select certain individual systems from the ensemble. In this way, it may be possible to see the coherences which are lost in the nonselective ensemble evolution [25, 22]. This could be looked upon as a very basic sort of feedback, in which the measured result is used to eliminate some members of the ensemble. As the reader will have gathered, the use of quantum trajectories in more active feedback schemes is a major topic of this thesis. This application would not be possible unless the quantum trajectories had the subjective reality referred to above.

5.1.4 Objectively Real Quantum Trajectories

The final motivation for using quantum trajectories is a belief in their objective existence, independent of any measurement scheme. This belief is presumably founded on mistrust of the standard interpretation of quantum mechanics, and a desire to replace subjective collapses due to measurements with objectively real collapses. The roots of this approach are the dynamical state reduction models of Gisin [57], Diosi [41], and Pearle [109], which in turn hark back to the original objective interpretation of the wavefunction by Schrödinger [118]. In this desire, proponents of objectively real quantum trajectory theory are akin to hidden variable theorists, but unlike the latter, they accept the wavefunction as a representation of reality. It is only when irreversible processes occur (described by a master equation) that stochasticity enters quantum mechanics. This interpretation is based on the analogy between the state matrix as an ensemble of state vectors and a classical distribution function as an ensemble of points in phase-space, as explained in the introduction (where the flaw in this interpretation was also described). Because the state vector is considered a basic element of reality in this interpretation, it only makes sense to consider quantum trajectories which preserve pure states. Thus it is that stochastic Schrödinger equations (SSEs) are essential only if one regards them as not real or objectively real.

There are two groups which have published on this theme. The first is Gisin and Percival [58, 59, 60], also with other authors [61]. As stated in the preceding chapter, the quantum trajectory which Gisin and Percival (henceforth abbreviated to GP) consider is that which I have called the heterodyne detection SSE. It is worth repeating briefly. If the master equation is assumed for simplicity to be

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \mathcal{D}[c]\rho, \tag{5.1}$$

then the GP SSE is

$$d|\bar{\psi}\rangle = \left\{-\mathrm{i}H dt - \tfrac{1}{2}c^\dagger c dt + \left[\langle\psi|c^\dagger|\psi\rangle dt + dV(t)\right] c\right\} |\psi\rangle, \tag{5.2}$$

followed by normalization. $V(t)$ is a complex Wiener process without interpretation. The master equation is restored for the ensemble average $\rho = \mathrm{E}[|\psi\rangle\langle\psi|]$. As GP point out, this equation can be used to numerically simulate irreversible evolution just as can the quantum jump SSEs. However, they attribute an objective reality to their SSE, saying that it "represents the evolution of an individual quantum system in interaction with its environment". They prove a number of "theorems" from their equation [60]. By adding *ad hoc* irreversible processes to a model of a three level atom, they are able to induce their equation to exhibit jump-like behaviour [61], as seen in electron shelving experiments [33]. In general, however, GP cannot associate their trajectories with the detection of photons for the very good reason that they actually correspond to heterodyne, not direct detection.

The second group is Teich and Mahler (to be referred to as TM) [135]. Their model for unraveling the master equation is completely different from any discussed so far. Briefly, they take an arbitrary master equation and solve it for all time for some initial condition. Next, they find the (in general time-dependent) eigenstates of this density operator at all times. The system state is then assumed to be in one of these eigenstates at any given time, and makes transitions between them at certain rates. These rates are determined by the necessity that the ensemble average evolution should reproduce the master equation. TM also consider their stochastic equations to be objectively real, saying that they "... govern the time evolution of an individual quantum system." They consider that their quantum jumps are "... connected with the spontaneous emission of a photon." If one equated the spontaneous emission of a photon with the detection of a photon, then the TM model should be the same as the direct detection theory developed in Ch. 4. This is manifestly not the case, although this seems to be misunderstood by some authors [59, 61]. If one cannot equate an emitted photon with a detected photon, then the model of TM would seem to be completely disconnected from reality. This is a serious problem for their model because, unlike all other quantum trajectories considered do far (including those of GP), the model of TM does not have the computational advantage of solving a master equation using state vectors. That is because the density operator solution to the master equation must be constructed before any stochastic trajectory can be simulated. These issues will be discussed further in Sec. 5.3.

The basic problem with the objective interpretation of quantum trajectories lies with the origin of irreversible evolution. In quantum mechanics, the irreversibility of the master equation for a system is an approximation to the exact reversible evolution for the universe as a whole. Neither GP nor TM try to define when the master equation evolution is truly irreversible. If there was some physical mechanism by which irreversibility entered the world, for example at some sufficiently large scale, then it is conceivable that either the GP or the TM scheme could be correct. The quantum measurement problem would then be solved, as measurement results would be determined by the stochastic quantum trajectories. As stated in the introduction, there is no evidence for such a violation of unitary quantum mechanics, but that does not rule it out. Irrespective of this issue, both the GP and the TM models cannot be objectively real for the systems to which the authors apply them [1]. Irreversible stochastic evolution does not take place at the level of individual atoms. The entanglement between an atom and its outgoing field is a fact which is not represented by the models of GP and TM. This entanglement is most simply shown by the fact that one can chose different detection schemes to observe the atom, as discussed in Ch. 4. Gisin and Percival's scheme only gives the correct results for heterodyne detection, while Teich and Mahler's scheme will not give the correct results for any detection scheme.

Gisin and Percival are aware of the other types of quantum trajectories based on direct detection. Regarding the quantum jump evolution, they say [58] that it " ... provides a *different* insight [into the behaviour of individual systems], and it remains to be seen which, if any, is preferable." It could be said that the main point of this discussion is that is does not remain to be seen which model is preferable. For the purposes of numerical solution of the master equation, any stochastic unraveling may be used (although that of TM is pointless, as explained above). However, if one wishes to attribute reality to the quantum trajectories, then the subjective reality of quantum measurement theory is the only option. Then the direct detection 'quantum jump' models are precisely as valid as the diffusive homodyne detection trajectories, or the model of GP, corresponding to heterodyne detection. The relevant model depends upon the experimental method by which information is to be extracted from the light leaving the system. That is to say, the state of the quantum system is always conditioned on (and in fact can be identified with) one's knowledge of the system obtained from a measuring apparatus which effectively behaves classically. This lesson is almost as old as quantum mechanics. In the words of Bohr [14], "... these conditions [which define the possible types of predictions regarding the future behaviour of the system] constitute an inherent element of the description of any phenomenon to which the term 'physical reality' can be properly attached." Quantum states, including those produced by quantum trajectories, can only be defined *relative* to the information one has about them.

In spite of the above remarks about the equality of all measurement schemes, it must be admitted that the direct and heterodyne SSEs are in a sense more natural ways to unravel the master equation than the homodyne SSE. The homodyne SSE requires the specification of the phase of the local oscillator and so is not unique. In the case of a driven system (as will be considered in the following section), there are two natural choices (in phase and in quadrature with the driving field) but in general this is not the case. The direct SSE results from measuring the intensity of the outgoing light, while the heterodyne SSE results from measuring its electric field — amplitude and phase. The former emphasizes the quantum nature of the dissipation (jumps due to individual photodetections),

[1] These criticism now appears to be accepted by Percival, as evidenced by Ref. [113]

while the latter presents a more classical (diffusive) behaviour. It might be expected that the heterodyne SSE would be a more general model, perhaps applicable to field measurements where photon detection is impractical. An example of this is with microwave radiation, although there the detection efficiency would be expected to be so low that a SSE would not apply even approximately, and one would have to use a stochastic master equation instead.

5.2 Quantum Trajectories on the Bloch Sphere

5.2.1 The Optical Bloch Equations

In this section, I illustrate the quantum trajectories of Ch. 4 by application to a simple quantum system: a classically driven, damped, detuned two-level atom. For simplicity, only efficient detection will be considered so that SSEs may be used. First, it is helpful to review the usual (nonselective) optical Bloch equations. Denoting the upper and lower atomic states $|2\rangle$ and $|1\rangle$ respectively, I use the following operators:

$$\sigma_x = |2\rangle\langle 1| + |1\rangle\langle 2|, \tag{5.3}$$

$$\sigma_y = -\mathrm{i}|2\rangle\langle 1| + \mathrm{i}|1\rangle\langle 2|, \tag{5.4}$$

$$\sigma_z = |2\rangle\langle 2| - |1\rangle\langle 1|, \tag{5.5}$$

$$\sigma = |1\rangle\langle 2| = \tfrac{1}{2}(\sigma_x - \mathrm{i}\sigma_y). \tag{5.6}$$

The master equation for the atom is

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \gamma\mathcal{D}[\sigma]\rho, \tag{5.7}$$

where γ is the spontaneous emission rate (Einstein A coefficient), and

$$H = \frac{\Omega}{2}\sigma_x + \frac{\Delta}{2}\sigma_z, \tag{5.8}$$

where Ω is the Rabi frequency (proportional to the classical field amplitude by the dipole coupling constant) and Δ is the atomic frequency minus the classical field frequency. Denoting the averages of the operators $\sigma_x, \sigma_y, \sigma_z$ by x, y, z respectively, the density operator for the atom can be simply expressed in terms of the Bloch vector (x, y, z) as

$$\rho(t) = \tfrac{1}{2}[1 + x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z]. \tag{5.9}$$

The master equation (5.7) can then be written in the following succinct form:

$$\dot{x} = -\Delta y - \frac{\gamma}{2}x, \tag{5.10}$$

$$\dot{y} = -\Omega z + \Delta x - \frac{\gamma}{2}y, \tag{5.11}$$

$$\dot{z} = +\Omega y - \gamma(z+1). \tag{5.12}$$

The stationary solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathrm{ss}} = \begin{pmatrix} -4\Delta\Omega \\ 2\Omega\gamma \\ -\gamma^2 - 4\Delta^2 \end{pmatrix} \left(\gamma^2 + 2\Omega^2 + 4\Delta^2\right)^{-1}. \tag{5.13}$$

In anticipation of the following subsections in which the master equation is unraveled as stochastic trajectories for a state vector, note that when ρ is pure, the Bloch vector is confined to the unit sphere $x^2 + y^2 + z^2 = 1$. In this case, it is possible to parameterize the state of the system by two Euler angles on the unit sphere, (θ, ϕ). These angles will obey coupled stochastic differential equations. Nevertheless, the probability distribution of states on the sphere surface $p(\phi, \theta, t)$ will obey a deterministic evolution equation, derived from these stochastic equations. Such an equation is equivalent to the master equation in that

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} p(\phi, \theta, t). \tag{5.14}$$

However, different unravelings of the master equation will give rise to different evolution equations for $p(\phi, \theta, t)$. For the three measurement schemes considered here, these evolution equations are members of the class of two-dimensional differential Chapman-Kolmogorov equations [50].

5.2.2 Direct Photodetection

The state of a classically driven two-level atom conditioned on direct photodetection of its resonance fluorescence has been considered in detail before [23, 37, 44]. The treatment here is thus restricted to formulating the stochastic evolution in the manner of Ch. 4, and giving a closed-form expression for the stationary probability distribution of states on the Bloch sphere, which has not been done before.

Consider a two-level atom situated in an experimental apparatus such that the light it emits is all collected and enters a detector. (In principle this could be achieved by placing the atom at the focus of a large parabolic mirror.) Then the direct detection theory of Sec. 4.1 can be applied, with $c = \sqrt{\gamma}\sigma$. The state vector of the atom conditioned on the photodetector count obeys the following SSE:

$$d|\psi_{\rm c}(t)\rangle = \left[dN_{\rm c}(t)\left(\frac{\sigma}{\sqrt{\langle\sigma^\dagger\sigma\rangle_{\rm c}(t)}} - 1\right) - dt\left(\frac{\gamma}{2}[\sigma^\dagger\sigma - \langle\sigma^\dagger\sigma\rangle_{\rm c}(t)] + {\rm i}H\right)\right]|\psi_{\rm c}(t)\rangle, \tag{5.15}$$

where H is as defined in Eq. (5.8) and the photocount increment $dN_{\rm c}(t)$ satisfies ${\rm E}[dN_{\rm c}(t)] = \gamma\langle\sigma^\dagger\sigma\rangle_{\rm c}(t)dt$. With the conditioned subscript understood, one can write the conditioned state in terms of the Euler angles (ϕ, θ) as defined in the previous subsection. These parameters then obey the following coupled nonlinear stochastic differential equations

$$d\phi(t) = A_\phi[\phi(t), \theta(t)]dt, \tag{5.16}$$

$$d\theta(t) = A_\theta[\phi(t), \theta(t)]dt + \frac{\gamma}{2}\sin\theta(t)dt + [\pi - \theta(t)]dN(t), \tag{5.17}$$

where the Hamiltonian drift terms are defined by

$$A_\phi(\phi, \theta) = -\Omega\cot\theta\cos\phi + \Delta, \tag{5.18}$$

$$A_\theta(\phi, \theta) = -\Omega\sin\phi, \tag{5.19}$$

and

$${\rm E}[dN(t)] = \gamma\cos^2[\theta(t)/2]dt. \tag{5.20}$$

Now to write these SDEs as a Differential Chapman-Kolmogorov equation for the probability distribution $p(\phi, \theta, t)$, the following non-negative function must exist:

$$W(\phi', \theta'|\phi, \theta) = \lim_{\delta t \to 0} p(\phi', \theta', t + \delta t|\phi, \theta, t)/\delta t, \tag{5.21}$$

where $p(\phi', \theta', t+\delta t|\phi, \theta, t)$ has its obvious meaning [50], and (ϕ', θ') is finitely separated from (ϕ, θ). In the above equations (5.16,5.17), the jump process is particularly simple: it always takes the state to the South pole (level $|1\rangle$ of the atom), with probability $\mathrm{E}[dN(t)]$. This implies

$$W(\phi', \theta'|\phi, \theta) = \frac{\delta(\pi - \theta')}{2\pi}\gamma \cos^2(\theta/2), \tag{5.22}$$

where the delta function is defined so as to give unity when integrated on a finite interval closed below at zero. From this, the equation for $p(\phi, \theta, t)$ under direct photodetection is seen to be

$$\begin{aligned} \dot{p}(\phi, \theta, t) \quad &= \quad \left\{ -\frac{\partial}{\partial \phi} A_\phi(\phi, \theta) - \frac{\partial}{\partial \theta}\left[A_\theta(\phi, \theta) + \frac{\gamma}{2}\sin\theta \right] - \gamma\cos^2(\theta/2) \right\} p(\phi, \theta, t) \\ &\quad + \gamma\frac{\delta(\pi - \theta)}{2\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \cos^2(\theta'/2) p(\phi', \theta', t). \end{aligned} \tag{5.23}$$

In practice, it is easier to find the steady-state solution of this equation by returning to the SSE (5.15), and ignoring normalization terms. Consider the evolution of the system following a photodetection at time $t = 0$ so that $|\psi(0)\rangle = |1\rangle$. Assuming that no further photodetections take place, and omitting the normalization terms in Eq. (5.15), the state evolves via

$$\frac{d}{dt}|\tilde{\psi}_{\mathrm{c}}(t)\rangle = -\left(\frac{\gamma}{2}\sigma^\dagger\sigma + \mathrm{i}H\right)|\tilde{\psi}_{\mathrm{c}}(t)\rangle. \tag{5.24}$$

Here, the state vector has a norm appropriate to the probability of it remaining in this uncollapsed state. Writing the unnormalized conditioned state vector as

$$|\tilde{\psi}_{\mathrm{c}}(t)\rangle = \tilde{c}_1(t)|1\rangle + \tilde{c}_2(t)|2\rangle, \tag{5.25}$$

the solution satisfying $\tilde{c}_j(0) = \delta_{j,1}$ is easily found to be

$$\tilde{c}_1(t) \quad = \quad \cos(\alpha t) + \frac{\gamma/2 + \mathrm{i}\Delta}{2\alpha}\sin(\alpha t), \tag{5.26}$$

$$\tilde{c}_2(t) \quad = \quad -\mathrm{i}\frac{\Omega}{2\alpha}\sin(\alpha t), \tag{5.27}$$

where

$$2\alpha = \left[(\Delta - \mathrm{i}\gamma/2)^2 + \Omega^2 \right]^{1/2} \tag{5.28}$$

is a complex number which reduces to the detuned Rabi frequency as $\gamma \to 0$. From these amplitudes, one can define the time-dependent angle variables

$$\phi(t) \quad = \quad \arg[\tilde{c}_1(t)\tilde{c}_2^*(t)], \tag{5.29}$$

$$\theta(t) \quad = \quad 2\arctan[|\tilde{c}_1(t)/\tilde{c}_2(t)|]. \tag{5.30}$$

Note that it has not been necessary to introduce normalization.

Now, denote the probability that there have been no photodetections in the interval $(0, t)$ by $S(t)$. The decrease in this survival probability from t to $t + dt$ is equal to the probability for a photodetection to occur (given that none have occurred so far) times the probability that none have occurred so far:

$$dS(t) = -\mathrm{E}[dN(t)]S(t). \tag{5.31}$$

With the initial condition $S(0) = 1$, the solution is thus

$$S(t) = \exp\left\{-\gamma \int_0^t ds \cos^2[\theta(s)/2]\right\}. \tag{5.32}$$

It can be verified that $S(t)$ is also given by the modulus squared of the unnormalized state:

$$S(t) = \langle\tilde{\psi}_c(t)|\tilde{\psi}_c(t)\rangle = |\tilde{c}_1(t)|^2 + |\tilde{c}_2(t)|^2, \tag{5.33}$$

as expected since the decay in the norm of the conditioned state is due to the discarding of that component which arises from a photodetection having taken place.

Whenever a photodetection does occur, the system returns to its state at $t = 0$. This gives the obvious interpretation for the antibunching predicted [26] and observed [84] in the resonance fluorescence of a two-level atom: following a detection, the atom must be re-excited before it can emit again. This re-excitation is always identical, so the stationary probability distribution on the Bloch sphere is confined to the curve parameterized by $\big(\phi(t), \theta(t)\big)$, weighted by the survival probability $S(t)$. Explicitly,

$$p_{\rm ss}(\phi,\theta) = \left(\int_0^\infty dt S(t)\delta\big(\theta - \theta(t)\big)\delta\big(\phi - \phi(t)\big)\right)\left(\int_0^\infty dt S(t)\right)^{-1}. \tag{5.34}$$

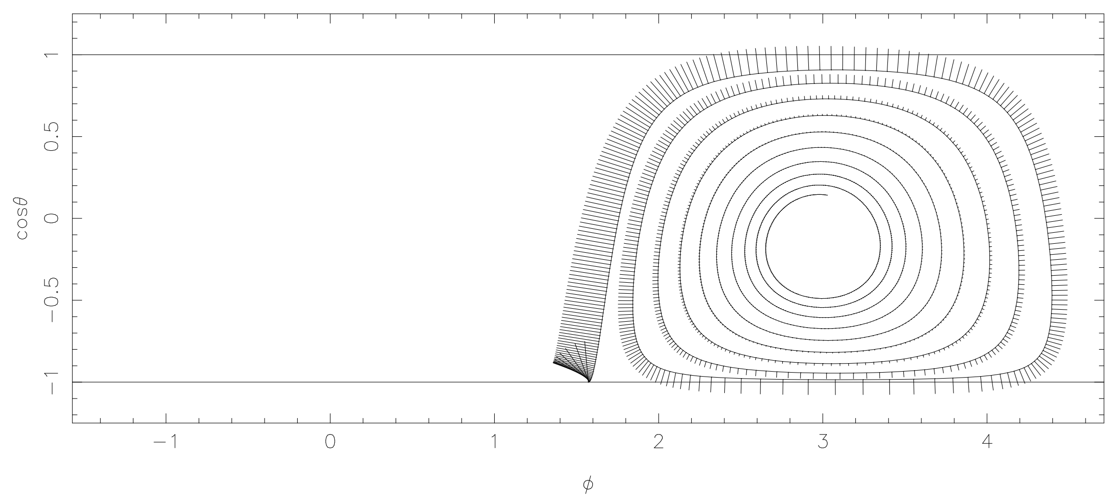

Figure 5.1: Equal-area projection of the Bloch sphere showing the stationary probability distribution of the state vector for a driven, damped, detuned atom whose fluorescence is subject to direct photodetection. For detailed explanation, see text. The parameters here are (in units of the spontaneous emission rate γ), driving $\Omega = 3$, and detuning $\Delta = 0.5$.

This probability distribution on the Bloch sphere is plotted in Fig. 5.1, for $\Omega = 3\gamma$, $\Delta = \gamma/2$. Here, as in subsequent figures, I am using an equal-area projection of the sphere onto the $\cos\theta, \phi$ plane. The solid curve (call it Γ) is a truncated representation of the one-dimensional submanifold to which $p_{\rm ss}(\phi,\theta)$ is confined. The probability density itself is approximated by a discrete distribution: the weight assigned to each small section of Γ is represented by the height of the line segment drawn orthogonal to Γ from the middle of that section. In fact, these line segments are drawn at regular intervals in time [the argument of the integral in (5.34)], so that their heights are simply given by $S(t)$. This figure thus also contains all of the information about the evolution of the atom conditioned on photodetection. For instance, the waiting time distribution between photodetections [23] could be determined from this graph. I have used this distribution only to calculate the average of the steady state Bloch vector. Using a time increment $dt = 10^{-2}\gamma^{-1}$ confirms the analytic result (5.13) to four decimal places; in this case $(x, y, z)_{\rm ss} = (-0.3, 0.3, 0.1)$.

5.2.3 Homodyne Detection

This subsection treats homodyne detection of the light emitted from the atom. Say the local oscillator has phase φ relative to the driving field (which is in phase with σ_x). Then, from Eq. (4.55), the system obeys the following SSE:

$$d|\bar{\psi}_{\rm c}(t)\rangle = \left\{-\left(\frac{\gamma}{2}\sigma^\dagger\sigma + {\rm i}H\right)dt + \left[\gamma dt\langle e^{-{\rm i}\varphi}\sigma + e^{{\rm i}\varphi}\sigma^\dagger\rangle_{\rm c}(t) + \sqrt{\gamma}\,dW(t)\right]e^{-{\rm i}\varphi}\sigma\right\}|\bar{\psi}_{\rm c}(t)\rangle, \quad (5.35)$$

where $dW(t)$ is a real infinitesimal Wiener increment. Transforming to the Euler angles gives the following set of SDEs:

$$\begin{aligned} d\phi(t) &= A_\phi[\phi(t),\theta(t)]dt + \cos\theta(t)\frac{1+\cos\theta(t)}{1-\cos\theta(t)}\sin\tilde{\phi}(t)\cos\tilde{\phi}(t)\gamma dt \\ &\quad - \frac{1+\cos\theta(t)}{\sin\theta(t)}\sin\tilde{\phi}(t)\sqrt{\gamma}dW(t), & (5.36) \\ d\theta(t) &= A_\theta[\phi(t),\theta(t)]dt + \frac{1+\cos\theta(t)}{\sin\theta(t)}\left\{1-\tfrac{1}{2}[1+\cos\theta(t)]\cos\theta(t)\cos^2\tilde{\phi}(t)\right\}\gamma dt \\ &\quad + \cos\tilde{\phi}(t)[1+\cos\theta(t)]\sqrt{\gamma}dW(t), & (5.37) \end{aligned}$$

where $\tilde{\phi}(t) = \phi(t) + \varphi$. In this case the noise terms are diffusive rather than jump processes, so that the probability distribution obeys the Fokker-Planck equation

$$\begin{aligned} \dot{p}(\phi,\theta,t) &= \left\{-\frac{\partial}{\partial\phi}\left[A_\phi(\phi,\theta) + \gamma\cos\theta\frac{1+\cos\theta}{1-\cos\theta}\sin\tilde{\phi}\cos\tilde{\phi}\right]\right. \\ &\quad - \frac{\partial}{\partial\theta}\left[A_\theta(\phi,\theta) + \gamma\frac{1+\cos\theta}{\sin\theta}\left(1 - \frac{1+\cos\theta}{2}\cos\theta\cos^2\tilde{\phi}\right)\right] \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\gamma\left[\frac{1+\cos\theta}{1-\cos\theta}\sin^2\tilde{\phi}\right] + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\gamma\left[(1+\cos\theta)^2\cos^2\tilde{\phi}\right] \\ &\quad \left. + \frac{\partial^2}{\partial\phi\partial\theta}\gamma\left[\frac{(1+\cos\theta)^2}{\sin\theta}\sin\tilde{\phi}\cos\tilde{\phi}\right]\right\}p(\phi,\theta,t), & (5.38) \end{aligned}$$

where $\tilde{\phi} = \phi + \varphi$. The steady-state solution to this equation can be approximated numerically by a long time average of trajectories on the Bloch sphere generated by the SDEs (5.36,5.37).

In Fig. 5.2, I have plotted this approximation to the stationary probability distribution for two values of φ. The average is taken over a time of $5 \times 10^5\gamma^{-1}$, and a total of 20 000 points are plotted (one every $0.025\,\gamma^{-1}$). The two values of φ are 0 and $\pi/2$, corresponding to measuring the quadrature of the spontaneously emitted light in phase and in quadrature with the driving field respectively. In both plots $\sqrt{2}\,\Omega = 7\gamma$ and $\Delta = 0$, so that the true distributions are symmetrical about the line $\phi = \pm\pi/2$. The effect of the measurement is dramatic and readily understandable. In terms of the Euler angles, the homodyne photocurrent from Eq. (4.51) is

$$I_{\rm c}^{\rm hom}(t) = [\gamma\sin\theta(t)\cos\tilde{\phi}(t) + \sqrt{\gamma}\xi(t)]. \quad (5.39)$$

When the local oscillator is in phase ($\varphi = 0$), the deterministic part of the photocurrent is proportional to $x(t)$. Under this measurement, the atom tends towards states with well defined σ_x. The

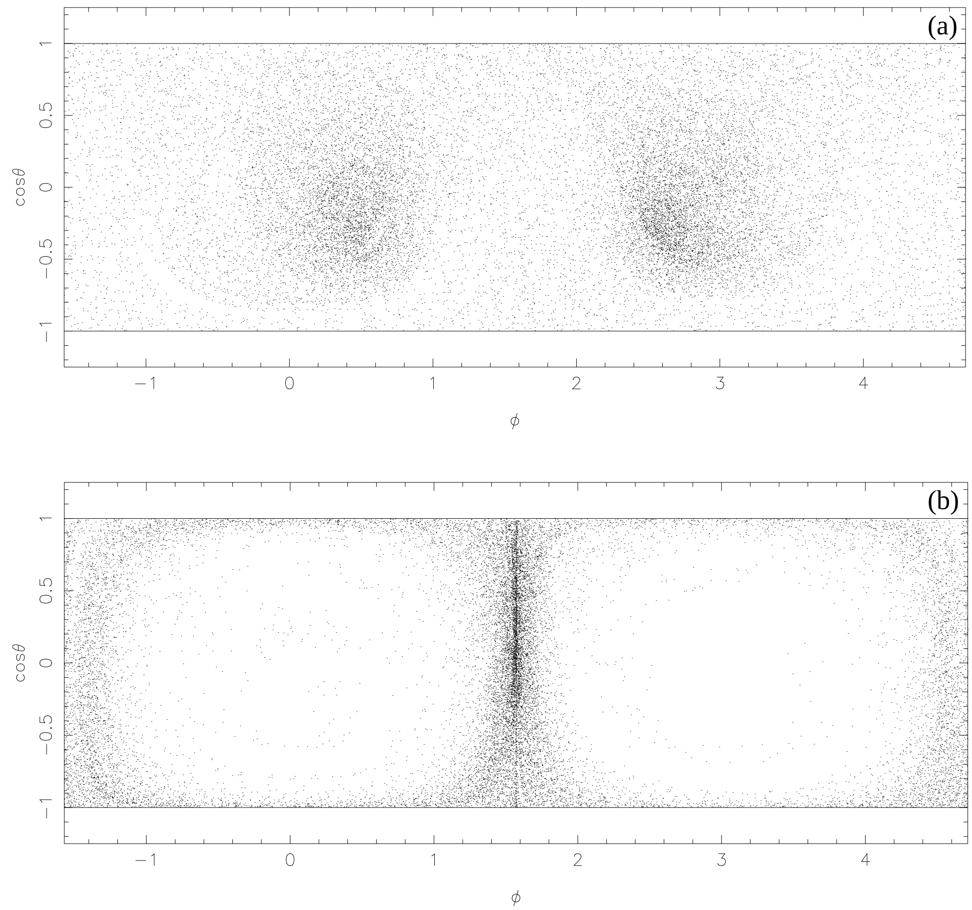

Figure 5.2: Stationary probability distribution of the state vector of an atom whose resonance fluorescence is subject to homodyne detection. The distribution is approximated by an ensemble of 20 000 points on the Bloch sphere. The phase ϕ of the local oscillator relative to the driving field is 0 in (a) and $\pi/2$ in (b). The parameters are $\sqrt{2}\Omega = 7\gamma$, $\Delta = 0$.

eigenstates of σ_x are stationary states of the driving Hamiltonian and so this leads to the probability distribution in Fig. 5.2(a) which has two circum-equatorial peaks near $\phi = 0$ and $\phi = \pi$. The steady-state bloch vector (5.13) points slightly Southwards and in the $\phi = \pi/2$ direction, so the two peaks are actually shifted somewhat in that direction also. In contrast, measuring the $\varphi = \pi/2$ quadrature tries to force the system into an eigenstate of σ_y. However, such an eigenstate will be rapidly spun around the sphere by the driving Hamiltonian. This effect is clearly seen on the steady-state distribution in Fig. 5.2(b) which is spread around the $\phi = \pm\pi/2$ great circle. As before, the $\phi = +\pi/2$ side is weighted somewhat more heavily. Both simulations confirm the analytic result (5.13) for the steady state Bloch vector of $(x, y, z)_{\rm ss} = (0, 7\sqrt{2}, -2)/100$.

The above explanation for the stationary probability distributions are also useful for understanding the noise spectra of the quadrature photocurrents in Eq. (5.39). The spectrum of resonance fluorescence of a strongly driven two-level atom has three peaks, the central one at the atomic frequency, and the two sidebands (of half the area) displaced by the Rabi frequency [105]. It is well known that the spectrum of the in-phase homodyne photocurrent gives the central peak, while the quadrature photocurrent gives the two sidebands [31] . This is readily explained qualitatively from the evolution of the atomic state under homodyne measurements. When σ_x is being measured, it varies slowly, remaining near one eigenvalue on a time scale like γ^{-1}. This gives rise to a simply decaying autocorrelation function for the photocurrent (5.39), or a Lorentzian with width scaling as γ in the frequency domain. When σ_y is measured, it undergoes rapid sinusoidal variation at frequency Ω, with noise added at a rate γ. This explains the side peaks.

5.2.4 Heterodyne Detection

If the atomic fluorescence enters a perfect heterodyne detection device, then from Eq. (4.81), the system evolves via the SSE

$$|\bar{\psi}_{\rm c}(t+dt)\rangle = \left\{1 - \left(\frac{\gamma}{2}\sigma^\dagger\sigma + {\rm i}H\right)dt + \left[\gamma dt\langle\sigma^\dagger\rangle_{\rm c}(t) + \sqrt{\gamma}dV(t)\right]\sigma\right\}|\bar{\psi}_{\rm c}(t)\rangle, \tag{5.40}$$

where here $dV(t)$ is a complex infinitesimal Wiener increment with independent real and imaginary parts. This SSE is equivalent to the following coupled SDEs for the angles on the Bloch sphere

$$\begin{aligned} d\phi(t) &= A_\phi[\phi(t),\theta(t)]dt - \frac{1+\cos\theta(t)}{\sin\theta(t)}\sqrt{\gamma}dW_\phi(t), &(5.41)\\ d\theta(t) &= A_\theta[\phi(t),\theta(t)]dt + \frac{1+\cos\theta(t)}{\sin\theta(t)}\left\{1 - \tfrac{1}{4}[1+\cos\theta(t)]\cos\theta(t)\right\}\gamma dt \\ &\quad + [1+\cos\theta(t)]\sqrt{\gamma}dW_\theta(t), &(5.42)\end{aligned}$$

where $dW_\phi = \Im\left(e^{{\rm i}\phi(t)}dV\right)$ and $dW_\theta = \Re\left(e^{{\rm i}\phi(t)}dV\right)$. The Fokker-Planck equation for $p(\phi,\theta,t)$ is then

$$\begin{aligned} \dot{p}(\phi,\theta,t) &= \left\{-\frac{\partial}{\partial\phi}\left[A_\phi(\phi,\theta)\right] - \frac{\partial}{\partial\theta}\left[A_\theta(\phi,\theta) + \gamma\frac{1+\cos\theta}{\sin\theta}\left(1 - \frac{1+\cos\theta}{4}\cos\theta\right)\right]\right. \\ &\quad \left. + \frac{1}{4}\frac{\partial^2}{\partial\phi^2}\gamma\left[\frac{1+\cos\theta}{1-\cos\theta}\right] + \frac{1}{4}\frac{\partial^2}{\partial\theta^2}\gamma\left[(1+\cos\theta)^2\right]\right\}p(\phi,\theta,t). \qquad (5.43)\end{aligned}$$

As for the homodyne detection case, the steady-state probability distribution can be approximated by a time-ensemble of points on the Bloch sphere found from the above SDEs. The result (using the same parameters as in the previous subsection) is shown in Fig. 5.3.

In this case, the stationary probability distribution is spread fairly well over the entire Bloch sphere. This can be understood as the result of the two competing measurements (σ_x and σ_y) combined with the driving Hamiltonian causing rotation around the x axis. The complex photocurrent

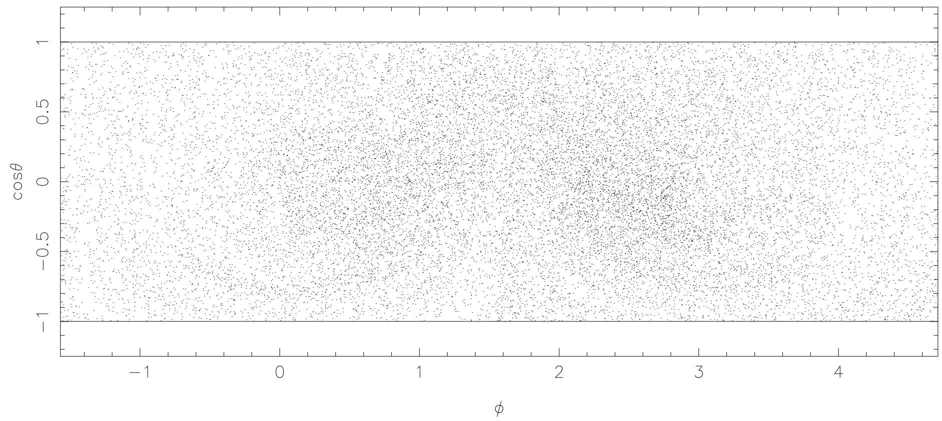

Figure 5.3: Stationary probability distribution of the state vector of an atom whose resonance fluorescence is subject to heterodyne detection. Other details as are in Fig. (5.2)

as defined in Eq. (4.83) is given simply in the Euler angles by

$$I_{\rm c}^{\rm het}(t) = \tfrac{1}{2}\gamma \sin\theta(t) \exp[i\phi(t)] + \sqrt{\gamma}\zeta(t), \tag{5.44}$$

where $\zeta(t) = dV(t)/dt$. The spectrum of this photocurrent [the Fourier transform of the two-time correlation function (4.89)] gives the complete Mollow triplet, as y is rotated at frequency Ω while x simply diffuses.

5.3 Other Quantum Jumps on the Bloch Sphere

In Sec. 5.1.4, it was stated that the stochastic state vector dynamics of Teich and Mahler did not correspond to any of the measurement schemes treated in Ch. 4, and hence that it could have no interpretation. Furthermore, the TM model is not even an algorithm for numerically solving the master equation because the master equation must be solved before the stochastic dynamics can be simulated. Thus it can be concluded that the TM model is useless at best, wrong and misleading at worst. It might be thought that this is a harsh judgment. While the continuous measurement theory of Ch. 4 does cover all cases allowed by exact quantum measurement theory, it is conceivable that there are approximate measurement schemes which the TM model may simulate, at least in some limit. If this were true then the TM simulations would at least provide a physical picture that has an approximate relation to reality. In particular, Teich and Mahler refer to the frequencies of the emitted photons which they associate with the jumps in their scheme. Thus, it might be thought that their scheme would apply to spectral detection, when the various peaks of the spectrum are sufficiently distinct to allow detected photons to be unambiguously assigned to a particular peak. This is the possibility to be investigated in this section.

5.3.1 Spectral Detection

By 'spectral detection', I do not mean a measurement scheme from which a spectrum can be computed (such as heterodyne detection), but rather measurements in which an individual photodetection may be identified with the emission of a photon from a certain part of the source spectrum (within some accuracy). This is more difficult to treat than previous detection schemes because it is necessarily involves the interference of light emerging from the system over some time period (related inversely to the accuracy of the spectral measurement). This can be achieved by using another system to store the emitted light. The most obvious candidate is a a Fabry-Perot etalon. A complete conventional (non quantum trajectory) treatment of such a filter is given in Ref. [35]. For measurements in a localized region of a spectrum, the etalon can be approximated as a single-mode cavity with two output mirrors of loss rate κ. From the theory of Sec. 3.4, it will have a Lorentzian lineshape of width 2κ and central frequency ω_1 . Detections can be made on the light which passes through the etalon, and the light which is reflected. The quantum trajectory for the combined system of source plus etalon will preserve pure states for unit detection efficiency. However, the quantum trajectory for the source alone will not preserves its purity in general, and will not even be Markovian. This is the essential difference from the other measurement schemes considered so far.

The non-Markovicity can be seen most easily in the Heisenberg picture. Recall from Sec. 3.4 that the field operator for the output of the source is

$$b_1 = \nu + c, \tag{5.45}$$

where I am assuming that the input to the source is the vacuum with annihilation operator ν. This will act as the input field operator for the 'near' mirror of the etalon, as in the cascaded systems theory of Sec 3.7. Ignoring the time difference between the two, the QSDE for the annihilation operator a of the etalon mode is, from Eq. (3.146) with no white noise,

$$\dot{a} = -\mathrm{i}\varpi a - \kappa a - \sqrt{\kappa} c - \sqrt{\kappa}\nu - \sqrt{\kappa}\mu, \tag{5.46}$$

where ϖ is the difference between the etalon mode frequency ω_1 and the system dipole frequency ω_0 (optical frequency rotation is being ignored), and μ is the vacuum input from the 'far' mirror. Call the field operator for the transmitted field (leaving the far mirror) f, and that of the reflected field from the near mirror r. Ignoring the vacuum fields (which play no part in direct detection), these fields are related to the source by

$$r(t) - c(t) = \sqrt{\kappa} a(t) = -f(t). \tag{5.47}$$

In the Heisenberg picture, again ignoring vacuum operators, a is given by

$$\sqrt{\kappa} a(t) = \int_0^t \kappa \exp[-(\kappa + \mathrm{i}\varpi)(t-s)] c(s) ds, \tag{5.48}$$

where the etalon is opened at time $t = 0$. In the frequency domain, this is a filtered signal with lineshape described above. In the time-domain (as here), it clearly shows why spectral detection with collapse operators r and f is non-Markovian.

There is one limit in which it is possible to obtain a Markovian (but not autonomous) quantum trajectory from spectral detection. That is the limit $\kappa \gg \gamma$, where γ is the damping rate for the source. In this limit, the irreversible evolution of the system will be negligible over the lifetime of the etalon. That is not to say that the system evolution can be ignored; the system Hamiltonian H may cause evolution at a rate much faster than γ. Nevertheless, the damping evolution can be ignored, and the etalon mode operator is given by

$$\sqrt{\kappa} a(t) = \int_0^t \kappa \exp[-(\kappa + \mathrm{i}\varpi)(t-s)] \hat{c}(s) ds + O(\gamma/\kappa), \tag{5.49}$$

where $\hat{c}(s)$ is the interaction picture operator

$$\hat{c}(s) = \exp[\mathrm{i}Hs]c\exp[-\mathrm{i}Hs]. \tag{5.50}$$

Because H and c are system operators, the etalon field operator is (to an error of order γ/κ) defined in terms of system operators. Thus, the reflected and transmitted fields, which may be subject to detection, may be written down in terms of system operators, via Eq. (5.47). They are related by

$$\hat{c}(t) = \hat{r}(t) + \hat{f}(t). \tag{5.51}$$

Then it is possible to write down the quantum trajectory for the system conditioned on detections of these fields. From the requirement that the nonselective evolution must reproduce the master equation for the interaction picture state matrix

$$d\hat{\rho}(t) = \mathcal{D}[\hat{c}(t)]\hat{\rho}(t)dt, \tag{5.52}$$

one obtains the following stochastic master equation:

$$d\hat{\rho}_{\rm c}(t) = \left(\mathcal{D}[\hat{c}(t)]dt + \sum_{p=f,r} \{dN_{\rm c}^{p}(t) - \mathrm{E}[dN_{\rm c}^{p}(t)]\}\, \mathcal{G}[\hat{p}(t)] \right) \hat{\rho}_{\rm c}(t), \tag{5.53}$$

where

$$\mathrm{E}[dN_{\rm c}^{p}(t)] = \mathrm{Tr}[\hat{p}(t)\hat{\rho}_{\rm c}(t)\hat{p}^{\dagger}(t)]dt. \tag{5.54}$$

Note that the SME (5.53) does not preserve pure states in general, because it cannot be rewritten as a SSE. Also, it is not obvious that it is a valid SME in preserving positivity. That is because it is only approximately valid, in the limit that $\kappa \gg \gamma$. In the following subsection I will consider a special case in which the SME evolution is approximately equivalent to a SSE, and so positivity will obviously be preserved.

5.3.2 The Mollow Spectrum

Consider the simplest system with a non-trivial spectrum: the two-level atom of Sec. 5.1. As noted in that section, its spectrum bears the name of its discoverer, Mollow [105]. In the limit of large driving ($\Omega \gg \gamma, \Delta$), the spectrum has three well-defined peaks. This is the limit which must be considered for spectral detection. For simplicity, I will set $\Delta = 0$. Then the master equation for the interaction picture state matrix is

$$d\hat{\rho}(t) = \gamma\mathcal{D}[\hat{\sigma}(t)]\hat{\rho}(t)dt. \tag{5.55}$$

From the system Hamiltonian

$$H = \frac{\Omega}{2}\sigma_x, \tag{5.56}$$

the interaction picture lowering operator is

$$\hat{\sigma}(t) = \tfrac{1}{2}\left[\sigma_x + |+\rangle\langle-|e^{\mathrm{i}\Omega t} + |-\rangle\langle+|e^{-\mathrm{i}\Omega t}\right]. \tag{5.57}$$

Here $|+\rangle, |-\rangle$ are the eigenstates of σ_x

$$\sigma_x|\pm\rangle = \pm|\pm\rangle. \tag{5.58}$$

Note that $\hat{\sigma}(t)$ consists of three contributions, rotating at frequencies $0, \pm\Omega$. These components correspond to the three peaks of the Mollow spectrum. In the limit that $\Omega \gg \gamma$, it is possible to construct etalons with linewidth satisfying $\kappa \gg \gamma$ (as required), but $\kappa \ll \Omega$. That is to say, an etalon can filter out all but one peak of the spectrum, while being broad enough to let through all the light of the selected peak. Say one etalon is tuned to the atom ($\varpi = 0$), to select the central peak. Then put the reflected beam $\hat{r}(t)$ into another etalon tuned to the Rabi frequency ($\varpi = \Omega$). The filtered light will be of the upper peak, while the reflected light will be of the lower peak. Thus it is possible in principle to separate the fluorescence of the atom into three beams

$$\sqrt{\gamma}\hat{\sigma}(t) = l(t) + m(t) + u(t), \tag{5.59}$$

where

$$\begin{aligned} l(t) &= \frac{\sqrt{\gamma}}{2}|+\rangle\langle-|e^{i\Omega t}, & (5.60)\\ m(t) &= \frac{\sqrt{\gamma}}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right), & (5.61)\\ u(t) &= \frac{\sqrt{\gamma}}{2}|-\rangle\langle+|e^{-i\Omega t}, & (5.62)\end{aligned}$$

with errors of order κ/Ω and γ/κ.

The conditioning master equation from detections from the three peaks, is, to zeroth order in κ/Ω and γ/κ,

$$\begin{aligned} d\hat{\rho}_c(t) &= \sum_{p=l,m,u} \left\{dN_c^p(t)\mathcal{G}[\hat{p}(t)] + \mathrm{E}[dN_c^p(t)]\right\}\hat{\rho}_c(t) - \tfrac{1}{2}\{\gamma\hat{\sigma}^\dagger(t)\hat{\sigma}(t), \hat{\rho}_c(t)\} \\ &\quad + dt\left[m(t)\hat{\rho}_c(t)l^\dagger(t) + m(t)\hat{\rho}_c(t)u^\dagger(t) + l(t)\hat{\rho}_c(t)m^\dagger(t)\right. \\ &\qquad \left. + \; l(t)\hat{\rho}_c(t)u^\dagger(t) + u(t)\hat{\rho}_c(t)m^\dagger(t) + u(t)\hat{\rho}_c(t)l^\dagger(t)\right]. \end{aligned} \tag{5.63}$$

In the limit $\kappa/\Omega, \gamma/\kappa \to 0$, one may ignore the errors, and also make a rotating wave approximation for the frequency Ω. That is to say, if one is only interested in times longer than $1/\Omega$, then one can ignore terms in Eq. (5.63) rotating at frequency Ω (or 2Ω). This includes all of the cross terms like $m(t)\hat{\rho}(t)l^\dagger(t)$. Also under this approximation it is simple to show that

$$\gamma\hat{\sigma}^\dagger(t)\hat{\sigma}(t) = l^\dagger(t)l(t) + m^\dagger(t)m(t) + u^\dagger(t)u(t) = \frac{\gamma}{2}. \tag{5.64}$$

Then the SME becomes

$$d\hat{\rho}_c(t) = \sum_{p=l,m,u} dN_c^p(t)\mathcal{G}[\hat{p}(t)]\hat{\rho}_c(t). \tag{5.65}$$

The nonselective master equation which comes from Eq. (5.65) is

$$d\hat{\rho}_c(t) = \frac{\gamma}{4}\left(\mathcal{J}[\sigma_x] + \mathcal{J}[|+\rangle\langle-|] + \mathcal{J}[|-\rangle\langle+|] - 2\right)\hat{\rho}(t)dt, \tag{5.66}$$

which could have been obtained directly from Eq. (5.7) in the same rotating wave approximation. It is seen to have the stationary solution

$$\hat{\rho}_{ss} = \frac{1}{2}, \tag{5.67}$$

which agrees with Eq. (5.13) in the limit $\gamma/\Omega \to 0$. From this steady state and Eqs. (5.60 – 5.62), it is easy to see that the rate of photodetections in the three peaks are given by

$$\langle m^\dagger m\rangle = 2\langle l^\dagger l\rangle = 2\langle u^\dagger u\rangle = \frac{\gamma}{4}. \tag{5.68}$$

That is to say, the rate of photo emissions into the middle peak is double that in the side peaks, and the total rate is $\gamma/2$. This is what would be predicted from the areas under the Mollow spectrum in this limit [105].

To obtain further information about the nature of this spectral detection, it is useful to turn the SME (5.65) into a SSE. As soon as one side-peak photon has been detected, the conditioned state will be in an eigenstate of σ_x. Thereafter, another emission into a side peak will cause the state to swap to the other σ_x eigenstate, and an emission into the middle peak will have no effect. Because the eigenstates of σ_x are unchanged by transformation to the interaction picture, one can now return to the Schrödinger picture. The conditioned state vector can thus be written

$$|\psi_{\rm c}(t)\rangle = \tfrac{1}{2}\left\{[1+x_{\rm c}(t)]|+\rangle + [1-x_{\rm c}(t)]|-\rangle\right\}, \tag{5.69}$$

where $x_{\rm c}(t) \in \{-1,1\}$ is a random variable equal to the conditioned value of σ_x. The evolution of $x_{\rm c}(t)$ is a random telegraph process [50], obeying

$$dx_{\rm c}(t) = dN_{\rm c}^{l}(t)[1-x_{\rm c}(t)] + dN_{\rm c}^{u}(t)[-1-x_{\rm c}(t)], \tag{5.70}$$

where

$$\begin{aligned} {\rm E}[dN_{\rm c}^{u}(t)] &= \frac{\gamma[1+x_{\rm c}(t)]dt}{8}, & (5.71)\\ {\rm E}[dN_{\rm c}^{l}(t)] &= \frac{\gamma[1-x_{\rm c}(t)]dt}{8}. & (5.72) \end{aligned}$$

The observational evidence that the atom is jumping between σ_x eigenstates as described above would be that between any two consecutive photodetections in one sideband (say l) there must be one (and only one) photodetection in the other sideband (u). Photodetections in the middle peak are uncorrelated to those in the other detectors, and to each other (they obey a Poisson process). This behaviour of the sideband photodetections can be expressed in more conventional language as saying that the fields $l(t), u(t)$ exhibit photon *antibunching* independently, but that they are correlated via photon *bunching* (in the manner of the correlated beams in the original Hanbury-Brown and Twiss experiment [69]). These correlations may be quantified by the normalized correlation functions for detections. In terms of the field operators, these are defined by

$$g^{(2)}_{a,b}(\tau) = \frac{\langle :a^\dagger(t+\tau)a(t+\tau)b^\dagger(t)b(t):\rangle_{\rm ss}}{\langle a^\dagger(t)a(t)\rangle_{\rm ss}\langle b^\dagger(t)b(t)\rangle_{\rm ss}}, \tag{5.73}$$

where $a,b \in \{u,l\}$. In terms of the quantum point processes,

$$g^{(2)}_{a,b}(\tau) = \frac{{\rm E}[dN_{\rm c}^{a}(t+\tau)dN_{\rm c}^{b}(t)]_{\rm ss}}{{\rm E}[dN_{\rm c}^{a}(t)]_{\rm ss}{\rm E}[dN_{\rm c}^{b}(t)]_{\rm ss}}, \tag{5.74}$$

where the delta function at $\tau=0$ for $a=b$ is ignored. From Eqs. (5.70–5.72), it is easy to see that

$$\begin{aligned} g^{(2)}_{l,u}(\tau) = g^{(2)}_{u,l}(\tau) &= 1+e^{-\gamma\tau/2}, & (5.75)\\ g^{(2)}_{l,l}(\tau) = g^{(2)}_{u,u}(\tau) &= 1-e^{-\gamma\tau/2}, & (5.76) \end{aligned}$$

which expresses the bunching and antibunching referred to above. These expressions were first derived by Cohen-Tannoudji and Reynaud [30], and observed by Aspect *et al* [3]. They were derived using the dressed states for atom and quantized driving field, which is the topic of the next subsection.

5.3.3 The Dressed Atom Model

The nature of spectral detection of a strongly driven two-level atom can be understood qualitatively by considering the dressed atom model [30]. If the Rabi frequency Ω is much larger than the spontaneous emission rate then the dynamics of the atom is dominated by the Hamiltonian $H = \frac{\Omega}{2}\sigma_x$ (again ignoring detuning for simplicity). Thus it is natural to work in the eigenstates of this Hamiltonian, which are simply the σ_x eigenstates as introduced above. These stationary states of the free Hamiltonian have energy separation $\hbar\Omega$.

Writing the Hamiltonian of the atom as $\frac{\Omega}{2}\sigma_x$ is based on the assumption that the driving field can be treated classically. If this assumption were not made, then the full atom-field Hamiltonian would be

$$H = \hbar\omega_0 \left(a^\dagger a + \tfrac{1}{2}\sigma_z\right) + \frac{g}{2}(a^\dagger \sigma + a\sigma^\dagger), \tag{5.77}$$

where a is the annihilation operator for the driving field and g is the dipole coupling constant also known as the one-photon Rabi frequency. This has eigenstates

$$|n, \pm\rangle = (|n\rangle|1\rangle \pm |n-1\rangle|2\rangle)/\sqrt{2}, \tag{5.78}$$

where $|n\rangle$ are number states of the driving field. These are known as dressed states of the atom. They have energies

$$E_{n,\pm} = \hbar(n\omega_0 \pm \sqrt{n}\, g/2). \tag{5.79}$$

For large coherent driving field and small coupling constant g the classical approximation is valid and one can replace $\sqrt{n}g$ by $\sqrt{\bar{n}}g = \Omega$. Then, for $n \sim \bar{n}$, the ladder of energy eigenstates (5.79) will consist of pairs of closely-spaced rungs, with an inter-pair separation of $\hbar\omega_0$, and an intra-pair separation of $\hbar\Omega$.

Now one can interpret the Mollow triplet in terms of spontaneous-emission-induced transitions between these stationary states. If the dressed atom is in one of the states $|n, \pm\rangle$, it can spontaneously emit a photon and drop down a rung on the ladder. If it drops to $|n-1, \pm\rangle$ (that is, the atom effectively remains in the same state, $|\pm\rangle$), then the change in energy of the cavity is $\hbar\omega_0$ and so the frequency of the emitted photon must be ω_0 — in the central peak of the triplet. If the atom changes state via a transition to $|n-1, \mp\rangle$, then the frequency of the emitted photon must be $\omega_0 \pm \Omega$ — in the sideband peaks. This leads to precisely the same predictions regarding spectral detection as from the quantum trajectory treatment of the preceding subsection. Thus the dressed atom model does provide a useful physical picture, which corresponds to experiments (spectral detection) in the limit of strong driving.

5.3.4 The Teich and Mahler Model

Finally, I return to the TM model to see whether it has any correspondence to reality as revealed by the quantum trajectories of spectral detection. Recall that Teich and Mahler treat photo-emissions as objectively real events, and refer to the frequencies of the emitted photons. Further, they refer to their work as "a generalization of the dressed-state approach to open quantum systems". In their treatment of the two-level atom in the strong driving limit, they associate their four possible jumps (between the two diagonal states) with the emission of photons of frequencies corresponding to the peaks of the Mollow triplet. Two 'jumps' (from one diagonal state back to the same one) produce elastically scattered photons of frequency ω_0, while the other two cause the atom to change its state and produce inelastically scattered photons of frequency $\omega_0 \pm \Omega$. The rates for each of these jumps is equal (as assumed in the dressed atom model), and the sum is $\gamma/2$. The ordering of emissions is also as predicted by the dressed atom model.

So far, this description of the TM model seems identical to the dressed state model, or to the spectral detection quantum trajectory model. It might seem that one has found at least one application for their model. However, there is one crucial difference. The diagonal states of TM, between which the atom is jumping, are the eigenstates of the steady-state density operator. In the limit of infinite driving, this is given by Eq. (5.67), for which all states are eigenstates. The correct way to find the diagonal states is to determine them for finite Ω/γ, and then take the limit to infinity. From the expression (5.13), the diagonal states with $\Delta = 0$ are obviously on the great circle $x = 0$. In the limit $\Omega \gg \gamma$, they become (in the notation of Teich and Mahler)

$$|\tilde{1}\rangle = |\oplus\rangle \ ; \ \ |\tilde{2}\rangle = |\ominus\rangle, \tag{5.80}$$

where these are eigenstates of σ_y

$$\sigma_y|\oplus\rangle = |\oplus\rangle \ ; \ \ \sigma_y|\ominus\rangle = -|\ominus\rangle. \tag{5.81}$$

These states are as different from the dressed states as it is possible to be. Teich and Mahler's model predict completely the wrong states for the state of the atom conditioned on spectral photodetection. That their model predicts the correct statistics (in this case) is best viewed as a coincidence which one has no reason to expect to be repeated. In this sense, the TM model is misleading because it gives apparently reasonable results which in fact have no relation to reality.

As well as having no relation to experiment, the TM scheme seems to be internally inconsistent. As described above, in their model a system at steady state will always be in one of the diagonal states of the steady-state density operator, and will jump between these states. TM call this incoherent evolution. Before it reaches steady state, the density operator will relax as usual (according to Teich and Mahler) so that its diagonal states will change in time. They call this coherent evolution, and is accompanied by incoherent jumping between the evolving diagonal states. Now presumably an observer should be able to know which state the system is in, otherwise the meaning of the state of the system is highly questionable. This state (call it $|\mu\rangle$) is thus objective knowledge. Say the first observer has been watching the system for some time and can therefore assume that it has reached stationarity. Let the first observer tell a second observer the state of the system ($|\mu\rangle$, which will be one of the stationary diagonal states), but not that the system is at steady-state. The second observer will then treat $|\mu\rangle$ as the initial state of the system which (according to this second observer) will then relax to steady state by smooth evolution of the diagonal states accompanied by jumps between these changing states. It is easy to verify for a simple system (such as the two-level atom) that such diagonal states *do* change during this relaxation process, so that one observer's diagonal states are different from another's. This undermines Teich and Mahler's claim that their model represents the stochastic dynamics of individual quantum systems. The quantum trajectory model for reality has none of these problems.

5.4 Quantum Trajectories for a Free Cavity

In Sec. 4.5, the relationship between projective measurements of photon number and field quadrature, and direct and homodyne detection respectively, was examined. For a freely decaying cavity and an infinite measurement time, the integrated photocurrent had the same statistics as a projective measurement of the relevant observable. Here the resemblance to projective measurements ends.

The conditioned state of the cavity at the end of the measurement interval is always the vacuum state, not the measured eigenstate. However, one would expect that direct or homodyne detection should, on some intermediate time scale, produce a conditioned state which is closer to an eigenstate than the unconditioned state. In this section, I show how this occurs.

5.4.1 Direct Detection

The master equation for a freely decaying cavity is of course

$$\dot{\rho} = \mathcal{D}[a]\rho, \tag{5.82}$$

where time is being measured in units of the cavity lifetime. The quantum trajectory appropriate for direct detection of efficiency η is

$$d\rho_{\rm c}(t) = \left(dN_{\rm c}(t)\mathcal{G}[a] - \tfrac{1}{2}\eta\mathcal{H}[a^\dagger a]dt + (1-\eta)\mathcal{D}[a]dt\right)\rho_{\rm c}(t), \tag{5.83}$$

where $\mathrm{E}[dN_{\rm c}(t)] = \eta\mathrm{Tr}[a^\dagger a\rho_{\rm c}(t)]$. First, consider the evolution of pseudoclassical states. These are states which are a mixture of coherent states and can thus be represented by a positive Glauber-Sudarshan P function [63, 130]. For the case $\eta = 1$, the conditioned P function obeys

$$dP_{\rm c}(\alpha,\alpha^*;t) = \left(dN_{\rm c}(t)\frac{|\alpha|^2 - \langle|\alpha|^2\rangle_{\rm c}(t)}{\langle|\alpha^2|\rangle_{\rm c}(t)} - \left[|\alpha|^2 - \langle|\alpha|^2\rangle_{\rm c}(t) - \tfrac{1}{2}\partial_\alpha\alpha - \tfrac{1}{2}\partial_{\alpha^*}\alpha^*\right]dt\right)P_{\rm c}(\alpha,\alpha^*;t), \tag{5.84}$$

where

$$\mathrm{E}[dN_{\rm c}(t)] = \langle|\alpha|^2\rangle_{\rm c}(t) = \int d^2\alpha|\alpha|^2P_{\rm c}(\alpha,\alpha^*;t). \tag{5.85}$$

From Eq. (5.84), it is evident that the positivity of the conditioned P function will be preserved for all time. If a detection occurs, $P_{\rm c}$ is multiplied everywhere by a non-negative function, If no detection occurs, the change in $P_{\rm c}$ is a first order derivative (which causes drift only), and a term which is zero when $P_{\rm c}$ is zero. Thus, a pseudoclassical initial state will not evolve into a nonclassical state. It is simple to show that this holds also if internal dynamics which do not generate nonclassical states are included. This is the first important point regarding optical quantum trajectories: they cannot produce nonclassical conditioned states.

Now consider the conditioned moments of photon number (as is appropriate for direct detection). First, the mean $\bar{n} = \langle a^\dagger a\rangle$ obeys

$$d\bar{n}_{\rm c}(t) = dN_{\rm c}(t)Q_{\rm c}(t) - \bar{n}_{\rm c}(t)[\eta Q_{\rm c}(t) + 1]dt. \tag{5.86}$$

Here $Q_{\rm c}(t)$ is the Mandel Q parameter [95], equal to the normally ordered photon number variance divided by the mean:

$$Q \equiv \frac{\langle a^{\dagger 2}a^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle}. \tag{5.87}$$

This is a measure of nonclassicality in photon number statistics. The value $Q = 0$ represents a state with Poissonian statistics. A value between 0 and -1 (the minimum) cannot be produced by a pseudo-classical state. The ensemble average evolution of $\bar{n}_{\rm c}(t)$ from Eq. (5.86) is

$$\frac{d}{dt}\bar{n} = -\bar{n}, \tag{5.88}$$

as would be obtained from the master equation (5.82).

It is evident from Eq. (5.86) that the behaviour of the conditioned mean differs markedly depending on the sign of $Q_{\rm c}(t)$. For $Q_{\rm c}(t)$ positive, the conditioned mean increases whenever a photon is detected. This is what would be expected from classical intuition. The rate of photodetections measures the intensity so that the observation of a detection would cause one to increase one's estimate for the mean intensity of the field. Conversely, if there are no detections the conditioned mean decreases. Pseudoclassically, if the field is completely known then it would be in a coherent state with $Q_{\rm c} = 0$. Then, a photodetection cannot provide any further information and hence the mean remains unchanged. If $Q_{\rm c}(t)$ is negative, then the mean photon number decreases when a detection is made. Here, quantum intuition is appropriate. The detection of a photon means that energy has escaped the cavity and hence the conditioned mean decreases. For a number state $Q_{\rm c} = -1$ and perfect efficiency $\eta = 1$, the intracavity photon number drops by one whenever a photon is detected externally, and otherwise is unchanged. This is as expected from the particle picture of photons.

Next, it is useful to consider the conditioned variance in the photon number. It is simplest to consider the normally ordered variance. To avoid introducing extra symbols, I will represent it by the product

$$\bar{n}Q = \langle a^{\dagger 2}a^2\rangle - \bar{n}^2. \tag{5.89}$$

From Eq. (5.83), this obeys

$$\begin{aligned} d[\bar{n}_{\rm c}Q_{\rm c}] &= dN_{\rm c}(t)\left[\frac{\langle a^{\dagger 3}a^3\rangle_{\rm c}}{\bar{n}_{\rm c}} - \langle a^{\dagger 2}a^2\rangle_{\rm c}\right] - dt\left[\eta\langle a^{\dagger 3}a^3\rangle_{\rm c} + 2\langle a^{\dagger 2}a^2\rangle_{\rm c} - \eta\langle a^{\dagger 2}a^2\rangle_{\rm c}\bar{n}_{\rm c}\right] \\ &\quad - \left[2\bar{n}_{\rm c}dN_{\rm c}(t)Q_{\rm c} - 2\bar{n}_{\rm c}(\eta Q + 1)dt + dN_{\rm c}(t)Q_{\rm c}^2\right]. \end{aligned} \tag{5.90}$$

Taking the ensemble average gives

$$\frac{d}{dt}{\rm E}[\bar{n}_{\rm c}Q_{\rm c}] = -{\rm E}[\bar{n}_{\rm c}Q_{\rm c}] - \eta{\rm E}[\bar{n}_{\rm c}Q_{\rm c}^2]. \tag{5.91}$$

Note that this equation cannot be obtained directly from the master equation (5.82). That is because, unlike Eq. (5.88), the quantity $\bar{n}_{\rm c}Q_{\rm c}$ contains products of conditioned means. The ensemble average of the product does not equal the product of the ensemble averages. The presence of the efficiency η in Eq. (5.91) specifically shows the effect due to the measurement. The measurement term is always negative, which shows that the conditioned variance will always be smaller than (or equal to if $Q_{\rm c} = 0$) the unconditioned variance, which would obey Eq. (5.91) with $\eta = 0$. However, the measurement term cannot cause $Q_{\rm c}$ to change sign from positive to negative. This is as expected because the measurement cannot produce a nonclassical conditioned state. Thus, although direct detection does not project the field into a photon number eigenstate, it will in general produce a conditioned state with a reduced photon number variance.

5.4.2 Homodyne Detection

Now consider homodyne measurements of the x quadrature of the decaying field. The master equation is the same as for direct detection (5.82), but now the conditioning equation is

$$d\rho_{\rm c}(t) = dt\mathcal{D}[a]\rho_{\rm c}(t) + \sqrt{\eta}dW(t)\mathcal{H}[a]\rho_{\rm c}(t), \tag{5.92}$$

where $W(t)$ is a real Wiener process. The Glauber-Sudarshan P function obeys

$$P_{\rm c}(\alpha,\alpha^*;t+dt) = P_{\rm c}(\alpha e^{dt/2},\alpha^* e^{dt/2};t)\{1+\sqrt{\eta}dW(t)[(\alpha+\alpha^*) - \langle\alpha+\alpha^*\rangle_{\rm c}(t)] + dt\}. \tag{5.93}$$

As with direct detection, if $P(\alpha, \alpha^*)$ is initially positive and nonsingular, then it remains so. That is, homodyne measurement cannot produce a nonclassical state from a classical one. In particular, it cannot produce a quadrature squeezed state (variance of x less than 1), and so the projection postulate (which would force the system into an x eigenstate) cannot hold in any limit.

Now consider the moments. From Eq. (5.92), it is easy to show that

$$\bar{x}_{\rm c}(t+dt) = e^{-dt/2}\bar{x}_{\rm c}(t) + \sqrt{\eta}dW(t)U_{\rm c}(t), \tag{5.94}$$

where I am denoting the normally ordered variance of x

$$U_{\rm c}(t) = \langle : x^2 : \rangle_{\rm c}(t) - \langle x \rangle_{\rm c}^2(t) = V_{\rm c}(t) - 1, \tag{5.95}$$

where V is the true variance in x. Like Q for direct detection, a negative U indicates nonclassicality, and the minimum is -1. Taking the ensemble average of (5.94) gives

$$d\bar{x} = -\tfrac{1}{2}\bar{x}dt, \tag{5.96}$$

as required by the master equation. Once again, the behaviour of the conditioned mean of x depends on the conditioned variance. If the conditioned variance is greater than one, then an upward fluctuation in the photocurrent $[dW(t) > 0]$ causes the estimated mean of x to be revised upward in accord with classical intuition. On the other hand, the conditioned mean of a squeezed state $(U_{\rm c} < 0)$ will move in the opposite direction to the measured photocurrent. This behaviour can be used to explain how squeezing is measured by a sub-shot noise homodyne photocurrent. A positive fluctuation in the photocurrent will cause the mean x to decrease, which lowers the photocurrent for later times. This anticorrelation between the shot noise and the later deterministic part of the photocurrent is what causes the overall noise to be lowered.

As with direct detection, it is also useful to consider the equation of motion for nonlinear conditioned moments. First, from Eq. (5.94), one finds

$$d[\bar{x}_{\rm c}^2] = -\bar{x}_{\rm c}^2 dt + \eta U_{\rm c}^2 dt + \sqrt{\eta}dW(t)(\cdots). \tag{5.97}$$

Here, the stochastic term has been left unspecified because it disappears in the ensemble average

$$d{\rm E}[\bar{x}_{\rm c}^2] = -\left({\rm E}[\bar{x}_{\rm c}^2] - \eta {\rm E}[U_{\rm c}^2]\right) dt. \tag{5.98}$$

From this equation and Eq. (5.92), one then finds

$$d{\rm E}[U_{\rm c}(t)] = -\left\{{\rm E}[U_{\rm c}(t)] + \eta {\rm E}[U_{\rm c}^2(t)]\right\} dt. \tag{5.99}$$

Some general features of (5.98) and (5.99) are worth noting. If the detector efficiency η goes to zero, then both equations describe the exponential decay predicted by the standard master equation (5.82). The effect of homodyne measurements on the ensemble mean square of the conditioned average of x is to cause it to increase, or slow its rate of decrease. This is expected, as measuring the x quadrature should to some extent force the system into a state with a well defined x even if initially $\bar{x}_{\rm c} = 0$. The effect on the normally ordered variance in x is to decrease it more rapidly (if initially positive) or to make it increase more slowly (if initially negative). Again, this is as expected; measuring x causes the variance in x to be smaller than otherwise.

5.4.3 Approximate Analytical Solution

At the start of the homodyne measurement, all elements in the ensemble are identical, so $\mathrm{E}[U_{\rm c}^2(0)] = \mathrm{E}[U_{\rm c}(0)]^2$. The factorization approximation (FA) that

$$\mathrm{E}[U_{\rm c}^2(t)] \simeq \mathrm{E}[U_{\rm c}(t)]^2 \tag{5.100}$$

will be valid for short times providing that $U_{\rm c}(0)$ is finite. Note however that for an initially coherent state [which has $U_{\rm c}(0) = 0$] the FA will be valid for all times. This is because it will remain a coherent state under homodyne detection [so that $U_{\rm c}(t) = 0$], which is obvious from the evolution equation for the P function (5.93).

Using this approximation, one can construct a differential equation from (5.99) for $u \equiv \mathrm{E}[U_{\rm c}]$:

$$\dot{u} = -(u + \eta u^2). \tag{5.101}$$

This has the solution

$$u(t) = \frac{u_0 e^{-t}}{1 + \eta u_0(1 - e^{-t})}. \tag{5.102}$$

Clearly this function is monotonic in time, asymptotically approaching zero. If $u_0 \gg 1$, this initially large variance in x is reduced to $(\eta)^{-1}$, which is of order unity, in one half life of the cavity ($t = \ln 2$). In the case when $\eta = 1$ (perfect detection), (5.102) can be more simply expressed in terms of $v = u+1$:

$$v(t) = v_0[1 + (v_0 - 1)(1 - e^{-t})]^{-1}. \tag{5.103}$$

In Fig. 5.4 I plot $\sqrt{v(t)}$ determined from an ensemble of 1000 trajectories initially in the Fock state $|8\rangle$. The numerical results agree well with the analytic solution for short times, as expected. However, for longer times, the ensemble mean variance falls below the classical limit of 1, a feature which the analytic solution fails to pick up. This indicates the failure of the FA for such times. Thus, although classical states cannot be squeezed by homodyne detection, conditional quadrature-squeezing can be typically produced from certain nonclassical (but not quadrature-squeezed) initial states.

Now, using the FA and substituting (5.102) into (5.98) yields

$$\dot{s} = -s + u_0 \frac{\eta u_0 e^{-2t}}{[1 + \eta u_0(1 - e^{-t})]^2}, \tag{5.104}$$

where I have defined $s \equiv \mathrm{E}[\bar{x}_{\rm c}^2]$. The solution is

$$s(t) = s_0 e^{-t} + u_0 e^{-t} \left(1 - \frac{1}{1 + \eta u_0(1 - e^{-t})}\right). \tag{5.105}$$

For short times, this solution exhibits linear growth (or inhibited decay if $s_0 > \eta u_0^2$)

$$s(t) \simeq s_0 + t(-s_0 + \eta u_0^2), \tag{5.106}$$

while for long times it decays exponentially:

$$s(t) \simeq \left(s_0 + \frac{\eta u_0^2}{1 + \eta u_0}\right) e^{-t}. \tag{5.107}$$

Numerical results for $\sqrt{s(t)}$, determined as for Fig. 5.4, are plotted in Fig. 5.5. They show good agreement with (5.105) in the short time regime (5.106), and qualtitative agreement into the exponential decay regime. The general features of the quantum effects of homodyne measurements thus seem well understood analytically.

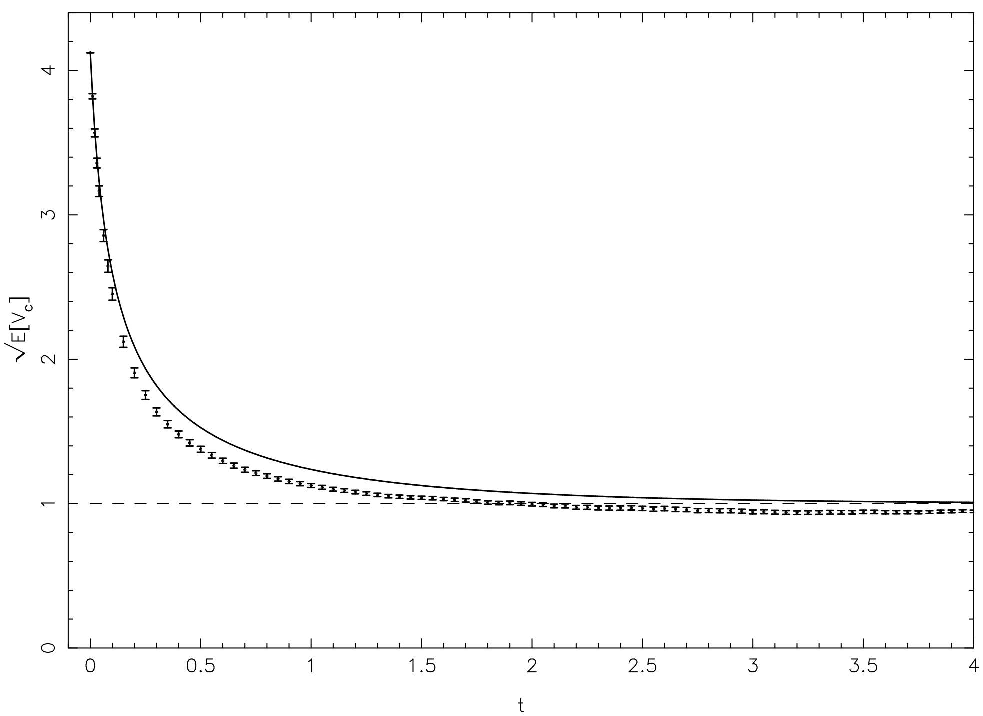

Figure 5.4: Plot of the square root of the ensemble average (over 1000 quantum trajectories) of the conditional variance in x in a freely decaying cavity under homodyne measurement for an initial Fock state $n = 8$. The solid line shows an approximate analytic solution, valid for short times. The dashed line indicates the classical limit. Error bars represent a 95% confidence interval.

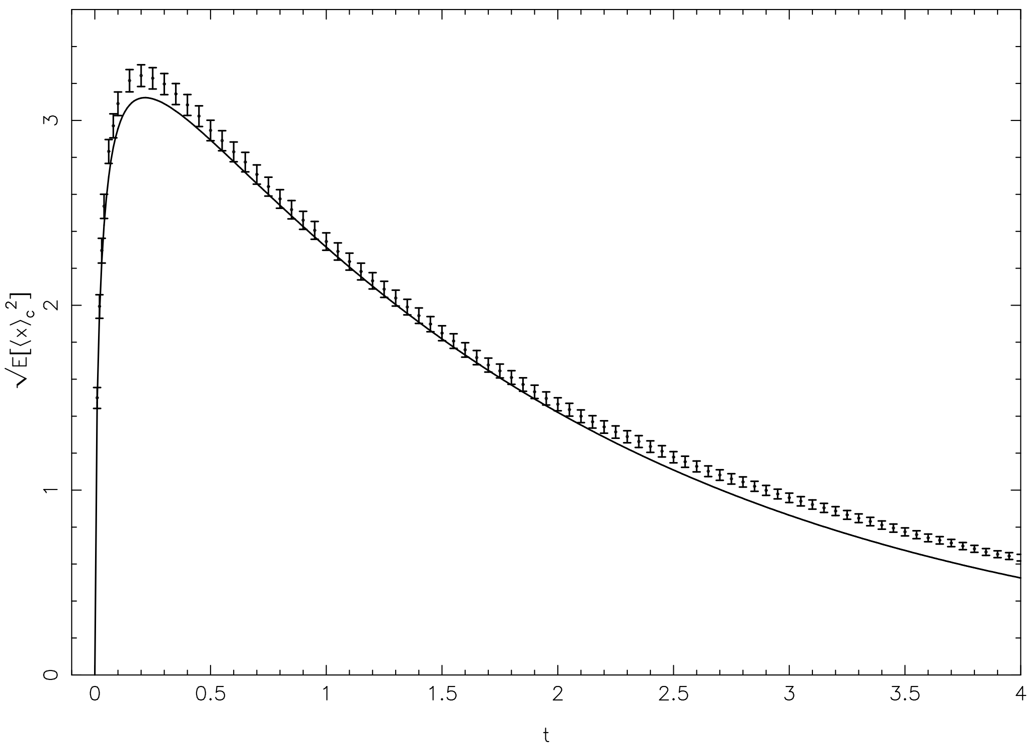

Figure 5.5: Plot of the square root of the ensemble average of the square of the conditional mean of x in a freely decaying cavity under homodyne measurement. Other details are as in Fig. (5.4)

Between the two time regimes (5.106) and (5.107), $s(t)$ reaches a maximum at some time t_M, providing $\eta u_0^2 > s_0$. From Eq. (5.105), this time is given by

$$\exp(-t_M) = 1 + \frac{1}{\eta u_0}\left[1 - \left(\frac{u_0(1+\eta u_0)}{s_0 + u_0}\right)^{1/2}\right]. \tag{5.108}$$

Substituting into (5.105) gives the maximum value of s to be

$$s_M = \left[1 + \eta u_0 - \left(\frac{u_0(1+\eta u_0)}{s_0+u_0}\right)^{1/2}\right]\frac{1}{\eta}\left[\frac{s_0}{u_0} + 1 - \left(\frac{s_0+u_0}{u_0(1+\eta u_0)}\right)^{1/2}\right]. \tag{5.109}$$

To elucidate these formulae, consider the special case $s_0 = 0$; $\eta u_0 \gg 1$. That is, the initial distribution for x has a zero mean and a large variance. This is the situation in which the effect of measuring x will be most dramatic. Examples include a large n Fock state, or a phase-diffused state such as produced by a laser. The result is

$$t_M \simeq (\eta u_0)^{-1/2}, \tag{5.110}$$

and

$$s_M \simeq u_0 - 2(u_0/\eta)^{1/2}. \tag{5.111}$$

From (5.102), the expected normally ordered variance at this time is

$$u(t_M) \simeq (u_0/\eta)^{1/2}. \tag{5.112}$$

Thus, the initial state in which x is poorly defined is rapidly collapsed into a state with a large mean x (of the order of the initial width of the distribution), and a much smaller uncertainty. The formula for t_M appears to be in good agreement with the numerical results plotted in Fig. (5.5). However, the analytical results marginally underestimate the localization caused by the measurement, with the true s_M greater than (5.109), and the true $u(t_M)$ less than (5.112). Nevertheles, the analytical results do agree qualitatively with these numerical results for a large n Fock state, and they are not unreasonable in general, illustrating the typical behaviour expected.

In Fig. 5.6, the behaviour described is illustrated by a particular quantum trajectory. The conditioned state of the cavity mode is represented by the Q function at various times. The initial condition is the Fock state $n = 8$. The system at $t = 1.0$ is most squeezed, with an x variance of 0.63 and a conditional mean of 3.0. The wealth of fine structure at later times can be interpreted as 'quantum phase space interference'. It is not unexpected because all of the states are pure, and the initial state is highly nonclassical. The final state at $t = \infty$ (not shown) would of course be the vacuum.

5.5 Quantum Trajectories for an Ideal Laser

The emergence of quantum optics as an important field of physics was precipitated by two important technological developments: the laser and the efficient photodetector [85]. Laser light is unlike other optical light sources in that it has a large coherent amplitude. To treat this quantum mechanically, it was found extremely useful to use coherent state representations, such as the Glauber-Sudarshan P function [63, 130]. Also, for dealing with open quantum systems, it was necessary to develop new techniques such as the master equation [92]. Both of these methods are related to photodetection. Coherent states are precisely those states which are not affected by conventional detection techniques. The relationship between the openness of quantum systems and their continuous monitoring is still being explored, as in this thesis. In this section, I revisit these early themes in quantum optics using the techniques of quantum trajectories.

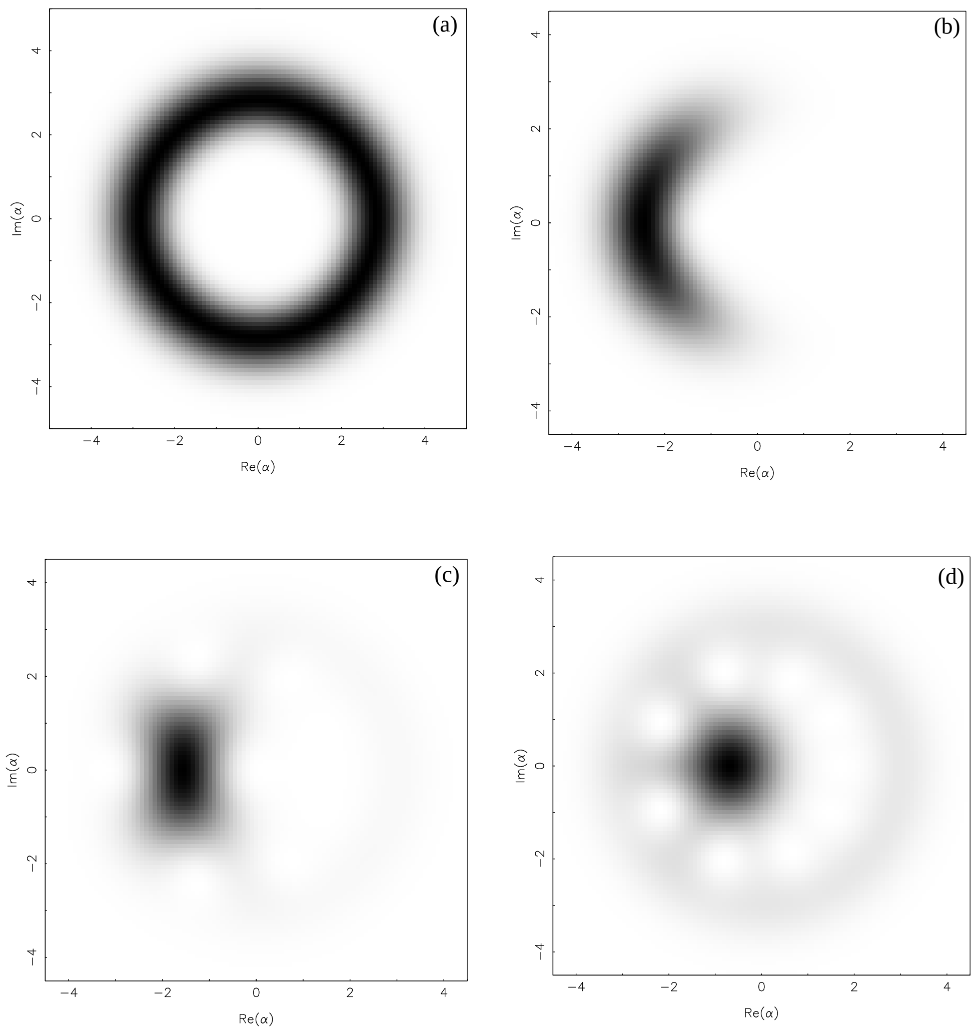

Figure 5.6: Grey-scale plot of the Q functions for the conditional states of the freely decaying cavity mode under a typical homodyne quantum trajectory. The state at $t = 0$ (a) is a Fock state $n = 8$, and the later times are $t = 0.25$ (b), $t = 1.0$ (c), and $t = 2.0$ (d).

5.5.1 Ideal Laser Model

By an ideal laser, I mean one with a Poissonian gain medium. That is to say, one in which photons are added to the field at a constant rate, independent of the state of the cavity. It is possible to consider a sub-Poissonian pumped laser [83, 96, 72, 116, 117]. I have treated quantum trajectories for this system elsewhere [144]; this thesis is restricted to an ideal laser. It is shown in App. C that such a laser obeys the master equation

$$\dot{\rho} = \left(\mu\mathcal{E}[a^\dagger] + \mathcal{D}[a]\right)\rho. \tag{5.113}$$

Here, time is again being measured relative to the cavity lifetime κ^{-1}, and μ is effectively the laser pump rate. The laser excitation superoperator is defined by

$$\mathcal{E} = \mathcal{J}\mathcal{A}^{-1} - 1, \tag{5.114}$$

where $\mathcal{J}$ is defined in Eq. (2.8) and $\mathcal{A}$ in Eq. (2.34). Compare this definition to the damping superoperator $\mathcal{D} = \mathcal{J} - \mathcal{A}$, where the rate of loss depends linearly on the mean photon number. For the laser at steady state, both loss and gain of photons are Poissonian, at rate μ, because the steady-state photon number distribution is exactly Poissonian. This can be seen from the Fock basis which puts Eq. (5.113) in the form

$$\dot{\rho}_{n,m} = \mu\left(\frac{2\sqrt{nm}}{n+m}\rho_{n-1,m-1} - \rho_{n,m}\right) + \sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1} - \tfrac{1}{2}(n+m)\rho_{n,m}. \tag{5.115}$$

It is easy to see that μ is the steady-state mean photon number by verifying that the steady-state state matrix satisfying Eq. (5.115) is

$$\rho_{n,m}(\infty) = \delta_{n,m}e^{-\mu}\frac{\mu^n}{n!}. \tag{5.116}$$

This photon number distribution is Poissonian, as in a coherent state of amplitude $\sqrt{\mu}$. A consequence of this is that direct detection has no effect on the stationary state of the laser. As noted in Sec. 5.4.1, a Poissonian distribution cannot be conditioned by direct detection.

Unlike a coherent state, the stationary distribution has a completely undefined phase as the off-diagonal elements are zero. This can be seen from an alternate representation of the steady-state density operator using coherent states

$$\rho(\infty) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi|\sqrt{\mu}e^{i\varphi}\rangle\langle\sqrt{\mu}e^{i\varphi}|. \tag{5.117}$$

Thus, the stationary density operator can be expressed either as a Poissonian mixture of number states (with maximally determined intensity and completely undefined phase), or as a mixture of fixed amplitude coherent states (with well defined intensity and phase). Why then is a laser often treated as being in a coherent state of possibly undetermined phase, but never as being in an imprecisely known Fock state? The basic answer to the above question is differential lifetimes, as pointed out by Gea-Banacloche [56]. If a laser were to be put into a Fock state, then under the master equation (5.115), the probability of it remaining in that state will decay at a rate of order μ. For a typical cavity decay rate $\kappa \sim 10^7\text{s}^{-1}$, and mean photon number $\mu \sim 10^9$, the lifetime of a number state is thus of the order of 10^{-16} seconds. On the other hand, the survival probability of a coherent state of amplitude $\sqrt{\mu}$ decays at a rate of order κ, and the phase changes macroscopically on a time scale of order κ/μ, which may be many minutes.

This phase diffusion can be seen by converting Eq. (5.113) into a Fokker-Planck equation (FPE) for the P function. Defining number and phase variables in terms of the complex amplitudes

$$n = |\alpha|^2 \, ; \quad \varphi = \frac{1}{2i} \log\left(\alpha^2/n\right), \tag{5.118}$$

and using the operator correspondences for the P function [52], one can demonstrate the following correspondences:

$$\mathcal{A}[a^\dagger]\rho \quad \to \quad n\,(1-\partial_n)\,P(n,\varphi), \tag{5.119}$$

$$\mathcal{D}[a^\dagger]\rho \quad \to \quad \left[-\partial_n n\,(1-\partial_n) + \frac{1}{4n}\partial_\varphi^2\right] P(n,\varphi). \tag{5.120}$$

Putting these into Eq. (5.113), and dropping terms of order $1/n^2$ and smaller gives the FPE

$$\dot{P} = \left(-\mu\partial_n + \frac{\mu}{4n^2}\partial_\varphi^2 + \partial_n n\right) P, \tag{5.121}$$

where the well known form of damping for the P function has also been used.

Near steady state, it is permissible to replace n in the diffusion term by μ. Furthermore, since there is no diffusion in the normally ordered photon number n, it will rapidly approach its stationary value μ and can thereafter be ignored. Then the dynamics of the laser is completely determined by the phase diffusion

$$\dot{P}(\varphi) = \frac{\Gamma}{2}\partial_\varphi^2 P(\varphi), \tag{5.122}$$

where Γ is the fundamental rate of phase diffusion of the laser, given by

$$\Gamma = \frac{1}{2\mu}. \tag{5.123}$$

This agrees with the expression derived by Louisell [92] if the same idealizations are made (such as a zero temperature bath *et cetera*), and with that derived heuristically by Loudon [93]. Experimentally, the rate of phase diffusion is measured by the laser linewidth, which is the width of the laser spectrum. This is determined from the Fourier transform of the two-time correlation function

$$\langle a^\dagger(t)a(0)\rangle_{\rm ss} = \mu \exp\left(-\Gamma t/2\right). \tag{5.124}$$

This decaying exponential implies a Lorentzian spectrum with a full width at half maximum equal to Γ. In reality, Γ will have many (possibly much larger) contributions apart from the fundamental one (5.123), which is why I will use the symbol Γ rather than the value (5.123).

5.5.2 Laser Phase and Heterodyne Detection

The fact that the inverse of the laser linewidth is of order minutes is often used as a justification for approximating the laser by a coherent state. On the time scale of many experiments, a laser initially in a coherent state will remain in that state to a very good approximation. However, this appears to beg the question of how the laser got into a coherent state in the first place. One obvious answer is that measuring the phase of the laser collapses it into a coherent state, (or alternatively, determines which coherent state it really was in). In practice, this phase measurement may be done using heterodyne detection. In this subsection, I examine the dynamics of a ideal laser under heterodyne detection to determine if the laser can be collapsed into a coherent state. The unanticipated conclusion is that it cannot. The residual phase variance is at best three times that of a coherent state.

Heterodyne detection of efficiency η adds the following stochastic term to the laser master equation

$$d\rho_c(t) = \sqrt{\eta}\{dV(t)[a\rho_c(t) - \langle a\rangle_c(t)\rho_c(t)] + \text{H.c.}\}, \tag{5.125}$$

where $V(t)$ is a complex Wiener process. As noted in the preceding subsection, the P function for the laser at steady state is a function of the phase φ only, with the amplitude fixed at $\sqrt{\mu}$. From Eq. (5.125) and Eq. (5.122), the conditioned P function will obey

$$dP_c(\varphi;t) = \left(dt\frac{\Gamma}{2}\partial_\varphi^2 + \sqrt{\eta\mu}\left\{dV(t)\left[e^{i\varphi} - \int d\varphi' e^{i\varphi'}P_c(\varphi';t)\right] + \text{c.c.}\right\}\right)P_c(\varphi;t). \tag{5.126}$$

Now I will show shortly that the long time solutions to this equation have a variance in φ of order $1/\mu \ll 1$. Keeping terms of lowest order in $1/\mu$ in Eq. (5.126) gives

$$dP_c(\varphi;t) = \left\{dt\frac{\Gamma}{2}\partial_\varphi^2 + \sqrt{2\eta\mu}\, dW_c^\phi(t)[\varphi - \phi_c(t)]\right\}P_c(\varphi;t). \tag{5.127}$$

Here, $dW_c^\phi(t)$ is a real infinitesimal Wiener increment defined by

$$dW_c^\phi(t) = -\Im[\sqrt{2}e^{i\phi_c(t)}dV(t)], \tag{5.128}$$

and $\phi_c(t)$ is the central angle of the distribution

$$\phi_c(t) = \int d\varphi P_c(\varphi;t)\varphi. \tag{5.129}$$

Since the variance in the phase is very small, one can ignore the periodicity requirement on $P_c(\varphi,t)$ and instead use the following ansatz for the solution of Eq. (5.127):

$$P_c(\varphi;t) = \frac{1}{\sqrt{2\pi U_c(t)}}\exp\{-[\varphi - \phi_c(t)]^2/2U_c(t)\}. \tag{5.130}$$

For the moment, ignore the deterministic diffusion term. By the rules of Itô calculus,

$$1 + \sqrt{\lambda}dW(t)[\varphi - \phi_c(t)] = \exp\left\{\sqrt{\lambda}dW(t)[\varphi - \phi_c(t)] - \frac{\lambda}{2}dt[\varphi - \phi_c(t)]^2\right\}. \tag{5.131}$$

Applying this to the ansatz (5.130), with λ standing for $2\eta\mu$, gives

$$\begin{aligned} P(\varphi, t+dt) &= \frac{1}{\sqrt{2\pi U(t)}}\exp\left\{-[\varphi - \phi_c(t)]^2\left[\frac{1}{2U(t)} + \frac{\lambda}{2}dt\right] + \sqrt{\lambda}dW[\varphi - \phi_c(t)]\right\} && (5.132)\\ &= \frac{1}{\sqrt{2\pi U(t+dt)}}\exp\{-[\varphi - \phi_c(t+dt)]^2/2U(t+dt)\}, && (5.133)\end{aligned}$$

where

$$\begin{aligned} \phi_c(t+dt) &= \phi_c(t) + \sqrt{\lambda}U(t)dW, && (5.134)\\ U(t+dt) &= U(t) - \lambda U(t)^2 dt. && (5.135)\end{aligned}$$

This confirms the Gaussian ansatz (5.130).

Adding the deterministic diffusion gives the following equations for the conditioned mean and normally ordered variance for the phase:

$$\begin{aligned} \dot{\phi}_c(t) &= \sqrt{2\eta\mu}\,\xi_c^\phi(t)U_c(t), && (5.136)\\ \dot{U}_c(t) &= \Gamma - 2\eta\mu U_c^2(t), && (5.137)\end{aligned}$$

where $\xi_{\rm c}^{\phi}(t) = dW_{\rm c}^{\phi}(t)/dt$. Note that the effect of the measurement on the variance is deterministic, and causes it to reduce as expected. If the laser state is taken to be initially in a coherent state, the normally ordered variance is $U_{\rm c}(0) = 0$. From Eq. (5.137), the variance increases as

$$U_{\rm c}(t) = U_\infty \tanh(2\eta\Gamma\mu t), \tag{5.138}$$

where the steady-state P function phase variance is

$$U_\infty = \sqrt{\Gamma/2\eta\mu}. \tag{5.139}$$

If there is no excess phase diffusion in the laser, then $\Gamma = 1/2\mu$, as derived above. Assuming in addition that $\eta = 1$, this gives the minimum steady-state conditioned P phase variance of

$$U_\infty = \frac{1}{2\mu}. \tag{5.140}$$

As the mean photon number μ goes to infinity, this goes to zero. Nevertheless, on a quantum scale, it is always significantly higher than the value of zero for a coherent state. In fact, it is equal to the Q function phase variance of a coherent state. The 'true' phase variance of $3/4\mu$ (which in this linearized regime would correspond to that of the Wigner function [7]) for the ideal conditioned state is three times that of a coherent state ($1/4\mu$). Thus it is not true that monitoring the phase of a laser collapses it to a coherent state.

Eq. (5.138) shows that the phase variance relaxes to its steady-state value at a rate of order the cavity linewidth, which is of order μ times the laser linewidth. Thus, one may replace $U_{\rm c}(t)$ in Eq. (5.136) by its steady-state value. This gives

$$\dot{\phi}_{\rm c}(t) = \sqrt{\Gamma}\xi_{\rm c}^{\phi}(t). \tag{5.141}$$

This equation is precisely of the form of the stochastic differential equation for the laser phase which would be derived from the original Fokker-Planck equation for the P function (5.122). That is not to say that the analysis of this subsection is simply a very lengthy and obscure derivation of a standard result. The phase $\phi_{\rm c}(t)$ is not a mathematical artefact. It has a physical interpretation as the mean phase of the laser, conditioned on the results of the continuous heterodyne measurement. The conditioning occurs via the noise in Eq. (5.141) which is physically derived from the photocurrent shot noise, not merely a formal device producing a stochastic equation equivalent to a Fokker-Planck equation. The dynamics of a continuously monitored standard laser can thus be pictured as follows. The laser state is a Gaussian mixture of equal-amplitude coherent states with a constant phase variance inversely proportional to the mean photon number. The mean phase of the mixture undergoes a random walk on a time scale inversely proportional to the laser linewidth.

Although these results for the conditioned state of a laser may be helpful pedagogically, it is reasonable to ask what practical use they have. Usually, the conditioned state of a system is of little interest to experimentalists and some might even doubt whether the concept is meaningful. In fact, the conditioned states such as I have investigated are meaningful experimentally, because they are precisely those states which can be produced (as unconditioned states) by using the measured photocurrent (on which they are conditioned) optimally in a feedback loop. Feedback is the topic of the remaining chapters in this thesis. For the ideal laser, it means that the steady-state conditioned variances which I derived here represent the best achievable steady-state (unconditioned) variances under feedback. The feedback-stabilized variance could be measured by dumping the cavity light into an independent heterodyne detection device. In summary, the practical significance of this work is that conditioning can be realized by feedback.

Chapter 6

Quantum Trajectories with Feedback

In this chapter I use quantum trajectories to describe feedback in open quantum systems. The stochastic photocurrent which conditions the system state is used to control the future evolution of the system. In the limit that the time delay in the feedback loop is small, a master equation describing the ensemble average effect of the feedback can be derived. The smallness condition on the time delay is derived from an approximate master equation describing non-Markovian feedback. Direct, homodyne, and heterodyne detection are treated, the latter two being generalized to cover feedback in the presence of white noise. The special case of feedback as self-excitation is also treated. The final three sections of the chapter deal with a linear system in which the quantum trajectories with feedback can be solved exactly. This is possible even with a time delay, which allows the results of the approximate master equation to be verified. The limits to noise reduction by linear feedback are derived in terms of the intracavity and extracavity squeezing. It is shown that feedback of a QND photocurrent can overcome the limitations of feedback of a homodyne photocurrent.

6.1 Direct Detection Feedback

6.1.1 Non-Markovian Feedback

In Ch. 2, quantum measurement theory was applied to systems obeying a Markovian master equation

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \mathcal{D}[c]\rho \tag{6.1}$$

to show that the measurement record for continuous observation is essentially a point process. It was then argued that immediately feeding back the information in this record to alter the system dynamics would change the master equation (6.1) to

$$\dot{\rho} = -\mathrm{i}[H, \rho] + e^{\mathcal{K}}\mathcal{J}[c]\rho - \mathcal{A}[c]\rho, \tag{6.2}$$

where $\mathcal{K}$ is a Liouville superoperator determined by the physical mechanism for the feedback. The purpose of this section is to rederive this equation in a way which more clearly shows the relation to laboratory experiments in quantum optics. The starting point is the quantum trajectory theory of Ch. 4, which gives the evolution of the system conditioned on a measured photocurrent. This photocurrent can then be used to control the system dynamics in an arbitrary way. Because the photocurrent is inherently noisy, the stochastic calculus of Ch. 3 must be used to determine the effect of this on the system.

The SME appropriate for direct detection is Eq. (4.12)

$$d\rho_{\rm c}(t) = \left\{ dN_{\rm c}(t)\mathcal{G}[c] - dt\mathcal{H}\left[{\rm i}H + \tfrac{1}{2}c^\dagger c\right] \right\} \rho_{\rm c}(t), \tag{6.3}$$

where the superoperators $\mathcal{G}, \mathcal{H}$ are as defined below that equation, and where the point process $dN_{\rm c}(t)$ is defined by

$$\begin{aligned} {\rm E}[dN_{\rm c}(t)] &= {\rm Tr}[c^\dagger c\rho_{\rm c}(t)], \qquad (6.4)\\ dN_{\rm c}(t)^2 &= dN_{\rm c}(t). \qquad (6.5) \end{aligned}$$

As explained in Ch. 4, using the first of these relations gives the original master equation for the ensemble average evolution. For the purposes of continuous feedback, experimentalists would consider a photocurrent rather than a photocount. In this theory, they are formally related by

$$I_{\rm c}(t) = dN_{\rm c}(t)/dt. \tag{6.6}$$

The current $I_{\rm c}(t)$ could be used to alter the system dynamics in many different ways. Some examples from quantum optics are: modulating the pump rate of a laser, the amplitude of a driving field, the cavity length, or the cavity loss rate. The last three examples could be effected by using an electro-optic modulator (a device with a refractive index controlled by a current), possibly in combination with a polarization-dependent beam splitter. The most general expression for the effect of the feedback (assumed not to depend explicitly on time) would be

$$[\dot\rho_{\rm c}(t)]_{\rm fb} = \mathcal{F}[I_{\rm c}(t-s)]\rho_{\rm c}(t), \tag{6.7}$$

where $\mathcal{F}[I_{\rm c}(t-s)]$ is a superoperator *functional* of the current for $s > 0$. This would include the response function of the feedback loop, which may even be nonlinear, and must include some smoothing in time. The complete description of this feedback is given simply by adding Eq. (6.7) to Eq. (6.3). However, this adds little to one's understanding of continuous quantum feedback, as the equation could only be solved by numerical simulation. To make progress towards understanding, it is necessary to make simplifying assumptions. Firstly, assume the response function is linear. That is to say, let the feedback evolution be of the form

$$[\dot\rho_{\rm c}(t)]_{\rm fb} = \int_0^\infty h(s)I_{\rm c}(t-s)\mathcal{K}\rho_{\rm c}(t), \tag{6.8}$$

where $\mathcal{K}$ is an arbitrary Liouville superoperator. Eventually, I wish to consider the Markovian limit in which the response function $h(s)$ goes to $\delta(s)$. To find this limit it is first useful to consider the case $h(s) = \delta(s-\tau)$, where the feedback has a fixed delay τ. Then the feedback evolution is

$$[\dot\rho_{\rm c}(t)]_{\rm fb} = I_{\rm c}(t-\tau)\mathcal{K}\rho_{\rm c}(t). \tag{6.9}$$

Because there is no smoothing response function in Eq. (6.9), the right hand side of the equation is a mathematically singular object, with $I_{\rm c}(t)$ a string of delta functions. If it is meant to describe a physical feedback mechanism, then it is necessary to interpret the equation as an implicit stochastic differential equation, as explained in Sec. 3.2. This is indicated already in my notation of using a fluxion on the left hand side. The reader may quickly verify that the alternative interpretation as the explicit equation

$$[d\rho_{\rm c}(t)]_{\rm fb} = dN_{\rm c}(t-\tau)\mathcal{K}\rho_{\rm c}(t) \tag{6.10}$$

yields nonsense. In order to combine Eq. (6.9) with Eq. (6.3), it is necessary to convert it from an implicit to an explicit equation. As already noted in Sec. 3.2, this is easy to accomplish because of the linearity (with respect to ρ) of Eq. (6.9). The result is

$$[d\rho_{\rm c}(t)]_{\rm fb} = \left(\exp[\mathcal{K}dN_{\rm c}(t-\tau)] - 1\right)\rho_{\rm c}(t). \tag{6.11}$$

Using the rule (6.5) and adding this evolution to that of the SME (6.3) gives the total conditioned evolution of the system

$$d\rho_{\rm c}(t) = \left\{ dN_{\rm c}(t)\mathcal{G}[c] - dt\mathcal{H}\left[{\rm i}H + \tfrac{1}{2}c^\dagger c\right] + dN_{\rm c}(t-\tau)\left(e^{\mathcal{K}} - 1\right)\right\}\rho_{\rm c}(t). \tag{6.12}$$

It is not possible to turn this stochastic equation into a master equation by taking an ensemble average, as was possible with Eq. (6.3). This is because the feedback noise term (with argument $t-\tau$) is not independent of the state at time t. Physically, it is not possible to derive a master equation because the feedback is not Markovian.

6.1.2 The Feedback Master Equation

In order to make Eq. (6.12) more useful, it would be desirable to take the Markovian limit ($\tau \to 0$) in order to derive a nonselective master equation. Simply putting $\tau = 0$ in this equation, and replacing $dN_{\rm c}(t)$ by its ensemble average (6.4) fails. The resultant evolution equation would be nonlinear in ρ and so could not be a valid evolution equation. The reason that the nonlinearity is admissible in Eq. (6.12) is that $\rho_{\rm c}$ is a conditioned state for an individual system, not the ensemble average ρ. (See Ref. [114] for a discussion of this point.) Putting $\tau = 0$ fails because the feedback must act after the measurement even in the Markovian limit. As explained in Sec. 3.3, this can be achieved by the equation

$$\rho_{\rm c}(t+dt) = \exp[dN_{\rm c}(t-\tau)\mathcal{K}]\left\{1 + dN_{\rm c}(t)\mathcal{G}[c] + dt\mathcal{H}\left[-{\rm i}H - \tfrac{1}{2}c^\dagger c\right]\right\}\rho_{\rm c}(t). \tag{6.13}$$

For τ finite, this reproduces Eq. (6.3). However, if $\tau = 0$, expanding the exponential gives

$$d\rho_{\rm c}(t) = \left\{ dN_{\rm c}(t)\left[e^{\mathcal{K}}\left(\mathcal{G}[c]+1\right) - 1\right] + dt\mathcal{H}\left[-{\rm i}H - \tfrac{1}{2}c^\dagger c\right]\right\}\rho_{\rm c}(t). \tag{6.14}$$

In this equation, it is possible to take the ensemble average because $dN_{\rm c}(t)$ can simply be replaced by its expectation value (6.4), giving

$$\dot\rho = e^{\mathcal{K}}\mathcal{J}[c]\rho - \mathcal{A}[c]\rho - {\rm i}[H,\rho]. \tag{6.15}$$

This master equation is of course the same as Eq. (6.2) derived from general principles. Now however, the relation of the superoperator $\mathcal{K}$ to experiment via Eq. (6.9) is known. In the special case where $\mathcal{K}\rho = -{\rm i}[Z,\rho]$, the conditioned SME with feedback can also be expresses as a SSE, which is

$$d|\psi_{\rm c}(t)\rangle = \left[\left(dN_{\rm c}(t)\frac{e^{-{\rm i}Z}c}{\sqrt{\langle c^\dagger c\rangle_{\rm c}(t)}} - 1\right) + dt\left(\frac{\langle c^\dagger c\rangle_{\rm c}(t)}{2} - \frac{c^\dagger c}{2} - {\rm i}H\right)\right]|\psi_{\rm c}(t)\rangle. \tag{6.16}$$

It has been noted in Ch. 5 that if the nonselective evolution of a cavity cannot produce nonclassical states, then the states conditioned on direct detection (or any other form of external photodetection) will not be nonclassical either. From the theory of feedback presented here, it is thus evident that if $\mathcal{K}$ is a classical superoperator, then the feedback cannot produce a nonclassical state, because it is simply turned on and off by the current, a $c-$number. That is to say, using a direct detection photocurrent to alter the classical dynamics of a cavity cannot produce nonclassical light. It is not difficult to verify this result for the special case of Markovian feedback presented here, by showing that the master equation (6.15) will preserve the positivity of the Glauber-Sudarshan P representation of the state matrix, with c equal to the annihilation operator for the cavity mode. Of course this result also holds if the Hamiltonian evolution of the cavity is generalized to that generated by a Liouville superoperator $\mathcal{L}_0$ (including the damping), in which case the feedback master equation can be rewritten as

$$\dot{\rho} = \left\{ \mathcal{L}_0 + \left(e^{\mathcal{K}} - 1 \right) \mathcal{J}[c] \right\} \rho \equiv \mathcal{L}\rho. \tag{6.17}$$

From the preceding paragraph, it is evident that controlling intracavity classical dynamics by an externally measured photocurrent cannot produce nonclassical (sub-Poissonian) photon statistics as measured by an independent (out-of-loop) detector. The in-loop detector may record such statistics, however. This does not mean that there is nonclassical light incident on the in-loop detector. If all of the superoperators are classical, then the entire feedback process may be described in terms of coherent states. The explanation for the possibility of sub-Poissonian in-loop statistics is that the two-time correlation function for the current no longer measures normally ordered intensity correlations, as in Sec. 4.1.3. Rather, it is easy to show (by the same method as used in that section) that

$$\begin{aligned} \mathrm{E}[I_c(t')I_c(t)] &= \mathrm{Tr}\left[e^{\mathcal{K}}\mathcal{J}[c]e^{\mathcal{L}(t'-t)}e^{\mathcal{K}}\mathcal{J}[c]\rho(t) \right] + \mathrm{Tr}\left[\mathcal{J}[c]\rho(t) \right] \delta(t'-t) && (6.18) \\ &= \mathrm{Tr}\left[c^\dagger c e^{\mathcal{L}(t'-t)} e^{\mathcal{K}} c\rho(t) c^\dagger \right] + \mathrm{Tr}\left[c^\dagger c \rho(t) \right] \delta(t'-t), && (6.19) \end{aligned}$$

where $\mathcal{L}$ is as defined in Eq. (6.17). This expression shows that it is the effect of the feedback specific to the in-loop current via $e^{\mathcal{K}}$, not the overall evolution including feedback via $e^{\mathcal{L}(t'-t)}$, which may cause sub-Poissonian statistics for the fed-back current.

6.1.3 The Effect of a Short Delay

In this section I attempt to answer the question of how fast the feedback mechanism must respond to justify the Markovian approximation. In so doing, I derive an approximate master equation which is applicable in the limit where the feedback time delay τ is small but not negligible. The approach used is a first order perturbation expansion in the parameter τ. That is, it is assumed as a zeroth order approximation that the evolution of the system can be described by the $\tau = 0$ result

$$\rho(t') = e^{\mathcal{L}(t'-t)}\rho(t) \qquad (t' > t), \tag{6.20}$$

where $\mathcal{L}$ is the superoperator defining the entire feedback process, as in Eq. (6.17).

Now consider feedback in the selective picture, as described by Eq. (6.12), with a finite time delay τ. Consider a hypothetical photodetection at time $t - \tau$. Let the state of the system at this time be $\rho(t-\tau)$. The use of an unconditioned state matrix to represent the system is consistent with the zeroth order approximation which describes nonselective evolution. The conditioned state matrix at time $t - \tau + dt$ is

$$\rho_c(t - \tau + dt) = \left\{ 1 + dN_c(t-\tau)\mathcal{G}[c] + O(dt) \right\} \rho(t-\tau). \tag{6.21}$$

This state is conditioned only on the hypothetical jump. All of the non-jump evolution has been included in the term of order dt. This includes the feedback from earlier jumps, which will be of order dt on average. Again, this nonselective treatment is what is implied by the zeroth order approximation. Now the secondary effect of this possible detection at time $t-\tau$ on the system due to feedback is delayed by τ. By that time, the state of the system has evolved to

$$\rho_{\rm c}(t) = e^{\mathcal{L}\tau}\left\{1 + dN_{\rm c}(t-\tau)\mathcal{G}[c] + O(dt)\right\}\rho(t-\tau), \tag{6.22}$$

where the zeroth order approximation to the evolution [Eq.(6.20)] has been used. Over the next infinitesimal time step, the feedback takes effect, so that the conditioned state is [from Eq. (6.12)]

$$\rho_{\rm c}(t+dt) = \left\{1 + dN_{\rm c}(t)\mathcal{G}[c] + dt\mathcal{H}[-\tfrac{1}{2}c^\dagger c - {\rm i}H]\right\}\rho_{\rm c}(t) + dN_{\rm c}(t-\tau)(e^{\mathcal{K}}-1)\rho_{\rm c}(t). \tag{6.23}$$

The first term comes from the usual evolution (including measurement), while the second comes from the feedback, which is now being treated selectively in order to see the effect of the delay τ.

The form of this equation has been chosen so that the nonselective equation is simple to see. In taking the ensemble average, the first term simply turns into the usual expression for the state matrix without feedback,

$$(1 + dt\mathcal{L}_0)\,\rho(t). \tag{6.24}$$

However, for the second term, it is necessary to use the expression (6.22), because of the feedback correlations, to get

$$E\left[(e^{\mathcal{K}}-1)e^{\mathcal{L}\tau}dN_{\rm c}(t-\tau)\left\{1 + dN_{\rm c}(t-\tau)\mathcal{G}[c] + O(dt)\right\}\rho(t-\tau)\right]. \tag{6.25}$$

Using the stochastic rules (6.4,6.5), this expression becomes

$$(e^{\mathcal{K}}-1)e^{\mathcal{L}\tau}\left\{\mathcal{J}[c] + O(dt)\right\}\rho(t-\tau)dt. \tag{6.26}$$

Adding the two terms together gives the first order approximation to the effect of a finite delay τ,

$$\dot\rho(t) = \mathcal{L}_0\rho(t) + (e^{\mathcal{K}}-1)e^{\mathcal{L}\tau}\mathcal{J}[c]\rho(t-\tau). \tag{6.27}$$

In the limit $\tau \to 0$, the zeroth order evolution is recovered as required. It must be emphasized that Eq. (6.27) is an approximate solution only. In fact, since it is a solution of first order in τ, it is only proper to expand its individual terms to first order in τ also. That is, one can approximate $e^{\mathcal{L}\tau}$ by $1+\mathcal{L}\tau$, and use

$$\rho(t-\tau) \simeq (1-\mathcal{L}\tau)\rho(t). \tag{6.28}$$

Substituting these into Eq. (6.27) gives

$$\dot\rho(t) = \left[\mathcal{L} + \tau(e^{\mathcal{K}}-1)\left(\mathcal{L}\mathcal{J}[c] - \mathcal{J}[c]\mathcal{L}\right)\right]\rho(t). \tag{6.29}$$

This final approximate master equation is equal to the instantaneous feedback master equation (6.17), plus a correction linear in τ. The condition for this correction to the Markovian feedback master equation to be negligible is obviously

$$\tau||(e^{\mathcal{K}}-1)\{\mathcal{L}\mathcal{J}[c] - \mathcal{J}[c]\mathcal{L}\}\rho|| \ll ||\mathcal{L}\rho||, \tag{6.30}$$

where ρ is a suitable density operator [perhaps the steady state solution of Eq. (6.17)], and the bounds $||$ indicate a suitable norm. To elucidate this expression, consider a typical quantum optical system, damped to the vacuum at a rate of unity, so that c represents the annihilation operator for the intracavity field. Let the intracavity photon number n have a large mean μ and a relatively small variance $\sim \mu$. In this case, the evolution can be successfully described by a Fokker-Planck equation for a distribution function such as the P or W function [52]. The magnitude of the damping evolution $||\mathcal{D}[c]\rho||$ can be seen to be of order $\partial_n n \sim 1$. If the feedback is to be of the same order of magnitude then $e^{\mathcal{K}} - 1 \sim \mathcal{K} \sim \mu^{-1}$. Now, $[\mathcal{L}, \mathcal{J}[c]] \sim [\partial_n n, n] \sim \mu$. Thus the condition (6.30) simply reduces to $\tau \ll 1$. That is to say, the feedback loop delay must be much less than the cavity lifetime. This is quite feasible, with loop delays of order 10^{-8} s and cavity lifetimes of order 10^{-7} s [134]. It is important to note that it is *not* necessary for τ to be much less than the time between detections, which is of order the cavity lifetime divided by μ. If the latter condition were necessary, then Markovian feedback would be quite impractical for most systems.

6.2 Self-Excited Quantum Point Processes

I have used the term quantum point process (QPP) to refer to the detections which arise in continuous monitoring of a quantum system. Classically, a Poissonian point process can be made non-Poissonian by making its rate depend on past detections. This is called a self-excited point process [74]. The quantum analog to this is a source of irreversibility whose strength is controlled by the rate of detections from that source. This can be called a self-excited quantum point process. This sort of feedback has not before been given a correct quantum treatment. Classically, the spectrum of the self-excited point process is of primary interest [74]. In the quantum case, the effect of the self-exciting QPP on the source is of more interest. The aim of this section is to derive a master equation which describes this effect in the Markovian limit. The concept of a self-excited QPP is interesting for a number of reasons. Firstly, it is a nontrivial extension of the general theory developed in Sec. 6.1. Secondly, it is an example in which the measurement theory approach of this chapter is clearly easier to apply than the quantum mechanical approach which will be developed in the following chapter. Thirdly, a self-excited QPP was the first feedback system for which a model was attempted using the measurement theory and master equation approach [148, 88]. It turns out that this early approach was flawed, which will be seen by comparison with the correct equation.

Consider a QPP with a time-variable rate $\kappa(t)$, so that the master equation is

$$\dot{\rho} = \kappa(t)\mathcal{D}[c]\rho - \mathrm{i}[H, \rho]. \tag{6.31}$$

If the QPP is to be a self-exciting QPP, then $\kappa(t)$ becomes $\kappa_{\mathrm{c}}(t)$, conditioned on the photocurrent

$$\kappa_{\mathrm{c}}(t) = 1 + \lambda I_{\mathrm{c}}(t), \tag{6.32}$$

where $I_{\mathrm{c}}(t) = dN_{\mathrm{c}}(t)/dt$ and it is necessary to use the selective master equation

$$d\rho_{\mathrm{c}}(t) = \left(dN_{\mathrm{c}}(t)\mathcal{G}[c] + dt\mathcal{H}\left[-\mathrm{i}H - \tfrac{1}{2}\kappa_{\mathrm{c}}(t)c^\dagger c\right]\right)\rho_{\mathrm{c}}(t), \tag{6.33}$$

where now

$$\mathrm{E}[dN_{\mathrm{c}}(t)] = \kappa_{\mathrm{c}}(t)\langle c^\dagger c\rangle_{\mathrm{c}}(t)dt. \tag{6.34}$$

Note that since $\kappa_{\mathrm{c}}(t)$ must always be positive, λ should strictly always be positive also, hence the self-excitation rather than self-inhibition. However, for λ negative but sufficiently small, the theory developed below may apply in particular cases.

The same considerations of physicality [causality and smoothness of $I_c(t)$] explained in the preceding section also apply to Eq. (6.32). Thus the general theory implies that the nonselective master equation must be of the form (6.15). Comparison of Eqs. (6.31,6.32) with Eqs. (6.9) would suggest $\mathcal{K} = \lambda\mathcal{D}[c]$. Under more careful consideration, it is obvious that this equation describes feedback controlling a second irreversible coupling, rather than a self-exciting process. Since the original superoperator has been modified from $\mathcal{D}$ to $e^{\mathcal{K}}\mathcal{J} - \mathcal{A}$, the action of the feedback superoperator should be

$$\mathcal{K} = \lambda(e^{\mathcal{K}}\mathcal{J} - \mathcal{A}). \tag{6.35}$$

Here, the argument $[c]$ to the superoperators $\mathcal{J}$, $\mathcal{A}$, and $\mathcal{D}$ is being omitted for convenience. The Markovian self-exciting QPP master equation can be written

$$\dot{\rho} = \lambda^{-1}\mathcal{K}\rho - \mathrm{i}[H,\rho]. \tag{6.36}$$

Unfortunately, there is no closed-form solution to the transcendental superoperator equation (6.35)

If the feedback were weak, then $\mathcal{K}$ could be approximated by iterating Eq. (6.35). However, this is not a very satisfactory solution, as the self-excitation does not even become evident until the second iteration. An alternate approximation may be found by constraining the nature of the system, rather than the strength of self-excitation. Let the state of the system have a well-defined value of $c^\dagger c$. That is to say, let the mean μ of $c^\dagger c$ be very large, and the variance be of the same order. Then the superoperator $\mathcal{D}$ is of order one, but $\mathcal{J}$ is of order μ. For the self-excitation to be of order one requires $\lambda\mu \sim 1$. This implies that λ is small, but the self-excitation strength is not small. Expanding the self-exciting QPP superoperator $\mathcal{K}$ to second order in $1/\mu$ is then a good approximation for the systems under consideration. Effectively, this can be achieved by assuming a solution of the form

$$\mathcal{K} = \lambda\mathcal{D}\mathcal{F} + \lambda^2\mathcal{D}^2\mathcal{S} + O(\lambda^3), \tag{6.37}$$

where $\mathcal{F}$ and $\mathcal{S}$ are superoperators of order one to be determined. This approximation also implies that the superoperator ordering in the second order term $\lambda\mathcal{D}^2\mathcal{S}$ is not really important.

Substituting the ansatz (6.37) into Eq. (6.35) and equating powers of λ yields

$$\mathcal{F} = 1 + \mathcal{F}\lambda\mathcal{J}, \tag{6.38}$$
$$\mathcal{S} = \mathcal{S}\lambda\mathcal{J} + \tfrac{1}{2}\mathcal{F}^2\lambda\mathcal{J}. \tag{6.39}$$

Formally evaluating these gives

$$\mathcal{F} = (1-\lambda\mathcal{J})^{-1}, \tag{6.40}$$
$$\mathcal{S} = \frac{\lambda\mathcal{J}}{2(1-\lambda\mathcal{J})^3}. \tag{6.41}$$

Thus the approximate expression for the self-excited QPP master equation is

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \mathcal{D}(1-\lambda\mathcal{J})^{-1}\rho + \lambda\mathcal{D}^2\frac{\lambda\mathcal{J}}{2(1-\lambda\mathcal{J})^3}\rho. \tag{6.42}$$

This evidently includes terms which indicate an arbitrarily large number of detections within an infinitesimal time interval. This is one reason why the calculation of the spectrum of the self-excited QPP is too difficult to attempt here.

In quantum optics, the approximations leading to Eq. (6.42) are often valid if $c = a$, the annihilation operator for the cavity mode. In that case, the self-excitation would correspond to an end mirror with variable transmittivity controlled by the current coming from a photodetector just outside that end. Assuming that the time delay in the feedback loop is negligible, the formalism developed above can be applied. It is useful to expand the master equation (6.42) explicitly in terms of a and $a^\dagger$.

$$\begin{aligned}\dot\rho \;=\; & \left\{\sum_{m=0}^{\infty}\lambda^m a^{m+1}\rho a^{\dagger m+1}\right\} - \tfrac{1}{2}\sum_{m=0}^{\infty}\lambda^m\left(a^\dagger a^{m+1}\rho a^{\dagger m} + a^m\rho a^{\dagger m+1}a\right)\\ & + \frac{\lambda}{2}\sum_{m=0}^{\infty}\frac{m(m+1)}{2}\lambda^m\Big[a^{m+2}\rho a^{\dagger m+2} - a^\dagger a^{m+2}\rho a^{\dagger m+1} - a^{m+1}\rho a^{\dagger m+2}a \\ & \qquad + \tfrac{1}{4}\left(a^{\dagger 2}a^{m+2}\rho a^{\dagger m} + 2a^\dagger a^{m+1}\rho a^{\dagger m+1}a + a^m\rho a^{\dagger m+2}a^2\right)\Big],\end{aligned} \tag{6.43}$$

where the operators in the $\mathcal{D}^2$ term have been put in normal order for simplicity.

In this form, the dynamics for the self-exciting photodetection process can be contrasted with those from a previous model also intended to apply to the apparatus described above. The master equation postulated in Ref. [148] was

$$\dot\rho = \left\{\sum_{m=0}^{\infty}\lambda^m a^{m+1}\rho a^{\dagger m+1}\right\} - \tfrac{1}{2}\sum_{m=0}^{\infty}\lambda^m\left(a^{\dagger m+1}a^{m+1}\rho + \rho a^{\dagger m+1}a^{m+1}\right). \tag{6.44}$$

The first feedback term in this equation (enclosed in curly brackets) is identical to the corresponding term in the correct equation (6.43). This part of Eq. (6.44) was in fact the only feedback term which was "derived" in Ref. [148]. The form of the remaining terms was assumed (wrongly as it turns out) to follow automatically from that of the first term. It was these erroneous terms which lead to the prediction of nonclassical light generation by the feedback loop. In contrast, the corresponding terms in the correct equation (6.43) cannot lead to nonclassical states. In addition, there are the extra final terms in Eq. (6.43) which increase the photon number variance without significantly affecting the mean.

Equation (6.42) can be expressed using the the Glauber-Sudarshan $P(\alpha,\alpha^*)$ function. Ignoring the phase dynamics, only the photon number $n = |\alpha|^2$ need be considered, and $P(n)$ obeys

$$\dot P(n) = \left(\frac{\partial}{\partial n}\frac{n}{1-\lambda n} + \frac{1}{2}\frac{\partial^2}{\partial n^2}\frac{(\lambda n)^2}{(1-\lambda n)^3}n\right)P(n), \tag{6.45}$$

where this is valid only for $\lambda n < 1$. In this regime, the classicality of the self-exciting loss source is evident. The nonlinearity in the incorrect equation (6.44) is of a quantum nature, and can produce sub-Poissonian light. The wrong model is more like an intrinsically nonlinear loss source, such as atoms with multiphoton transitions [147], which was the analogy on which it was built. Nevertheless, it can be shown [145] that the self exciting loss source described by Eq. (6.45) can reduce classical intensity noise, such as produced by a laser with a noisy pump. However, it is unlikely that it would be a practical noise reduction device because it also depresses the photon number, and would be difficult to implement efficiently. More practical uses of feedback for noise reduction will be considered in Secs. 6.4 – 6.6.

6.3 Homodyne Detection Feedback

6.3.1 The Homodyne Feedback Master Equation

As shown in Sec. 4.2, the SME for homodyne detection of efficiency η is

$$d\rho_{\rm c}(t) = -{\rm i}[H,\rho_{\rm c}(t)]dt + \mathcal{D}[c]\rho_{\rm c}(t)dt + \sqrt{\eta}dW(t)\mathcal{H}[c]\rho_{\rm c}(t). \tag{6.46}$$

The homodyne photocurrent, normalized so that the deterministic part does not depend on the efficiency, is

$$I_{\rm c}^{\rm hom}(t) = \langle x\rangle_{\rm c}(t) + \xi(t)/\sqrt{\eta}, \tag{6.47}$$

where $\xi(t) = dW(t)/dt$ and $x = c+c^\dagger$ as usual. Unlike the direct detection photocurrent, this current may be negative because the constant local oscillator background has been subtracted. That means that if one were to feed back this current as in Sec. 6.1, with

$$[\dot\rho_{\rm c}(t)]_{\rm fb} = I_{\rm c}^{\rm hom}(t-\tau)\mathcal{K}\rho_{\rm c}(t), \tag{6.48}$$

then the superoperator $\mathcal{K}$ must be such as to give valid evolution irrespective of the sign of time. That is to say, it must give reversible evolution with

$$\mathcal{K}\rho \equiv -{\rm i}[F,\rho] \tag{6.49}$$

for some Hermitian operator F.

Because the stochasticity in the measurement (6.46) and the feedback (6.48) is Gaussian white noise, it is possible to directly apply the feedback theory developed for such processes in Sec. 3.4. Bearing in mind that the feedback must act after the measurement, and that Eq. (6.48) is a Stratonovich equation, the result for the total conditioned evolution of the system is

$$\begin{aligned}\rho_{\rm c}(t+dt) &= \left\{1+\mathcal{K}[\langle c+c^\dagger\rangle_{\rm c}(t-\tau)dt + dW(t-\tau)/\sqrt{\eta}] + \frac{1}{2\eta}\mathcal{K}^2dt\right\} \\ &\quad \times\left\{1+\mathcal{H}[-{\rm i}H]dt + \mathcal{D}[c]dt + \sqrt{\eta}dW(t)\mathcal{H}[c]\right\}\rho_{\rm c}(t).\end{aligned} \tag{6.50}$$

For τ finite, this becomes

$$\begin{aligned}d\rho_{\rm c}(t) &= dt\left\{\mathcal{H}[-{\rm i}H] + \mathcal{D}[c] + \langle c+c^\dagger\rangle_{\rm c}(t-\tau)\mathcal{K} + \frac{1}{2\eta}\mathcal{K}^2\right\}\rho_{\rm c}(t) \\ &\quad + dW(t-\tau)\mathcal{K}\rho_{\rm c}(t)/\sqrt{\eta} + \sqrt{\eta}dW(t)\mathcal{H}[c]\rho_{\rm c}(t).\end{aligned} \tag{6.51}$$

On the other hand, putting $\tau = 0$ in Eq. (6.50) gives

$$\begin{aligned}d\rho_{\rm c}(t) &= dt\left\{-{\rm i}[H,\rho_{\rm c}(t)] + \mathcal{D}[c]\rho_{\rm c}(t) - {\rm i}[F, c\rho_{\rm c}(t) + \rho_{\rm c}(t)c^\dagger] + \mathcal{D}[F]\rho_{\rm c}(t)/\eta\right\} \\ &\quad + dW(t)\mathcal{H}[\sqrt{\eta}c - {\rm i}F/\sqrt{\eta}]\rho_{\rm c}(t).\end{aligned} \tag{6.52}$$

For $\eta = 1$, this can be alternatively be expressed as a SSE. Ignoring normalization, this is simply

$$d|\bar\psi_{\rm c}(t)\rangle = dt\left[-{\rm i}H - \tfrac{1}{2}\left(c^\dagger c + 2{\rm i}Fc + F^2\right) + I_{\rm c}^{\rm hom}(t)\left(c - {\rm i}F\right)\right]|\bar\psi_{\rm c}(t)\rangle. \tag{6.53}$$

The nonselective evolution of the system is easier to find from the SME (6.52). This is a true Itô equation, so that taking the ensemble average simply removes the stochastic term. This gives the homodyne feedback master equation

$$\dot\rho = -{\rm i}[H,\rho] + \mathcal{D}[c]\rho - {\rm i}[F, c\rho + \rho c^\dagger] + \frac{1}{\eta}\mathcal{D}[F]\rho. \tag{6.54}$$

This equation was actually derived before Eq. (6.15), because the Gaussian nature of the noise was more familiar. Equation (6.54) is in fact identical with that derived by Caves and Milburn [29] for an idealized model for position measurement plus feedback, if c is replaced by x and η set to 1. Their model did not involve a stochastic continuous conditioning equation, but rather treated successive inaccurate position measurements, and took the continuum limit only after the feedback had been determined. The relationship of that measurement model to quantum trajectories was explored in Ref. [49], referred to in Sec. 5.1. In any case, the interpretation of the terms is the same. The first feedback term, linear in F, is the desired effect of the feedback which would dominate in the classical regime. The second feedback term causes diffusion in the variable conjugate to F. It can be attributed to the inevitable introduction of noise by the measurement step in the quantum-limited feedback loop. The lower the efficiency, the more noise introduced.

The homodyne feedback master equation can be rewritten in the Lindblad form (2.32) as

$$\dot{\rho} = -\mathrm{i}\left[H + \tfrac{1}{2}(c^\dagger F + Fc), \rho\right] + \mathcal{D}[c - \mathrm{i}F]\rho + \frac{1-\eta}{\eta}\mathcal{D}[F]\rho \equiv \mathcal{L}\rho. \tag{6.55}$$

In this arrangement, the effect of the feedback is seen to replace c by $c - \mathrm{i}F$, and to add an extra term to the Hamiltonian, plus an extra diffusion term which vanishes for perfect detection. In what follows, η will be assumed to be one unless otherwise stated, as the generalization is usually obvious from previous examples. The two-time correlation function of the current can be found from Eq. (6.52) using the method of Sec. 4.2 to be

$$\mathrm{E}[I_{\mathrm{c}}^{\mathrm{hom}}(t')I_{\mathrm{c}}^{\mathrm{hom}}(t)] = \mathrm{Tr}\left\{(c + c^\dagger)e^{\mathcal{L}(t'-t)}[(c - \mathrm{i}F)\rho(t) + \rho(t)(c^\dagger + \mathrm{i}F)]\right\} + \delta(\tau). \tag{6.56}$$

Again, note that the feedback affects the term in square brackets, as well as the evolution by $\mathcal{L}$ for time $t' - t$. This means that the in-loop photocurrent may have a sub-shot-noise spectrum, even if the light in the cavity dynamics is classical. From the same reasoning as in Sec. 6. 1, the feedback will not produce non-classical dynamics if F is a 'classical' operator, meaning that F is proportional to x, y, or $c^\dagger c$. This can be verified directly from Eq. (6.55).

6.3.2 Feedback with White Noise

From one point of view, the results just obtained are simply a special case of those of Sec. 6.1. Consider the quantum jump SME for homodyne detection with finite local oscillator, as in Eq. (4.44). Now add feedback according to

$$[\dot{\rho}_{\mathrm{c}}(t)]_{\mathrm{fb}} = -\mathrm{i}[F, \rho_{\mathrm{c}}(t)]\frac{dN_{\mathrm{c}}(t) - \gamma^2 dt}{\gamma dt}, \tag{6.57}$$

where this is understood to be the $\tau \to 0$ limit. Using Sec. 6.1, the feedback master equation is

$$\dot{\rho} = -\mathrm{i}\left[H + \mathrm{i}\tfrac{1}{2}(-c\gamma + c^\dagger\gamma) - F\gamma, \rho\right] + \mathcal{D}\left[e^{-\mathrm{i}F/\gamma}(c + \gamma)\right]\rho. \tag{6.58}$$

Expanding the exponential to second order in $1/\gamma$ and then taking the limit $\gamma \to \infty$ reproduces (6.55). The correlation functions follow similarly as a special case. However, there is another sense in which feedback based on homodyne detection is more general. That is, that it is possible to treat detection, and hence feedback, in the presence of thermal or squeezed white noise, as in Sec. 4.4.

The relevant conditioning equation for homodyne detection with a white noise bath parameterized by N and M as in Ref. [52] is Eq. (4.115)

$$d\rho_{\mathrm{c}}(t) = \left\{dt\mathcal{L} + \frac{1}{\sqrt{L}}dW(t)\mathcal{H}\left[(N + M^* + 1)c - (N + M)c^\dagger\right]\right\}\rho_{\mathrm{c}}(t), \tag{6.59}$$

where $L = 2N + M + M^* + 1$, and the photocurrent is

$$I_{\rm c}^{\rm hom}(t) = \langle c + c^\dagger \rangle_{\rm c}(t) + \sqrt{L}\xi(t). \tag{6.60}$$

Adding feedback as in Eq. (6.48), which is the same as introducing a feedback Hamiltonian

$$H_{\rm fb}(t) = F I_{\rm c}(t), \tag{6.61}$$

and following the method of Sec. 6.3.1 yields the master equation

$$\begin{aligned} \dot\rho &= (N+1)\left\{\mathcal{D}[c]\rho - {\rm i}[F, c\rho + \rho c^\dagger]\right\} + N\left\{\mathcal{D}[c^\dagger]\rho + {\rm i}[F, c^\dagger\rho + \rho c]\right\} \\ &\quad + M\left\{\tfrac{1}{2}[c^\dagger,[c^\dagger,\rho]] + {\rm i}[F,[c^\dagger,\rho]]\right\} + M^*\left\{\tfrac{1}{2}[c,[c,\rho]] - {\rm i}[F,[c,\rho]]\right\} \\ &\quad + L\mathcal{D}[F]\rho - {\rm i}[H,\rho]. \end{aligned} \tag{6.62}$$

For the case $N = M = 0$ this reduces to Eq. (6.54).

For completeness, I will briefly give the equivalent results for heterodyne detection. Here, there are two photocurrents, one for each quadrature, given by

$$I_{\rm c}^x(t) = \langle x \rangle_{\rm c}(t) + \sqrt{2L_x}\,\xi_x(t), \tag{6.63}$$
$$I_{\rm c}^y(t) = \langle y \rangle_{\rm c}(t) + \sqrt{2L_y}\,\xi_y(t), \tag{6.64}$$

where $L_x = L$ and $L_y = 2N+1-M-M^*$ as in Sec. 4.4. Let these currents be fed back simultaneously via the feedback Hamiltonian

$$H_{\rm fb}(t) = I_{\rm c}^x(t)F + I_{\rm c}^y(t)G, \tag{6.65}$$

where F and G are Hermitian. Then using Eq. (4.121) [analogous to Eq. (6.59)], one obtains the general heterodyne detection feedback master equation

$$\begin{aligned} \dot\rho &= (N+1)\left\{\mathcal{D}[c]\rho - {\rm i}[F, c\rho + \rho c^\dagger] - {\rm i}[G, -{\rm i}c\rho + {\rm i}\rho c^\dagger]\right\} \\ &\quad + N\left\{\mathcal{D}[c^\dagger]\rho + {\rm i}[F, c^\dagger\rho + \rho c] + {\rm i}[G, {\rm i}c^\dagger\rho - {\rm i}\rho c]\right\} \\ &\quad + M\left\{\tfrac{1}{2}[c^\dagger,[c^\dagger,\rho]] + {\rm i}[F,[c^\dagger,\rho]] - {\rm i}[G,[{\rm i}c^\dagger,\rho]]\right\} \\ &\quad + M^*\left\{\tfrac{1}{2}[c,[c,\rho]] - {\rm i}[F,[c,\rho]] + {\rm i}[G,[-{\rm i}c,\rho]]\right\} \\ &\quad + 2L_x\mathcal{D}[F]\rho + 2L_y\mathcal{D}[G]\rho - {\rm i}[H,\rho]. \end{aligned} \tag{6.66}$$

This expression for simultaneous measurement and feedback of both quadratures will be of interest when contrasted with an all-optical feedback scheme of the complex amplitude in Sec. 7.5.

6.4 Markovian Feedback in a Linear System

6.4.1 The Linear System

In order to understand the nature of quantum limited feedback, it is useful to consider an exactly solvable system. In the remainder of this chapter, I will be considering the case of a linear optical system, with linear feedback based on homodyne detection. By a linear system, I mean that the equations of motion for the two quadrature operators are linear. This is approximately the case for many quantum optical systems, in the limit of large photon numbers. For specificity, I will chose a system which is exactly linear. If, as in the remainder of this chapter, one is interested in the behaviour of one quadrature only (here the x quadrature), then all linear dynamics can be composed of damping, driving, and parametric driving. Damping will be assumed to be always present (as necessary to do feedback or obtain an output from the cavity) and will have rate 1. Constant linear driving simply shifts the origin away from $x = 0$, and will be ignored. Stochastic linear driving in the white noise approximation causes diffusion in the x quadrature, at a rate l. Finally, if the strength of the parametric driving ($H \sim xy$) is χ (where $\chi = 1$ would represent a degenerate parametric oscillator at threshold), then the master equation for the system is

$$\dot{\rho} = \mathcal{D}[a]\rho + \tfrac{1}{4}l\mathcal{D}[a^\dagger - a]\rho + \tfrac{1}{4}\chi[a^2 - a^{\dagger 2}, \rho] \equiv \mathcal{L}_0\rho, \tag{6.67}$$

where a is the annihilation operator for the cavity mode.

An alternative definition for the linearity of the x quadrature dynamics is that the marginal distribution of the Wigner function for x (which is the true probability distribution for x) obeys an Ornstein-Uhlenbeck equation. That is to say,

$$\dot{W}(x) = \left(\partial_x kx + \tfrac{1}{2}D\partial_x^2\right) W(x), \tag{6.68}$$

where k and D are constants. The solution of this equation is a Gaussian with variance

$$V = \frac{D}{2k}. \tag{6.69}$$

For the particular master equation above (the properties of which will be denoted by the subscript 0), the drift and diffusion constants are

$$k_0 = \tfrac{1}{2}(1+\chi), \tag{6.70}$$
$$D_0 = 1 + l. \tag{6.71}$$

In this case, $V_0 = (1+l)/(1+\chi)$. If this is less than unity, the system exhibits squeezing of x. It is more useful to work with the normally ordered variance, as in Sec. 5.4. Here, I will denote it

$$U = V - 1, \tag{6.72}$$

which for this system takes the value

$$U_0 = \frac{l - \chi}{1 + \chi}. \tag{6.73}$$

If the system is to stay below threshold (so that the y quadrature does not become unbounded), then the maximum value for χ is one. At this value, $U_0 = -1/2$ when the x diffusion rate $l = 0$. Therefore the minimum value of squeezing which this linear system can attain as a stationary value is half of the theoretical minimum of $U_0 = -1$.

In quantum optics, the output light is generally more useful than the intracavity light. Therefore it is necessary to compute the output noise statistics. For squeezed systems, the relevant quantity is the spectrum of the homodyne photocurrent [82]. This is defined as the Fourier transform of the autocorrelation function defined in Sec. 4.2.3:

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \mathrm{E}[I_{\mathrm{c}}^{\mathrm{hom}}(t+\tau) I_{\mathrm{c}}^{\mathrm{hom}}(t)] e^{-i\omega\tau}, \tag{6.74}$$

where the t dependence disappears at steady state. This spectrum is easily calculated for linear systems as I am considering here [52]. Given the Ornstein-Uhlenbeck coefficients for the Wigner function in Eq. (6.68), the spectrum is

$$S(\omega) = 1 + \frac{D - 2k}{\omega^2 + k^2}. \tag{6.75}$$

This consists of a constant term representing shot noise (recall that time is being measured in units of the decay time) plus a Lorentzian which will be negative for squeezed systems. The spectrum is related to the intracavity squeezing by

$$\int_{-\infty}^{\infty} d\omega[S(\omega)-1] = \frac{D-2k}{2k} = U. \tag{6.76}$$

That is, the total squeezing across all frequencies in the output is equal to the intracavity squeezing. However, the minimum squeezing (which for a linear system will occur at zero frequency) may be greater than or less than U. It is useful to define it by another parameter

$$R = S(0) - 1 = 2U/k. \tag{6.77}$$

For the particular system considered above,

$$R_0 = \frac{l-\chi}{\frac{1}{4}(1+\chi)^2}. \tag{6.78}$$

In the ideal limit ($\chi \to 1$, $l \to 0$), the zero frequency squeezing approaches the minimum value of -1.

6.4.2 Adding Linear Feedback

Now consider adding feedback to try to reduce the fluctuations in x. Restricting the feedback to classical processes suggests the feedback operator

$$F = -\lambda y/2. \tag{6.79}$$

As a separate Hamiltonian, this translates a state in the negative x direction for λ positive. By controlling this Hamiltonian by the homodyne photocurrent one thus has the ability to change the statistics for x and perhaps achieve better squeezing. Substituting Eq. (6.79) into the general homodyne feedback master equation (6.54) and adding the free dynamics (6.67) gives

$$\dot{\rho} = \mathcal{L}_0\rho + \frac{\lambda}{2}[a-a^\dagger, a\rho + \rho a^\dagger] + \frac{\lambda^2}{4\eta}\mathcal{D}[a-a^\dagger]\rho. \tag{6.80}$$

Here η is the proportion of output light used in the feedback loop, multiplied by the efficiency of the detection. For the Wigner function $W(x)$ one finds that the Fokker-Planck equation is still linear, but now with

$$\begin{aligned} k &= k_0 + \lambda, & (6.81)\\ D &= D_0 + 2\lambda + \lambda^2/\eta. & (6.82) \end{aligned}$$

Provided $\lambda + k_0 > 0$, there will exist a stable Gaussian solution to the master equation (6.80). The new intracavity squeezing parameter is

$$U_\lambda = (k_0+\lambda)^{-1}\left(k_0 U_0 + \frac{\lambda^2}{2\eta}\right). \tag{6.83}$$

An immediate consequence of this expression is that U_λ can only be negative if U_0 is. That is to say, the feedback cannot produce squeezing, as explained in Sec. 6.1. Minimizing U_λ with respect to λ one finds

$$U_{\min} = \eta^{-1}\left(-k_0 + \sqrt{k_0^2 + 2\eta k_0 U_0}\right), \tag{6.84}$$

when

$$\lambda = -k_0 + \sqrt{k_0^2 + 2\eta k_0 U_0}. \tag{6.85}$$

Note that this λ has the same sign as U_0. That is to say, if the system produces squeezed light, then the best way to enhance the squeezing is to add a force which displaces the state in the direction of the difference between the measured photocurrent and the desired mean photocurrent. This is the opposite of what would be expected classically, and can be attributed to the effect of homodyne measurement on squeezed states, as will be explained in Sec. 6.4.3. Obviously, the best intracavity squeezing will be when $\eta = 1$, in which case the intracavity squeezing can be simply expressed as

$$U_{\rm min} = k_0\left(-1 + \sqrt{1+R_0}\right). \tag{6.86}$$

Although classical feedback cannot produce squeezing, this does not mean that it cannot reduce noise. In fact, it can be proven that $U_{\rm min} \leq U_0$, with equality only if $\eta = 0$ or $U_0 = 0$. I use the result $\sqrt{1+R_0} \leq 1 + R_0/2$ since $R_0 \geq -1$. Recalling that $R_0 = 2U_0/k_0$, and comparing to Eq. (6.86) gives

$$U_{\rm min} \leq U_0 \tag{6.87}$$

for $\eta = 1$. Using the mean value theorem, it is easy to show that this is true for all η. This result implies that the intracavity variance in x can always be reduced by classical homodyne-mediated feedback, unless it is at the classical minimum. In particular, intracavity squeezing can always be enhanced. For the parametric oscillator defined originally in Eq. (6.67), with $l = 0$, $U_{\rm min} = -\chi/\eta$. For $\eta = 1$, the (symmetrically ordered) x variance is $V_{\rm min} = 1 - \chi$. The y variance, which is unaffected by feedback, is seen from Eq. (6.67) to be $(1-\chi)^{-1}$. Thus, with perfect detection, it is possible to produce a minimum uncertainty squeezed state with arbitrarily high squeezing as $\chi \to 1$. This is not unexpected as a parametric amplifier (in an undamped cavity) also produces minimum uncertainty squeezed states. The feedback removes the noise which was added by the damping which is necessary to do the measurement used in the feedback.

Next, I turn to the calculation of the output squeezing. Here, it must be remembered that at least a fraction η of the output light is being used in the feedback loop. Thus, the fraction θ of cavity emission available as an output of the system is at best $1-\eta$. Integrated over all frequencies, the total available output squeezing is thus θU_λ. I will show that

$$\theta U_{\rm min} \geq U_0 \;\; \text{for } U_0 < 0. \tag{6.88}$$

That is, dividing the cavity output and using some in a feedback loop produces worse squeezing in the remaining output than was present in the original, undivided output. Note however, that if the cavity output is inherently divided (which is often the case, with two output mirrors), then using one output in the feedback loop would enhance squeezing in the other output. This is because the squeezing in the system output of interest would have changed from θU_0 to $\theta U_{\rm min}$.

The proof of the Eq. (6.88) is as follows. Assume $U_0 \neq 0$, and let $\theta = 1 - \eta$ (since imperfect detection can only make the results worse). The condition $(1-\eta)U_{\rm min} = U_0$ gives $\sqrt{1+\eta R_0} = 1 + \eta R_0/[2(1-\eta)]$. For $R_0 \geq -1$ this has only one solution, namely $\eta = 0$. The condition $(1-\eta)U_{\rm min} = 0$ implies $\eta = 1$. Thus, by the intermediate value theorem, $(1-\eta)U_{\rm min}$ lies between 0 and U_0. Eq. (6.88) follows when $U_0 < 0$. For the other case, when $U_0 > 0$, the result implies that the feedback does reduce the noise in the output. Part of this reduction is simply due to reducing the fraction of light used as the system output, but part is due to the reduction of U_0 to $U_{\rm min}$.

As explained above, the quantity θU_λ represents the total output squeezing over all frequencies. Often experimentalists are more interested in the minimum noise reduction, which is at zero frequency here. With no feedback, this is given by R_0. With feedback, it is given by

$$R_\lambda = \theta 2U_\lambda/(k_0+\lambda) = \frac{\theta}{(k_0+\lambda)^2}\left(2k_0U_0 + \lambda^2/\eta\right). \tag{6.89}$$

In all cases, R_λ is minimized for a different value of λ from that which minimizes U_λ. One finds

$$R_{\rm min} = R_0\frac{\theta}{1+R_0\eta} \tag{6.90}$$

when

$$\lambda = 2\eta U_0. \tag{6.91}$$

Again, λ has the same sign as U_0. It follows immediately from Eq. (6.90) that, since $R_0 \geq -1$ and $\theta \leq 1-\eta$,

$$R_{\rm min} \geq R_0 \ \text{ for } R_0 < 0. \tag{6.92}$$

That is to say, dividing the cavity output to add a homodyne-mediated classical feedback loop cannot produce better output squeezing at any frequency than would be available from an undivided output with no-feedback. These "no-go" theorems are not surprising given the approximate traveling-wave results of Shapiro *et al* [126], and the experimental results of Yamamoto, Imoto and Machida [155].

There is one case in which adding feedback does not degrade the output squeezing: if the no-feedback output noise is zero ($R_0 = -1$). In that case, the noise added by dividing the output can be exactly offset by the feedback, providing that the detection efficiency is unity. However, the bandwidth of the squeezing is reduced from k_0 to $k_0 + 2\eta U_0 = 0$. In all other cases ($-1 < R_0 < 0$), the noise added is greater and squeezing is degraded. Of course, if the original output had classical noise ($R_0 > 0$), then the feedback can reduced these fluctuations. It is a matter of minor interest that, regardless of the system state, the following relation holds:

$$R_\lambda = R_{\rm min} \implies U_\lambda = U_0. \tag{6.93}$$

That is to say, when the output noise is minimized, there is no change in the intracavity noise from the no-feedback case.

6.4.3 Understanding Feedback as Conditioning

The preceding section gave the limits to noise reduction by classical feedback for a linear system, both intracavity and extracavity. In this section, I give an explanation for the intracavity results, in terms of the conditioning of the state by the measurement on which the feedback is based. This is the link between conditioning and feedback referred to in the final paragraph of Sec. 5.5.2. To find this link, it is necessary to return to the selective stochastic master equation (6.50) for the conditioned state matrix $\rho_{\rm c}(t)$

$$d\rho_{\rm c}(t) = dt\left(\mathcal{L}_0\rho_{\rm c}(t) + \mathcal{K}[a\rho_{\rm c}(t)+\rho_{\rm c}(t)a^\dagger] + \frac{1}{2\eta}\mathcal{K}^2\rho_{\rm c}(t)\right) + dW(t)\left(\sqrt{\eta}\mathcal{H}[a] + \mathcal{K}/\sqrt{\eta}\right)\rho_{\rm c}(t). \tag{6.94}$$

Here, $\mathcal{L}_0$ is as defined in Eq. (6.67), and $\mathcal{K}\rho = -{\rm i}[F,\rho]$ where F is defined in Eq. (6.79). Changing this to a stochastic Liouville equation for the conditioned Wigner function gives

$$\begin{aligned} dW_{\rm c}(x) \;&=\; dt\left[\partial_x(k_0+\lambda)x + \frac{1}{2}\partial_x^2\left(D_0 + 2\lambda + \lambda^2/\eta\right)\right]W_{\rm c}(x) \\ &\quad + dW(t)\left[\sqrt{\eta}\left(x - \bar{x}_{\rm c}(t) + \partial_x\right) + (\lambda/\sqrt{\eta})\partial_x\right]W_{\rm c}(x), \end{aligned} \tag{6.95}$$

where $\bar{x}_{\rm c}(t)$ is the mean of the distribution $W_{\rm c}(x)$ and $dW(t)$ is as usual.

This equation is obviously no longer a simple Ornstein-Uhlenbeck equation. Nevertheless, it still has a Gaussian as an exact solution. This is similar to the case of the conditioned phase distribution for the laser considered in Sec. 5.5.2. Using the method of that section, the mean $\bar{x}_{\rm c}$ and variance $V_{\rm c}$ of the conditioned distribution are found to obey

$$\dot{\bar{x}}_{\rm c} = -(k_0+\lambda)\bar{x}_{\rm c} + \xi(t)\left[\sqrt{\eta}\,(V_{\rm c}-1) - (\lambda/\sqrt{\eta})\right], \tag{6.96}$$

$$\dot{V}_{\rm c} = -2k_0 V_{\rm c} + D_0 - \eta\,(V_{\rm c}-1)^2. \tag{6.97}$$

Two points about the evolution equation for $V_{\rm c}$ are worth noting. It is completely deterministic (no noise terms), and it is not influenced by the presence of feedback. Furthermore, for this linear system, it is independent of $\bar{x}_{\rm c}$. Thus, the stochasticity and feedback terms in the equation for the mean do not even enter that for the variance indirectly.

The equation for the conditioned variance is more simply written in terms of the conditioned normally ordered variance $U_{\rm c} = V_{\rm c} - 1$

$$\dot{U}_{\rm c} = -2k_0 U_{\rm c} - 2k_0 + D_0 - \eta U_{\rm c}^2. \tag{6.98}$$

On a time scale as short as a cavity lifetime, $U_{\rm c}$ will approach its stable steady-state value of

$$U_{\rm c} = \eta^{-1}\left(-k_0 + \sqrt{k_0^2 + \eta(-2k_0+D_0)}\right) \tag{6.99}$$

Note that this is equal to the minimum unconditioned $U_{\rm min}$ (6.84) with feedback. The explanation for this will become evident shortly. Substituting the steady-state conditioned variance into Eq. (6.96) gives

$$\dot{\bar{x}}_{\rm c} = -(k_0+\lambda)\bar{x}_{\rm c} + \xi(t)\frac{1}{\sqrt{\eta}}\left[-k_0 + \sqrt{k_0^2+\eta(-2k_0+D_0)} - \lambda\right]. \tag{6.100}$$

If one were to choose $\lambda = -k_0 + \sqrt{k_0^2+\eta(-2k_0+D_0)}$ then there would be no noise at all in the conditioned mean and so one could set $\bar{x}_{\rm c} = 0$. This value of λ is precisely that value derived as Eq. (6.85) to minimize the unconditioned variance under feedback. Now one can see why this minimum unconditioned variance is equal to the conditioned variance. The feedback works simply by suppressing the fluctuations in the conditioned mean.

In general, the unconditioned variance will consist of two terms, the conditioned quantum variance in x plus the classical (ensemble) average variance in the conditioned mean of x:

$$U_\lambda = U_{\rm c} + {\rm E}[\bar{x}_{\rm c}^2]. \tag{6.101}$$

The latter term is found from Eq. (6.100) to be

$${\rm E}[\bar{x}_{\rm c}^2] = \eta^{-1}\frac{1}{2(k_0+\lambda)}\left[-(k_0+\lambda) + \sqrt{k_0^2+\eta(-2k_0+D_0)}\right]^2. \tag{6.102}$$

Adding Eq. (6.99) gives

$$U_\lambda = \eta^{-1}\frac{1}{2(k_0+\lambda)}[\lambda^2 + \eta(-2k_0+D_0)]. \tag{6.103}$$

It can easily be verified that this is identical to the expression (6.83) derived in the preceding subsection using the unconditioned master equation. In this context, the explanation for the feedback is obvious. The homodyne measurement reduces the conditioned variance (except when it is equal to the classical minimum of 1). The more efficient the measurement, the greater the reduction. Ordinarily, this reduced variance is not evident because the measurement gives a random shift to the conditional mean of x, with the randomness arising from the shot noise of the photocurrent. By appropriately feeding back this photocurrent, it is possible to precisely counteract this shift and thus observe the conditioned variance.

The sign of the feedback parameter λ is determined by the sign of the shift which the measurement gives. This was investigated in Sec. 5.4.2. For classical statistics ($U \geq 0$) , a higher than average photocurrent reading [$\xi(t) > 0$] leads to the conditioned mean x increasing (except if $U = 0$ in which case the measurement has no effect). However, for nonclassical states with $U < 0$, the classical intuition fails as a positive photocurrent fluctuation causes $\bar{x}_c$ to decrease. This explains the counterintuitive negative value of λ required in squeezed systems, which naïvely would be thought to destabilize the system and increase fluctuations. However, the value of the positive feedback required (6.85) is such that the overall restoring force $k_0 + \lambda$ is still positive.

Succinctly, one can state that conditioning can be made practical by feedback. The intracavity noise reduction produced by classical feedback can be precisely as good as that produced by conditioning. This reinforces the simple explanation as to why homodyne-mediated classical feedback cannot produce nonclassical states: because homodyne detection cannot. Nonclassical feedback (such as using the photocurrent to influence nonlinear intracavity elements) may produce nonclassical states, but such elements can produce nonclassical states without feedback, so this is hardly surprising. In order to produce nonclassical states by classical feedback, it would be necessary to have a nonclassical measurement scheme. That is to say, one which does not rely on measurement of the extracavity light to procure information about the intracavity state. Intracavity measurements (in particular, quantum non-demolition measurements) are not limited by the random process of damping to the external continuum. The extra term which the measurement introduces into the nonselective master equation will not produce nonclassical states, but may allow the measurement to produce nonclassical *conditioned* states. One would thus expect that intracavity QND measurements would enable feedback to overcomes the classical limit. In Sec. 6.6, I show this explicitly.

6.5 Non-Markovian Feedback in a Linear System

The results obtained in the preceding section are valid only when the delay time is negligible. It is useful to consider the more general case of non-Markovian feedback for two reasons. First, it is necessary to determine more precisely what a 'negligible' delay is. Secondly, many experiments may operate with a non-negligible delay, and indeed the overall response of the feedback loop may be important. There are two approaches to solving this problem. The first is to derive an approximate master equation which incorporates the effect of a small but non-negligible delay, as in Sec. 6.1.3. This will satisfy the first reason given above, but is limited to short delays and so cannot treat the general case of arbitrary linear feedback. The second approach is to attempt an exact solution using non-Markovian quantum trajectories. The fact that the conditioned equations for feedback in the Markovian limit can be solved exactly (as in Sec. 6.4.3) suggests that this could be possible also for non-Markovian feedback. In this section I consider both of these approaches, and show that they agree in the limit of short delays. The overall effect of such a delay is to degrade the noise-reducing ability of the feedback loop.

6.5.1 The Short Delay Approximation

Consider the general master equation for homodyne feedback in the Markovian ($\tau = 0$) limit

$$\dot{\rho} = \mathcal{L}_0\rho - \mathrm{i}[F, c\rho + \rho c^\dagger] + \eta^{-1}\mathcal{D}[F]\rho \equiv \mathcal{L}\rho, \tag{6.104}$$

where $\mathcal{L}_0$ is the Liouville superoperator in the absence of feedback. Using the same argument as in Sec. 6.1.3, the approximate master equation for small but non-zero delay τ is found to be

$$\begin{aligned}\dot{\rho} &= \mathcal{L}_0\rho - \mathrm{i}[F, c\rho + \rho c^\dagger] + \eta^{-1}\mathcal{D}[F]\rho \\ &\quad + \tau(-\mathrm{i})[F, \mathcal{L}(c\rho + \rho c^\dagger) - c(\mathcal{L}\rho) - (\mathcal{L}\rho)c^\dagger].\end{aligned} \tag{6.105}$$

For the case of linear feedback on a cavity mode considered above, $F = -\lambda y/2$ and $\mathcal{L}_0$ is given in Eq. (6.67).

For the system considered in the preceding section, the superoperator generating the Markovian feedback evolution (6.104) acts on the marginal probability distribution for x according to the Liouville differential operator $\mathcal{L}$ defined by

$$\mathcal{L}W(x) = \left[(k_0 + \lambda)\partial_x x + \tfrac{1}{2}(D_0 + 2\lambda + \lambda^2/\eta)\partial_x^2\right] W(x). \tag{6.106}$$

The correspondence

$$a\rho + \rho a^\dagger \to (x + \partial_x)\, W(x) \tag{6.107}$$

then allows one to write down the equation of motion for $W(x)$ corresponding to the approximate master equation (6.105)

$$\dot{W}(x) = (\mathcal{L} + \tau\lambda\partial_x[\mathcal{L}, x + \partial_x])\, W(x). \tag{6.108}$$

Evaluating the commutators of the differential operators yields another Ornstein-Uhlenbeck equation for W, but this time with

$$k = (k_0 + \lambda)(1 + \lambda\tau), \tag{6.109}$$

$$D = (D_0 + 2\lambda + \lambda^2/\eta)(1 + 2\lambda\tau) - 2\lambda\tau(k_0 + \lambda). \tag{6.110}$$

Evidently, the condition that the loop delay τ be negligible is simply

$$\lambda\tau \ll 1. \tag{6.111}$$

These equations give the new expression for the intracavity squeezing as

$$U_{\lambda,\tau} = \frac{(D_0 + 2\lambda + \lambda^2/\eta)(1 + 2\lambda\tau) - 2\lambda\tau(k_0 + \lambda) - 2(k_0 + \lambda)(1 + \lambda\tau)}{2(k_0 + \lambda)(1 + \lambda\tau)}. \tag{6.112}$$

To first order in $\lambda\tau$, this is related to the zero delay result by

$$U_{\lambda,\tau} = U_\lambda(1 + \lambda\tau). \tag{6.113}$$

For squeezed systems, with $U_0 < 0$, the optimum value of U_λ occurs for λ negative, as shown in Sec. 6.4.2. Thus, the time delay reduces the total squeezing by the factor $(1 + \lambda\tau)$. On the other hand, classical noise is reduced to $U_\lambda > 0$ with λ positive, so that the total noise is increased by the factor $(1 + \lambda\tau)$. Overall, the time delay degrades the effectiveness of the feedback, as expected.

For the output spectrum, the expression (6.75) can still be used. The result is

$$S_\tau(\omega) = 1 + \theta\frac{(D_0 - 2k_0 + \lambda^2/\eta)(1 + 2\lambda\tau)}{\omega^2 + [(k_0 + \lambda)(1 + \lambda\tau)]^2}. \tag{6.114}$$

Again to first order in $\lambda\tau$, this is simply related to the spectrum $S(\omega)$ with zero delay by

$$S_\tau(\omega) = S\big(\omega(1-\lambda\tau)\big). \tag{6.115}$$

That is to say, if λ is positive then the bandwidth of the spectrum is increased, if negative it is decreased. It is not difficult to see that this implies that the noise is always increased by the time delay. The zero frequency noise $R = S(0) - 1$ is unaffected by the time delay. This is as expected, since it corresponds to integration of the photocurrent over long times for which a finite time delay should have no effect.

6.5.2 Exact Solution by Quantum Trajectories

The second method for calculating the effect of a loop delay is to use quantum trajectories to describe the feedback, with the delay explicitly present. The quantity which is of most interest to calculate is the output squeezing spectrum, as this is what is easily measured by homodyne detection. If one is to use conditioned states to describe the feedback mechanism, then it is also necessary to include the effect of conditioning by this second, independent homodyne detection. The efficiency of this measurement is equal to θ, where θ is at most $1-\eta$, as before. The division of the cavity output into the system output and the feedback loop is most easily visualized as arising from a cavity with two end mirrors. The feedback and free photocurrents are denoted respectively by

$$\begin{aligned} I_{\rm c}(t) &= \langle a + a^\dagger \rangle_{\rm c}(t) + \xi(t)/\sqrt{\eta}, & (6.116)\\ J_{\rm c}(t) &= \langle a + a^\dagger \rangle_{\rm c}(t) + \chi(t)/\sqrt{\theta}, & (6.117)\end{aligned}$$

where $\xi(t)$ and $\chi(t)$ are two independent Gaussian white noise terms. Using the photocurrent $I_{\rm c}(t)$ in a feedback loop with time delay τ gives the general conditioning master equation for the cavity mode

$$d\rho_{\rm c}(t) = dt\left\{\mathcal{L}_0 + [\sqrt{\eta}\xi(t) + \sqrt{\theta}\chi(t)]\mathcal{H}[a] + [\langle a+a^\dagger\rangle_{\rm c}(t-\tau) + \xi(t-\tau)/\sqrt{\eta}]\mathcal{K} + \frac{1}{2\eta}\mathcal{K}^2\right\}\rho_{\rm c}(t). \tag{6.118}$$

The intent is to use this equation to calculate the spectrum of fluctuations in the system output photocurrent $J_{\rm c}(t)$.

Such a calculation would be very difficult in general. However, for a linear system it turns out to be quite tractable. In this case, the stochastic master equation (6.118) can be replaced by the stochastic Liouville equation for the Wigner function

$$\begin{aligned} dW_{\rm c}(x;t) &= \Big\{\partial_x k_0 x + \tfrac{1}{2}\partial_x^2 D_0 + \Big[\sqrt{\eta}\xi(t) + \sqrt{\theta}\chi(t)\Big]\left[x - \bar{x}_{\rm c}(t) + \partial_x\right] \\ &\quad + \left[\bar{x}_{\rm c}(t-\tau) + \xi(t-\tau)/\sqrt{\eta}\right]\lambda\partial_x + \tfrac{1}{2}(\lambda^2/\eta)\partial_x^2\Big\}W_{\rm c}(x;t)dt. & (6.119)\end{aligned}$$

Using the usual Gaussian ansatz, one finds the following equations for the conditional mean and variance in x

$$\begin{aligned} \dot{\bar{x}}_{\rm c}(t) &= -k_0\bar{x}_{\rm c}(t) + \Big[\sqrt{\eta}\xi(t) + \sqrt{\theta}\chi(t)\Big]\left[V_{\rm c}(t) - 1\right] \\ &\quad -\lambda\left[\bar{x}_{\rm c}(t-\tau) + \xi(t-\tau)/\sqrt{\eta}\right], & (6.120)\\ \dot{V}_{\rm c}(t) &= -2k_0V_{\rm c}(t) + D_0 - (\eta+\theta)\left[V_{\rm c}(t) - 1\right]^2. & (6.121)\end{aligned}$$

Once again, the differential equation for the conditioned variance (6.121) is closed and deterministic, and in addition it is Markovian. In fact, it is identical to Eq. (6.97) with the replacement of η

by $\eta+\theta$. The steady-state solution is defined as follows, using $U_{\rm c} = V_{\rm c} - 1$

$$(\eta+\theta)U_{\rm c}^2 + 2k_0 U_{\rm c} = 2k_0 U_0, \tag{6.122}$$

where U_0 is as before (6.73). Substituting the steady-state solution into Eq. (6.120) gives the closed, stochastic, non-Markovian differential equation for the conditioned mean

$$\dot{\bar{x}}_{\rm c}(t) = -k_0\bar{x}_{\rm c}(t) + \left[\sqrt{\eta}\xi(t) + \sqrt{\theta}\chi(t)\right]U_{\rm c} - \lambda\left[\bar{x}_{\rm c}(t-\tau) + \xi(t-\tau)/\sqrt{\eta}\right]. \tag{6.123}$$

This is easily solved by converting to the frequency domain. Denoting Fourier transforms by a tilde, so

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} e^{{\rm i}\omega t}\bar{x}_{\rm c}(t)dt, \tag{6.124}$$

Eq. (6.123) gives

$$\tilde{x}(\omega) = \frac{(\sqrt{\eta}U_{\rm c} - \lambda e^{{\rm i}\omega\tau}/\sqrt{\eta})\tilde{\xi}(\omega) + \sqrt{\theta}U_{\rm c}\tilde{\chi}(\omega)}{-{\rm i}\omega + k_0 + \lambda e^{{\rm i}\omega\tau}}. \tag{6.125}$$

Here $\tilde{\xi}(\omega)$ is a complex white noise term satisfying

$$\begin{aligned}\tilde{\xi}(-\omega) &= \tilde{\xi}(\omega)^*, &(6.126)\\ {\rm E}[\tilde{\xi}(\omega)\tilde{\xi}(\omega')] &= 2\pi\delta(\omega+\omega'), &(6.127)\end{aligned}$$

and $\tilde{\chi}(\omega)$ is an independent white noise term satisfying identical relations.

Now, the output squeezing spectrum for x which I wish to determine is

$$S(\omega) = 2\theta\int_0^{\infty} dt'\cos\omega t'{\rm E}[J_{\rm c}(t+t')J_{\rm c}(t)], \tag{6.128}$$

which is normalized to be one at high frequencies. By the Fourier transform theorem, this can be written as

$$S(\omega) = e^{-{\rm i}\omega t}\frac{\theta}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{{\rm i}\omega' t}{\rm E}[\tilde{J}(\omega)\tilde{J}(\omega')^*], \tag{6.129}$$

where, from Eq. (6.117),

$$\tilde{J}(\omega) = \frac{(\sqrt{\eta}U_{\rm c} - \lambda e^{{\rm i}\omega\tau}/\sqrt{\eta})\tilde{\xi}(\omega) + \sqrt{\theta}U_{\rm c}\tilde{\chi}(\omega)}{-{\rm i}\omega + k_0 + \lambda e^{{\rm i}\omega\tau}} + \tilde{\chi}(\omega)/\sqrt{\theta}. \tag{6.130}$$

Using the relations (6.126,6.127) gives

$$S(\omega) = 1 + \theta\frac{(\eta+\theta)U_{\rm c}^2 + \lambda^2/\eta + 2k_0U_{\rm c}}{(k_0+\lambda\cos\omega\tau)^2 + (\omega - \lambda\sin\omega\tau)^2}. \tag{6.131}$$

Substituting the expression for $U_{\rm c}$ (6.122) gives finally

$$S(\omega) = 1 + \theta\frac{2k_0U_0 + \lambda^2/\eta}{(k_0+\lambda\cos\omega\tau)^2 + (\omega - \lambda\sin\omega\tau)^2}. \tag{6.132}$$

Putting $\tau = 0$ into this formula reproduces the Lorentzian noise spectrum generated in Sec. 6.4.2. As with the approximate master equation approach of the preceding subsection, the zero frequency noise is unaffected by the delay. Furthermore, in the small τ limit, one obtains

$$S(\omega) = 1 + \theta\frac{2k_0U_0 + \lambda^2/\eta}{(k_0+\lambda)^2 + [\omega(1-\lambda\tau)]^2} \tag{6.133}$$

for $\omega\tau \ll 1$, which will be satisfied by all values of ω for which the fraction is significant. This is identical to the result obtained in the preceding section, and thus justifies the approximate master equation approach. The effect of the delay in degrading the noise reduction is shown in Fig. 6.1. There I have plotted the output noise spectrum for the squeezed quadrature of a below threshold parametric oscillator. In terms of the original parameters of Eq. (6.67), $l = 0$ and $\chi = 2/3$. The four cases are: (a) with an undivided output and no feedback; (b) with an output divided in half but still no feedback (so that the squeezing is degraded by a factor of one half); (c) with the loop completed ($\eta = \theta = 1/2$) with optimal feedback and no time delay (showing that much of the squeezing is restored, especially at low frequencies); and (d) with optimal feedback but a time delay of $\tau = 2.5$ (showing that the bandwidth of the squeezing is reduced and the spectrum is non-Lorentzian).

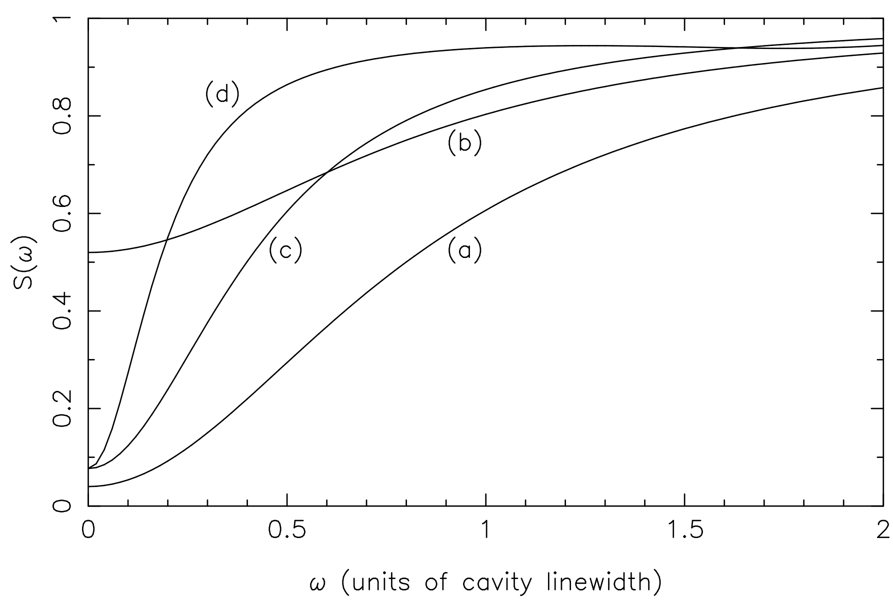

Figure 6.1: Squeezing spectra for a degenerate parametric oscillator with a classical pump amplitude of two-thirds the threshold amplitude, under various conditions: (a) free running in a single sided cavity; (b) free running in cavity with two mirrors of equal transmittivities; (c) in such a two-sided cavity with the instantaneous homodyne photocurrent from one end used optimally to control the amplitude of a coherent driving field; and (d) as in (c) but with a finite time delay in the feedback loop of 2.5 times the cavity lifetime.

So far, I have considered feeding back a time-delayed, but otherwise unchanged photocurrent. In fact, analogous results hold for the feedback of any signal linear in the photocurrent. Assuming a feedback term of the form

$$[\dot{\rho}_{\rm c}(t)]_{\rm fb} = \mathcal{K}\rho_{\rm c}(t)\int_0^\infty h(s)I_{\rm c}(t-s)ds, \tag{6.134}$$

where $\int_0^\infty h(s)ds = 1$, the equation for the stationary conditioned variance (6.122) is unchanged, while the conditioned mean obeys

$$\dot{\bar{x}}_{\rm c}(t) = -k_0\bar{x}_{\rm c}(t) + \left[\sqrt{\eta}\xi(t) + \sqrt{\theta}\chi(t)\right]U_{\rm c} - \lambda\int_0^\infty h(s)\left[\bar{x}_{\rm c}(t-s) + \xi(t-s)/\sqrt{\eta}\right]ds. \tag{6.135}$$

Proceeding as above yields the free spectrum

$$S(\omega) = 1 + \theta \frac{2k_0 U_0 + \lambda^2 |\tilde{h}(\omega)|^2/\eta}{|-\mathrm{i}\omega + k_0 + \lambda \tilde{h}(\omega)|^2}. \tag{6.136}$$

This reproduces the short delay expression (6.133) for

$$\tilde{h}(\omega) \simeq 1 + \mathrm{i}\omega\tau \simeq \exp(\mathrm{i}\omega\tau) \simeq (1 - \mathrm{i}\omega\tau)^{-1}. \tag{6.137}$$

This indicates that the parameter τ could represent a time delay, or, equally as well, the time constant for a decaying exponential response function in time.

6.6 QND Detection Feedback

6.6.1 The QND Detection Model

As explained at the end of Sec. 6.4, one would expect that the "no-go" theorems for the production of squeezing via linear feedback could be overcome by using intracavity QND measurements. In this section, I verify that this is the case, and find the limits to squeezing via feedback based on QND measurements. The natural choice of quantum non-demolition variable is the quadrature to be squeezed, say x as before. I present a simple model for an intracavity QND measurement of x, based on the model of Ref. [1]. In addition to the system mode with annihilation operator a, the cavity is assumed to support a second distinct mode with annihilation operator b. The two modes are coupled by the following nonlinear Hamiltonian (in the interaction picture):

$$H = gx\tfrac{1}{4}(-\mathrm{i}b + \mathrm{i}b^\dagger). \tag{6.138}$$

This could be achieved by, for example, a crystal with a $\chi^{(2)}$ nonlinearity in which two processes driven by classical fields, amplification ($\varepsilon_s^* ab$ + H.c., with $\omega_s = \omega_a + \omega_b$) and frequency conversion ($\varepsilon_d^* ab^\dagger$ + H.c., with $\omega_d = \omega_a - \omega_b$) have equal strengths. Mode b is damped through an output mirror at rate γ, and may be detected. The internal dynamics of mode a are defined as before by its Liouville superoperator $\mathcal{L}_0$. The density operator for both modes thus obeys the following master equation:

$$\dot{R} = \mathcal{L}_0 R - \frac{g}{4}[x(b - b^\dagger), R] + \gamma\mathcal{D}[b]R. \tag{6.139}$$

In order to treat mode b as part of the apparatus rather than part of the system, it is necessary to eliminate its dynamics. This can be done by assuming that it is heavily damped, with γ much larger than all other rates. Then, apart from initial transients, it will have few photons and will be slaved to mode a. Almost all of the probability will reside in the vacuum state for mode b. Thus it is possible to express the total state matrix R as an expansion around this state in the small parameter γ^{-1} (*cf* App. C)

$$R = \rho_0 \otimes |0\rangle_b\langle 0| + (\rho_1 \otimes |1\rangle_b\langle 0| + \mathrm{H.c.}) + \rho_2 \otimes |1\rangle_b\langle 1| + (\rho_{2'} \otimes |2\rangle_b\langle 0| + \mathrm{H.c.}) + O(\gamma^{-3}). \tag{6.140}$$

Substituting this expansion into Eq. (6.139) yields

$$\begin{aligned}
\dot{\rho}_0 &= \mathcal{L}_0\rho_0 - \frac{\mathrm{i}g}{4}(x\rho_1 - \rho_1^\dagger x) + \gamma\rho_2 + O(\gamma^{-2}), && (6.141)\\
\dot{\rho}_1 &= -\frac{\mathrm{i}g}{4}x\rho_0 - \frac{\gamma}{2}\rho_1 + O(\gamma^{-1}), && (6.142)\\
\dot{\rho}_2 &= -\frac{\mathrm{i}g}{4}(x\rho_1^\dagger - \rho_1 x) - \gamma\rho_2 + O(\gamma^{-2}). && (6.143)
\end{aligned}$$

Here I have ignored the equation for $\rho_{2'}$ because $\rho_{2'}$ does not appear in the equations shown.

Examination of Eq. (6.142) reveals that ρ_1 can be slaved to ρ_0 as

$$\rho_1 = -\frac{ig}{2\gamma} x\rho_0 + O(\gamma^{-2}). \tag{6.144}$$

This confirms the expansion (6.140) above. Also, because ρ_1 now evolves with ρ_0, ρ_2 will quickly damp to its stationary value with respect to ρ_1. Substituting Eq. (6.144) into Eq. (6.143) yields

$$\rho_2 = \frac{g^2}{4\gamma^2} x\rho_0 x + O(\gamma^{-3}). \tag{6.145}$$

Substituting this and Eq. (6.144) into Eq. (6.141) gives the master equation for $\rho_0 \simeq \rho$ the density operator for mode a alone

$$\dot\rho = \mathcal{L}_0\rho + \Gamma\mathcal{D}[x/2]\rho, \tag{6.146}$$

where the measurement strength parameter is $\Gamma = g^2/\gamma$. The superoperator $\mathcal{D}[x/2]$ has the form of a double commutator which routinely arises in QND measurements.

Now add homodyne measurement of the b mode with efficiency η. The conditioned density operator $R_{\rm c}$ obeys

$$dR_{\rm c} = dt\left(\mathcal{L}_0 R_{\rm c} - \frac{g}{4}[x(b-b^\dagger), R_{\rm c}] + \gamma\mathcal{D}[b]R_{\rm c}\right) + dW(t)\sqrt{\gamma\eta}\mathcal{H}[b]R_{\rm c}, \tag{6.147}$$

with the homodyne photocurrent

$$I_{\rm c}(t) = \langle b + b^\dagger\rangle_{\rm c}(t) + \xi(t)/\sqrt{\eta\gamma}. \tag{6.148}$$

Substituting in the solution (6.140) with ρ_1 given by its slaved value (6.144) gives the conditioning master equation for $\rho_{\rm c}$

$$d\rho_{\rm c} = dt\mathcal{L}_0\rho_{\rm c} + dt\Gamma\mathcal{D}[x/2]\rho_{\rm c} + \sqrt{\eta\Gamma}dW(t)\mathcal{H}[x/2]\rho_{\rm c}. \tag{6.149}$$

Normalizing the homodyne photocurrent so that the deterministic part is the same as in preceding sections gives

$$I_{\rm c}(t) = \langle x\rangle_{\rm c}(t) + \xi(t)/\sqrt{\mathrm{H}}. \tag{6.150}$$

Here I am using H (a capital η) for $\eta\Gamma$ as the effective efficiency of the measurement. Note that this is not bounded above by unity, since it is possible for g^2/γ to be much greater than one even with g much less than γ. Recall that all rates are measured in units of the a mode linewidth.

6.6.2 QND Feedback in a Linear System

The photocurrent (6.150) can be used in feedback onto the a mode just as in preceding sections. A feedback term of the form

$$[\dot\rho_{\rm c}]_{\rm fb} = I_{\rm c}(t-\tau)\mathcal{K}\rho_{\rm c} \tag{6.151}$$

gives, in the limit $\tau \to 0$, the nonselective master equation

$$\dot\rho = \mathcal{L}_0\rho + \Gamma\mathcal{D}[x/2]\rho + \mathcal{K}\tfrac{1}{2}[x\rho + \rho x] + \frac{1}{2\mathrm{H}}\mathcal{K}^2\rho. \tag{6.152}$$

This differs from Eq. (6.80) in that it has a QND measurement term, and a different feedback drift term. The feedback diffusion term retains its previous form, but the efficiency of the feedback loop η is replaced by H. Because H can be arbitrarily large, the noise associated with the feedback can be arbitrarily small. As will be shown soon, this allows the production of arbitrarily squeezed states. Of course, the quantum noise has not been eliminated but rather redistributed. For H to be large requires Γ to be large also, so that the variance in the unsqueezed quadrature is greatly increased by the measurement term in Eq. (6.152). This ensures that Heisenberg's uncertainty principle is not violated. In what follows, I am concerned only with the statistics of the squeezed quadrature x.

With superoperators $\mathcal{L}_0$ and $\mathcal{K}$ defined as in Sec. 6.4, Eq. (6.152) gives the following Ornstein-Uhlenbeck equation for the probability distribution for x:

$$\dot{W}(x) = \left[\partial_x (k_0 + \lambda) x + \frac{1}{2}\partial_x^2 \left(D_0 + \frac{\lambda^2}{\mathrm{H}}\right)\right] W(x). \tag{6.153}$$

From this, the intracavity squeezing parameter is

$$U_\lambda = -1 + \frac{D_0 + \lambda^2/\mathrm{H}}{2(k_0 + \lambda)}. \tag{6.154}$$

Minimizing with respect to λ yields

$$U_{\min} = -1 + \mathrm{H}^{-1}\left(-k_0 + \sqrt{k_0^2 + \mathrm{H}D_0}\right) \tag{6.155}$$

when

$$\lambda = -k_0 + \sqrt{k_0^2 + \mathrm{H}D_0}. \tag{6.156}$$

In the limit $\mathrm{H} \to \infty$, it is easy to see that $U_{\min}$ approaches the theoretical minimum value of -1. That is, perfect squeezing can be produced inside the cavity by QND mediated feedback. In this limit, one requires the feedback to be very strong, with $\lambda \simeq \sqrt{\mathrm{H}D_0}$. Unlike the homodyne mediated feedback case, λ should always be positive, as in accord with classical intuition. Indeed, all of the features of QND mediated feedback conform to a classical theory of feedback with measurements of finite accuracy (related to H). The quantum nature of the feedback is manifest only in the increased fluctuations in y due to the measurement back-action not present classically.

Outside the cavity, fluctuations in the x quadrature are not necessarily suppressed to the same extent as they are inside; the degree of extracavity noise reduction depends on the intracavity dynamics. Once again, I am using the parameter R of Eq. (6.77) to quantify output noise minus shot noise. From Eq. (6.153), it is easy to find

$$R = \frac{-2(k_0 + \lambda) + D_0 + \lambda^2/\mathrm{H}}{(k_0 + \lambda)^2}. \tag{6.157}$$

Here I am assuming that all light lost from the cavity goes into its output, which is possible as the QND feedback loop does not consume emitted light. This expression (6.157) is minimized when

$$\lambda = \left(1 + \frac{k_0}{\mathrm{H}}\right)^{-1}(-k_0 + D_0). \tag{6.158}$$

For H small, this gives λ small and hence negligible noise reduction as in the intracavity case. For H very large, one finds the best possible low frequency output noise reduction is

$$R_{\min} = -D_0^{-1}, \tag{6.159}$$

with $\lambda = -k_0 + D_0$ so that the bandwidth of the squeezing is D_0, and $U_\lambda = -\frac{1}{2}$.

It is thus always possible to produce a sub-shot-noise x homodyne photocurrent, but the degree of nonclassicality is determined by D_0. Recall from Sec. 6.4 that $D_0 = 1 + l$, where l is any excess noise in the x quadrature above that produced by damping. For example, simply driving the cavity does not introduce any excess noise, so $l=0$ and it is possible to achieve perfect noise reduction on resonance with $R_{\rm min} \to -1$. For an ideal laser (as investigated in Sec. 5.5), with x representing the amplitude fluctuations), $l=1$. Thus the output can only be squeezed to $R_{\rm min} = -\frac{1}{2}$. The reason that a perfect QND measurement does not necessarily allow perfect noise reduction in the output light is that knowing the intracavity x is not the same as knowing the output quadrature. A normal homodyne measurement does give information about the output, but destroys that output at the same time. To obtain perfect output squeezing by feedback one would need, in general, a QND measurement of the output light. This will be investigated in Sec. 7.4.

The above analysis could have been carried out using the selective evolution of the system, just as in Sec. 6.4. The stochastic master equation for the conditioned density operator is

$$\begin{aligned} d\rho_{\rm c} &= dt\left(\mathcal{L}_0\rho_{\rm c} + \Gamma\mathcal{D}[x/2]\rho_{\rm c} + \mathcal{K}\tfrac{1}{2}[x\rho_{\rm c}+\rho_{\rm c}x] + \frac{1}{2{\rm H}}\mathcal{K}^2\rho_{\rm c}\right) \\ &\quad + dW(t)\left(\sqrt{\rm H}\mathcal{H}[x/2] + \mathcal{K}/\sqrt{\rm H}\right)\rho_{\rm c}. \end{aligned} \tag{6.160}$$

The probability distribution for the x quadrature obeys

$$\begin{aligned} dW_{\rm c}(x) &= dt\left[\partial_x(k_0+\lambda)x + \frac{1}{2}\partial_x^2\left(D_0+\lambda^2/{\rm H}\right)\right]W_{\rm c}(x) \\ &\quad + dW(t)\left[\sqrt{\rm H}[x-\bar{x}_{\rm c}(t)] + (\lambda/\sqrt{H})\partial_x\right]W_{\rm c}(x). \end{aligned} \tag{6.161}$$

The mean and variance of this conditioned distribution obey

$$\dot{\bar{x}}_{\rm c} = -(k_0+\lambda)\bar{x}_{\rm c} + \xi(t)\left(\sqrt{\rm H}V_{\rm c} - \lambda/\sqrt{\rm H}\right), \tag{6.162}$$

$$\dot{V}_{\rm c} = -2k_0V_{\rm c} + D_0 - {\rm H}V_{\rm c}^2. \tag{6.163}$$

These equations are identical to the corresponding equations for homodyne mediated feedback (6.96,6.97) apart from the replacement of $(V_{\rm c}-1)$ by $V_{\rm c}$ and η by H in the measurement terms. In the limit ${\rm H}\to\infty$, Eq. (6.163) predicts an arbitrarily small steady-state conditioned variance. This is characteristic of a good QND measurement. Choosing the value of λ in Eq. (6.156) eliminates the stochastic element in Eq. (6.162). In this case, the unconditioned variance [as given by Eq. (6.155)] agrees with the stationary conditioned variance from Eq. (6.163). The output spectrum could also have been calculated from the selective evolution equation (6.161), and a finite delay (or more general frequency response) incorporated precisely as in Sec. 6.5. The result is

$$S(\omega) = 1 + \frac{-2\left(k_0+\lambda\Re[\tilde{h}(\omega)]\right) + D_0 + \lambda^2|\tilde{h}(\omega)|^2/{\rm H}}{|-{\rm i}\omega + k_0 + \lambda\tilde{h}(\omega)|^2}. \tag{6.164}$$

Chapter 7

Feedback Without Quantum Trajectories

This chapter shows how feedback can be treated within quantum mechanics, without the quantum trajectories used in the preceding chapter. The first part of this chapter, Secs. 7.1 and 7.2, reformulates feedback of direct and homodyne photocurrents in terms of the operator representations of those observables. The second part, Secs. 7.3 to 7.5, considers all-optical feedback. This is truly quantum mechanical feedback in that the main elements of the apparatus (light beams) can be treated fully quantum mechanically. The output of the source cavity is reflected into a second cavity coupled to the source by a nonlinear crystal. I first show that all-optical feedback can produce the same source dynamics as electro-optical (direct and homodyne detection) feedback. Unlike the electro-optical case, all-optical feedback loop has an output beam (reflected off the second cavity). I show that this may be squeezed even for the analog of classical electro-optical feedback. Lastly, I consider a form of all-optical feedback where both quadratures are fed back simultaneously. This cannot be reproduced at all by electro-optic means, because of the impossibility of measuring both quadratures simultaneously. This illustrates an important difference between all-optical and electro-optical feedback.

7.1 Direct Photocurrent Feedback

In Ch. 4 it was shown that observable quantities, such as two-time photocurrent correlations functions and integrated photocurrents, may be calculated in two ways: from operator expressions for the output field, or in terms of the source alone using quantum trajectories as the measurement theory for continuously monitored systems. Both methods give the same result even though they treat the basic measurement result (the photocurrent) quite differently. In Ch. 6 I used the quantum trajectory method, in which the photocurrent is treated as a stochastic $c-$number, to describe quantum-limited feedback. The analysis of Ch. 4 suggest that it should be possible to give an equivalent description of such feedback using an operator to represent the photocurrent. That such a description should exist, not requiring the concept of quantum measurement, is also what was implied by the arguments of Ch. 2. There, in Sec. 2.4, it was shown that it is possible to describe any feedback process within quantum mechanics. When continuous feedback was considered in Sec. 2.4.3, it was stated that the quantum mechanical (as opposed to quantum measurement theoretical) treatment would be delayed until the physical basis of the master equation had been explored. It is now time to take up this issue.

7.1.1 Non-Markovian Feedback

In treating continuous feedback within quantum mechanics, it is most convenient to use the Heisenberg picture. This allows the time-dependent direct detection photocurrent to be represented by the output photon flux operator for the system. As explained in Sec. 3.6, the photon flux will not be well defined if the input is contaminated by white noise. For simplicity, consider an input in the vacuum state. Then, under the linear system-bath coupling of Sec. 3.4.2, an arbitrary system operator obeys the quantum Langevin equation (QLE)

$$ds = \mathrm{i}[H, s]dt + \left(c^\dagger sc - \tfrac{1}{2}sc^\dagger c - \tfrac{1}{2}c^\dagger cs\right) dt - [dB_0^\dagger c - c^\dagger dB_0, s]. \tag{7.1}$$

Here $dB_0 = b_0(t)dt$, where b_0 is the annihilation operator for the input field which is delta-correlated in time. Because this is in the vacuum state, $dB_0 dB_0^\dagger = dt$, and all other second order moments vanish. The output field is

$$b_1(t) = b_0(t) + c(t). \tag{7.2}$$

The output photon flux operator (equivalent to the photocurrent derived from a perfect detection of that field) is $I_1(t) = b_1^\dagger(t)b_1(t)$. This suggests that the feedback considered in Sec. 6.1 could be treated in the Heisenberg picture by using the Hamiltonian

$$H_{\rm fb}(t) = I_1(t-\tau)Z(t), \tag{7.3}$$

where each of these quantities is an operator. Here, the feedback superoperator $\mathcal{K}$ used in Sec. 6.1 would be defined by $\mathcal{K}\rho = -\mathrm{i}[Z, \rho]$. The generalization to arbitrary $\mathcal{K}$ is easy to understand in principle (as argued in Sec. 2.4).

It might be thought that there is an ambiguity of operator ordering in Eq. (7.3), because I_1 contains system operators. In fact, the ordering is not important because $b_1(t)$ commutes with all system operators at a later time [53], and so $I_1(t)$ does also. Of course, $b_1(t)$ will not commute with system operators for times after $t + \tau$ (when the feedback acts), but $I_1(t)$ still will because it is not changed by the feedback interaction. (It commutes with the feedback Hamiltonian.) This fact would allow one to use the formalism developed here to treat feedback of a photocurrent smoothed by time-averaging. That is to say, there is still no operator ambiguity in the expression

$$H_{\rm fb}(t) = Z(t) \int_0^\infty h(s)I_1(t-s)ds, \tag{7.4}$$

or even for a general Hamiltonian functional of the current, as in Eq. (6.7). For a sufficiently broad response function $h(s)$, there is no need to use stochastic calculus for the feedback; the explicit equation of motion due to the feedback would simply be

$$ds(t) = \mathrm{i}[H_{\rm fb}(t), s(t)]dt. \tag{7.5}$$

However, this approach makes the Markovian limit difficult to find. Thus, as in Sec. 6.1, the response function will be assumed to consist of a time delay only, as in Eq. (7.3).

In order to treat Eq. (7.3) it is necessary to use the stochastic calculus theory of Ch. 3 to find the explicit effect of the feedback. The necessary stochastic calculus is that of the point process, since

$$dN_1(t)^2 = dN_1(t), \tag{7.6}$$

where $dN_1(t) = I_1(t)dt$. This is exactly the problem solved in Sec. 3.6. There it was posed as a problem involving photon flux pressure, but the form of the Hamiltonian (3.120) is identical. The fact that the photon flux in Eq. (7.3) comes from the system it is affecting is unimportant, because it commutes in the same way as if it came from a bath. Proceeding as in the photon pressure case, the total QLE including feedback is

$$ds = \mathrm{i}[H,s]dt + dN_1(t-\tau)\left(e^{\mathrm{i}Z}se^{-\mathrm{i}Z} - s\right) + \left(c^\dagger sc - \tfrac{1}{2}sc^\dagger c - \tfrac{1}{2}c^\dagger cs\right)dt - [dB_0^\dagger c - c^\dagger dB_0, s]. \quad (7.7)$$

Here all time arguments are t unless otherwise indicated. This should be compared to Eq. (6.12). The obvious difference is that Eq. (6.12) explicitly describes direct photodetection, followed by feedback, whereas the irreversibility in Eq. (7.7) does not specify that the output has been detected. Indeed, the original Langevin equation (7.1) is unchanged if the output is subject to homodyne, rather than direct, detection. This is the essential difference between the virtual quantum fluctuations of Eq. (7.7) and the fluctuations due to information gathering in Eq. (6.12). Expanding $dN_1(t)$ gives

$$\begin{aligned} ds &= \mathrm{i}[H,s]dt + [c^\dagger(t-\tau) + b_0^\dagger(t-\tau)]\left(e^{\mathrm{i}Z}se^{-\mathrm{i}Z} - s\right)[c(t-\tau) + b_0(t-\tau)]dt \\ &+ \left(c^\dagger sc - \tfrac{1}{2}sc^\dagger c - \tfrac{1}{2}c^\dagger cs\right)dt - [dB_0^\dagger c - c^\dagger dB_0, s]. \end{aligned} \quad (7.8)$$

It can be verified that this is a valid non-Markovian quantum Langevin equation, in the sense explained in the section on photon flux pressure. That is to say, $d(s_1s_2)$ is correctly given by $(ds_1)s_2 + s_1(ds_2) + (ds_1)(ds_2)$.

7.1.2 Markovian Feedback

In Eq. (7.8), the vacuum field operators $b_0(t)$ have been deliberately moved to the outside [using the fact that $b_1(t-\tau)$ commutes with system operators at time t]. This has been done for convenience, because in this position, they disappear when the trace is taken over the bath density operator. Taking the total trace over system and bath density operators gives

$$\langle ds\rangle = \left\langle \mathrm{i}[H,s] + c^\dagger(t-\tau)\left(e^{\mathrm{i}Z}se^{-\mathrm{i}Z} - s\right)c(t-\tau) + \left(c^\dagger sc - \tfrac{1}{2}sc^\dagger c - \tfrac{1}{2}c^\dagger cs\right)\right\rangle dt. \quad (7.9)$$

In the limit $\tau \to 0$, so that $c(t-\tau)$ differs negligibly from $c(t)$, this gives

$$\langle ds\rangle = \left\langle c^\dagger\left(e^{\mathrm{i}Z}se^{-\mathrm{i}Z} - s\right)c + \mathrm{i}[H,s] + \left(c^\dagger sc - \tfrac{1}{2}sc^\dagger c - \tfrac{1}{2}c^\dagger cs\right)\right\rangle dt. \quad (7.10)$$

In terms of the system density operator,

$$\langle ds\rangle = \mathrm{Tr}\left[s\left(-\mathrm{i}[H,\rho] + \mathcal{D}[e^{-\mathrm{i}Z}c]\rho\right)dt\right]. \quad (7.11)$$

This is precisely what would have been obtained from the Markovian feedback master equation (6.15) for $\mathcal{K}\rho = -\mathrm{i}[Z,\rho]$.

Moreover, it is possible to set $\tau = 0$ in Eq. (7.8) and still obtain a valid Langevin equation:

$$\begin{aligned} ds &= \mathrm{i}[H,s]dt - [s,c^\dagger]\left(\tfrac{1}{2}c + b_0\right)dt + \left(\tfrac{1}{2}c^\dagger + b_0^\dagger\right)[s,c]dt \\ &+ (c^\dagger + b_0^\dagger)\left(e^{\mathrm{i}Z}se^{-\mathrm{i}Z} - s\right)(c + b_0)dt. \end{aligned} \quad (7.12)$$

This equation is quite different from Eq. (7.8) because it is Markovian. This implies that in this equation, it is no longer possible to freely move $b_1 = (c+b_0)$, as it now has the same time argument as the other operators, rather than an earlier one. In this case, it is b_0 rather than b_1 which commutes with all system operators. This must be borne in mind when proving that Eq. (7.12) is a valid Heisenberg equation of motion. This trick with time arguments and commutation relations enables the correct quantum Langevin equation describing feedback to be derived without worrying about the method of dealing with the $\tau \to 0$ limit used in the Sec. 6.1. That method is more difficult to apply in the Heisenberg picture. The subtleties involved will become apparent in Sec. 7.2, where I will use both methods to treat quadrature feedback in the Heisenberg picture. In any case, there is no disputing that Eq. (7.12) is the correct quantum Langevin equivalent to the feedback master equation,

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \mathcal{D}[e^{-\mathrm{i}Z}c]\rho. \tag{7.13}$$

It is useful to have these two different formulations, because some problems are more easily attacked using one method or the other.

As emphasized in the preceding subsection, the feedback theory given here is not simply the Heisenberg picture equivalent to the Schrödinger picture theory of Ch. 6. The real difference is that in the theory of this chapter the photocurrent remains unrealized as a measurement result. The system remains always entangled with the bath, even after the bath has interacted with the system again to produce the feedback. This distinction can be seen clearly by performing the Markovian feedback calculation in the Schrödinger picture, without resolving the measurement. From Sec. 4.1.1, the combined state matrix for the system and the bath at time t after an interaction of length dt is

$$\begin{aligned} R(t+dt) &= |0\rangle\langle 0|\otimes\rho(t) + \sqrt{dt}\left[|1\rangle\langle 0|\otimes c\rho(t) + |0\rangle\langle 1|\otimes \rho(t)c^\dagger\right] - \mathrm{i}dt|0\rangle\langle 0|\otimes[H,\rho(t)] \\ &\quad + dt\left\{|1\rangle\langle 1|\otimes c\rho(t)c^\dagger - \tfrac{1}{2}|0\rangle\langle 0|\otimes\left[c^\dagger c\rho(t) + \rho(t)c^\dagger c\right]\right\}. \end{aligned} \tag{7.14}$$

Here, the Fock states are eigenstates of $a^\dagger a = b^\dagger b dt$. The bath, entangled with the system, can now play the rôle of the measurement and feedback apparatus. Of course I am not claiming that this is a reasonable description of an actual feedback apparatus. In reality, the electromagnetic field bath would be embedded in a much larger physical system which would include elements which can interact with the system (although I will show in Sec. 3 that the embedding system is not necessarily as large as one might expect). Nevertheless, the field can formally play that rôle without any essential change in the calculation.

From the preceding section, the feedback Hamiltonian which causes the bath and system to interact again is

$$H_{\mathrm{fb}} = b^\dagger b Z. \tag{7.15}$$

This generates the unitary evolution operator

$$U_{\mathrm{fb}} = \exp[-\mathrm{i}Zb^\dagger b dt] = \exp[-\mathrm{i}Za^\dagger a]. \tag{7.16}$$

which acts on the combined system immediately following the initial interaction. This causes the combined system to evolve to

$$\begin{aligned} R(t+dt^+) &= |0\rangle\langle 0|\otimes\rho(t) + \sqrt{dt}\left[|1\rangle\langle 0|\otimes e^{-\mathrm{i}Z}c\rho(t) + |0\rangle\langle 1|\otimes \rho(t)c^\dagger e^{\mathrm{i}Z}\right] - \mathrm{i}dt|0\rangle\langle 0|\otimes[H,\rho(t)] \\ &\quad + dt\left\{|1\rangle\langle 1|\otimes e^{-\mathrm{i}Z}c\rho(t)c^\dagger e^{\mathrm{i}Z} - \tfrac{1}{2}|0\rangle\langle 0|\otimes\left[c^\dagger c\rho(t) + \rho(t)c^\dagger c\right]\right\}. \end{aligned} \tag{7.17}$$

Note that all possible measurement results are still present in this entangled state. Now taking the trace over the bath yields the usual feedback master equation for the system (7.13). It would be possible to treat non-Markovian quantum mechanical feedback in the Schrödinger picture also, but it is obvious that the equations would appear far more unwieldy than in the Heisenberg picture.

7.1.3 Comparison with Earlier Treatments

As mentioned in the Introduction to this thesis, the first attempts to treat quantum-limited feedback were done using QLEs. These date from the mid 1980s, and came from two groups. The first group, of Yamamoto and co-workers, presented theoretical and experimental analyses [73, 155]. Theoretically, they considered QND measurements as well as direct and homodyne detection, but their experiments used only direct detection. In all of their analysis, they used linearized QLEs for the system. For example, in Ref. [155] they considered feedback on the pump rate of a diode laser. Since the intracavity photon number is very large, a linearized approximation gives an excellent description of the device. The linearization approximation would probably be insignificant compared to many other approximations which are needed to make the problem tractable. The second group, of Shapiro *et al* [126], also consistently worked in the linearized regime. Rather than feedback on the source of light, they considered feedback on a beam splitter in the path from source to detector. They also considered QND as well as direct detection, and used Heisenberg operators to describe the photocurrents, as well as the beam splitter transmittivities [1]. Again, their theory gives excellent agreement with experimental results [134].

Although a linearized quantum feedback theory is, to my knowledge, adequate for all experiments performed so far, there are a number of reasons that it is preferable to have the exact theory, which I have given above. Firstly, from a theoretical point of view, the linearized feedback theory is too cheap. By linearizing, one always avoids products of operators in any expression. This is the *raison d'être* of the method, as it makes the problems easy to solve. However, in so doing it discards much of what makes quantum mechanics interesting. As shown above, the issue of operator ordering is particularly important in finding the Markovian limit of feedback. The quantum noise in a linearized treatment of the problem is indistinguishable from the classical fluctuations in the anti-quantum theory of stochastic optics [99]. This theory is based on treating the Wigner function (assumed to be always positive) as a true probability distribution for the field. Also, a linearized treatment means that one does not have to worry about the stochastic calculus, as used above. This calculus is necessary to generate exact quantum Langevin equations which preserve the commutation relations. Thus, it is not obvious *a priori* that the linearized theory of quantum feedback is a valid quantum theory. Having done the exact analysis, it is now evident that the approach of Yamamoto and co-workers and Shapiro *et al* is essentially correct, but far from complete. The simplicities that follow from a linearized treatment will be seen in Sec. 7.4 where I consider an idealized system which is exactly linear.

A second reason for wishing to have an exact quantum theory of feedback is that there will be a time when experiments will be done for which the linearized theory based on a system size expansion will fail. In such a small system, it is unlikely that the Markovian approximation to feedback would be adequate either. In that case, it would probably be necessary to solve for the feedback numerically. The best method for doing this would be the quantum trajectory approach of Ch. 6, because the fed-back current is an easily stored $c-$number rather than an operator. In the case of the traveling-wave feedback of Shapiro *et al*, an exact treatment would have to be numerical. The exact quantum theory I have given above cannot apply to such feedback. The reason for this is that it assumed that the feedback can make an arbitrarily large change to the system dynamics over an arbitrarily short time. This is conceivable for feedback onto a cavity, because an arbitrarily short time really means a time much shorter than a cavity lifetime. However, with modulation of

[1] In the case of direct detection from a classical source, a pseudoclassical analysis of the feedback is all that is necessary, as pointed out in Ref. [126].

a beam splitter in the path of a traveling wave, there is no such time scale, and it is impossible to increase the transmittivity of a beam splitter arbitrarily, because it is bounded by unity. An exact treatment of such feedback would require a detailed knowledge of the electronics and electro-optics of the feedback loop. As stated above, this is not necessary at present, and there are many other interesting applications of feedback for which a linear theory is the most sensible approach.

It would be wrong to finish this section without mentioning some other workers in quantum optics who have considered feedback. The first good experiment demonstrating the use of feedback to achieve nonclassical noise reduction was by Tapster, Rarity and Satchell [133]. They used the correlations between the twin beams generated by parametric down conversion (which individually display thermal photon statistics) to produce one sub-Poissonian beam. This was done by controlling the pump rate of the $\chi^{(2)}$ crystal from the photocurrent derived from the other beam. For their theoretical analysis, they used linearized equations based on a phenomenological model of photon creation and detection. A similar approach, using a semiclassical formulation of the fluctuations, for feedback and feedforward with twin beams, is found in Ref. [100]. Mertz and Heidmann have also presented numerical simulations in which each photodetection is treated as an individual event [101, 77]. However, these are not quantum trajectory simulations; there is no quantum state for the source. Rather, as they state [77] "The basic premise of the photon model is that a light beam is considered a stream of discrete photons that are localized in time." As with all of the other works mentioned in this paragraph, this model is quite adequate to describe the experiments discussed, despite its foundation on a classical rather than quantum description of the phenomenon. It is probably this fact which has delayed a complete quantum theory of feedback until now.

7.2 Homodyne Photocurrent Feedback

7.2.1 Vacuum Input

The quantum Langevin treatment of quadrature flux feedback (corresponding to homodyne detection) is relatively straightforward, because of the Gaussian nature of the noise. The homodyne photocurrent is identified with the quadrature of the outgoing field

$$I^{\rm hom}(t) = b_1(t) + b_1^\dagger(t) = c(t) + c^\dagger(t) + b_0(t) + b_0^\dagger(t). \tag{7.18}$$

The feedback Hamiltonian is defined as

$$H_{\rm fb}(t) = F(t) I^{\rm hom}(t-\tau). \tag{7.19}$$

The time delay τ ensures that the output quadrature operator $I^{\rm hom}(t)$ commutes with all system operators at the same time. Thus, it will commute with $F(t)$ and there is no ambiguity in the operator ordering in Eq. (7.19). Treating the equation of motion generated by this Hamiltonian as an implicit equation, the explicit equation is

$$[ds(t)]_{\rm fb} = {\rm i} I^{\rm hom}(t-\tau)[F(t), s(t)]dt - \tfrac{1}{2}[F(t),[F(t),s(t)]]dt. \tag{7.20}$$

Adding in the the non-feedback evolution gives the total explicit equation of motion

$$\begin{aligned} ds \;=\; & {\rm i}[H,s]dt + {\rm i}[c^\dagger(t-\tau)dt + dB_0^\dagger(t-\tau)][F,s] + {\rm i}[F,s][c(t-\tau)dt + dB_0(t-\tau)] \\ & - \tfrac{1}{2}[F,[F,s]]dt + \left(c^\dagger s c - \tfrac{1}{2} s c^\dagger c - \tfrac{1}{2} c^\dagger c s\right) dt - [dB_0^\dagger c - c^\dagger dB_0, s]. \end{aligned} \tag{7.21}$$

Here, all time arguments are t unless indicated otherwise.

In Eq. (7.21), I have once again used the commutability of the output operators with system operators to place them suitably on the exterior of the feedback expression. This ensures that when an expectation value is taken, the input noise operators annihilate the vacuum and hence give no contribution. This is the same trick as used in Sec. 7.1, and putting $\tau = 0$ in Eq. (7.21) also gives a valid Heisenberg equation of motion. That equation is the counterpart to the homodyne feedback master equation (6.55). However, this trick will not work if the input field is not in the vacuum state, but is for example in a thermal state. For direct detection, it is impossible to treat feedback in the presence of white noise, so the operator ordering trick is perfectly legitimate. However, for quadrature-based feedback, as explained in the Sec. 6.3.2, it is possible to treat white noise. Thus, it is necessary to give a method of treating the Markovian ($\tau \to 0$) limit in this general case (although I will leave the addition of the white noise terms until the following subsection). The necessary method is essentially the same as that of Ch. 6, ensuring that the feedback acts later than the measurement. In applying it to Heisenberg equations of motion, it will be seen that one has to be quite careful with operator ordering.

If $\tau = 0$ then the feedback Hamiltonian (7.19) does have an ordering ambiguity. A symmetric ordering would seem a sensible starting point

$$H_{\rm fb} = \tfrac{1}{2}\{F, c + c^\dagger + b_0 + b_0^\dagger\}, \tag{7.22}$$

where the curly brackets denote an anticommutator. Although the time argument of all the operators in this expression is supposedly t, the operator F must actually be of a slightly later time, after the bath operators b_0 and $b_0^\dagger$ have interacted with the system. That is to say, the actual expression for the feedback Hamiltonian should be

$$H_{\rm fb} = \tfrac{1}{2}\{F - dB_0^\dagger[c, F] + dB_0[c^\dagger, F] + O(dt), c + c^\dagger + b_0 + b_0^\dagger\}, \tag{7.23}$$

where here the time arguments really are all t. Using the Itô rules gives

$$H_{\rm fb}dt = (c^\dagger + b_0^\dagger)F + F(c + b_0), \tag{7.24}$$

The total evolution of the system is thus

$$\begin{aligned} s(t+dt) &= \exp[\mathrm{i}H_{\rm fb}dt]\big\{s + \mathrm{i}[H, s]dt + \left(c^\dagger s c - \tfrac{1}{2}s c^\dagger c - \tfrac{1}{2}c^\dagger c s\right) dt \\ &\qquad - [dB_0^\dagger c - c^\dagger dB_0, s]\big\} \exp[-\mathrm{i}H_{\rm fb}dt], \end{aligned} \tag{7.25}$$

where all time arguments are t. Expanding the exponentials using the quantum Itô rules gives

$$\begin{aligned} ds &= \mathrm{i}[H, s]dt - [s, c^\dagger]\left(\tfrac{1}{2}c dt + dB_0\right) + \left(\tfrac{1}{2}c^\dagger dt + dB_0^\dagger\right)[s, c] \\ &\quad + \mathrm{i}[c^\dagger dt + dB_0^\dagger][F, s] + \mathrm{i}[F, s][c dt + dB_0] - \tfrac{1}{2}[F, [F, s]]dt. \end{aligned} \tag{7.26}$$

This equation is a valid Markovian quantum Langevin equation, equivalent to the homodyne feedback master equation (6.54). It is a true quantum Itô equation, in the sense of Ref. [53].

7.2.2 White Noise Input

Equation (7.25) can be rewritten as

$$s(t+dt) = U^\dagger(t, t+dt)s(t)U(t, t+dt) \tag{7.27}$$

where

$$U(t,t+dt) = U_1(t,t+dt)\exp[-\mathrm{i}\{(c^\dagger + b_0^\dagger)F + F(c+b_0)\}dt], \tag{7.28}$$

where $U_1(t,t+dt)$ is, as in Sec. 3.7,

$$U_1(t,t+dt) = \exp\left\{\sqrt{\gamma_1}\left[dB_0^\dagger c_1 - c_1^\dagger dB_0\right]\right\}. \tag{7.29}$$

In Eq. (7.25), the feedback acts on the operator after the first "measurement" interaction U_1, because the operators in the feedback Hamiltonian are those appropriate for an interaction after the "measurement" interaction. That is to say, the operator ordering was carefully determined as above, and the operator for the field quadrature is the output operator $b_1 + b_1^\dagger$ rather than the input operator $b_0 + b_0^\dagger$. These corrections are unnecessary if the unitary operator (7.28) is rewritten as

$$U(t,t+dt) = \exp[-\mathrm{i}(b_0^\dagger F + F b_0)dt]U_1(t,t+dt). \tag{7.30}$$

In this expression, one sees that the feedback appears to act on the system first, and the "measurement" later. This ordering is in fact the same as that in the classical expression for the effect of noisy feedback (3.50) in Sec. 3.3.2. Also, the above argument on ordering of the evolution operators is identical to that in Sec. 3.7 on cascaded systems theory. Here the second system operator $c_2(t)$ is replaced by $-\mathrm{i}F(t)$. Thus it appears that feeding back the quadrature output of the system is a special case of the cascaded systems theory where

- The driven system is coupled to the bath by an Hermitian operator.
- The driven system is not distinct from the source system.

However, it should be noted that a nonlinear bath-system coupling, as arising in photon flux pressure, and as necessary in describing direct detection feedback, has not been considered prior to this thesis.

In the cascaded systems theory of Sec. 3.7, I considered an arbitrary input bath with white noise statistics. That analysis can be simply carried over to the case of quadrature feedback in the presence of white noise. The resulting QLE in the Markovian limit is

$$\begin{aligned} ds \;=\;& (N+1)\left[\tfrac{1}{2}(2c^\dagger s c - s c^\dagger c - c^\dagger c s) - \tfrac{1}{2}[F,[F,s]] + (\mathrm{i}[F,s]c + \mathrm{i}c^\dagger[F,s])\right] \\ &+N\left[\tfrac{1}{2}(2csc^\dagger - scc^\dagger - cc^\dagger s) - \tfrac{1}{2}[F,[F,s]] + (-\mathrm{i}[F,s]c^\dagger - \mathrm{i}c[F,s])\right] \\ &+M\tfrac{1}{2}\left([c^\dagger,[c^\dagger,s]] - [F,[F,s]] + 2\mathrm{i}[c^\dagger,[F,s]]\right) \\ &+M^*\tfrac{1}{2}\left([c,[c,s]] - [F,[F,s]] - 2\mathrm{i}[c,[F,s]]\right) \\ &-[dB_0^\dagger(c-\mathrm{i}F) - (c^\dagger + \mathrm{i}F)dB_0, s] + \mathrm{i}[H,s]dt. \end{aligned} \tag{7.31}$$

Here all operators are at equal times, and dB_0 is a true Itô increment, independent of s. With $M = N = 0$ this is identical to Eq. (7.26). Converting it into a master equation yields the general homodyne feedback master equation (6.62). The QLE corresponding to the general heterodyne feedback master equation (6.66) is fairly much self-evident from Eq. (7.31).

7.3 Feedback Without Measurement

7.3.1 All-Optical Feedback

From the preceding two sections, it is evident that the quantum Langevin equation approach to feedback is completely equivalent to that based on quantum trajectories. For linearizable systems, which covers most applications, the QLE approach is generally easier to solve (as will be seen by comparing the analysis of Sec. 7.4 with that of Sec. 6.5), and so must be judged more practical. The main drawback of this approach is a conceptual one. Describing the feedback apparatus in terms of Heisenberg operators may give the impression that one has at hand a quantum mechanical treatment of a macroscopic system. This is of course a false impression. The only features of the operators which are important are their stochastic nature, and their correlations with the source. The macroscopic apparatus has not been quantized in the ordinary sense of the word; it is simply correlated to the fluctuations in the observed photocurrent which are represented by an operator. In the quantum trajectory description of feedback this false impression would never arise. The feedback apparatus is treated as a completely classical system. This is of course how experimentalists would naturally regard it. To reiterate: both approaches are quite valid, but quantum trajectories perhaps give a better feel for what is "really" going on.

Having said all of this, it is now necessary to qualify it. As hinted at in Sec. 1.2, the feedback apparatus need not be as macroscopic and classical as I have assumed so far. In fact, it is possible to build a feedback device entirely from light beams, interacting through nonlinear crystals. Such feedback may be called all-optical feedback, to distinguish from electro-optical feedback, which involves detecting the light and thereby producing a current which will influence the source by some electro-optical device. With all-optical feedback it is in fact possible to give a quantum mechanical description of the essential elements of the feedback loop (the light beams). In this section, I will show that all-optical feedback can reproduce, at least approximately, the results of electro-optical feedback based on direct or homodyne detection. Yet in all-optical feedback there is never a measurement step in the usual sense. For this no-measurement feedback, the quantum trajectory description is obviously inappropriate. Nevertheless, it will be seen in Sec. 7.4 that the analogy between electro-optical and all-optical feedback does help in understanding some results.

The basic idea of all-optical feedback is to reflect the output light from the source cavity around a loop and back into the source cavity, or, more fruitfully, into another cavity which is coupled to the first cavity in some way. Of course, for this mechanism to be considered feedback, the feedback loop must be one-way, otherwise I would be describing a pair of doubly-coupled cavities. The mechanism for achieving the required unidirectionality could be the Faraday isolator which utilizes Faraday rotation and polarization-sensitive beam splitters. This is described by the quantum theory of cascaded systems, as discussed in Sec. 3.7. In fact, the only difference from that section is that the Hamiltonian for the source and driven system does not split up, because the two modes are assumed coupled by some nonlinear crystal. The general configuration under consideration is shown schematically in Fig. 7.1. The nature of the feedback depends on the intracavity interaction Hamiltonian V. In particular, as I claimed above, a suitable choice of V can reproduce direct or homodyne detection feedback.

7.3.2 Intensity Feedback

In Fig. 7.1, the source cavity has annihilation operator c_1, and the driven cavity operator c_2. If the second cavity is heavily damped compared to the first cavity then the driven cavity will be slaved to the output of the first cavity. For instance, the instantaneous photon number of the driven cavity will reproduce approximately the output photon flux of the source cavity by the lifetime of the driven cavity. Thus, the all-optical analogue to electro-optic feedback of the direct photocurrent will be produced by an interaction Hamiltonian proportional to the photon number in the driven

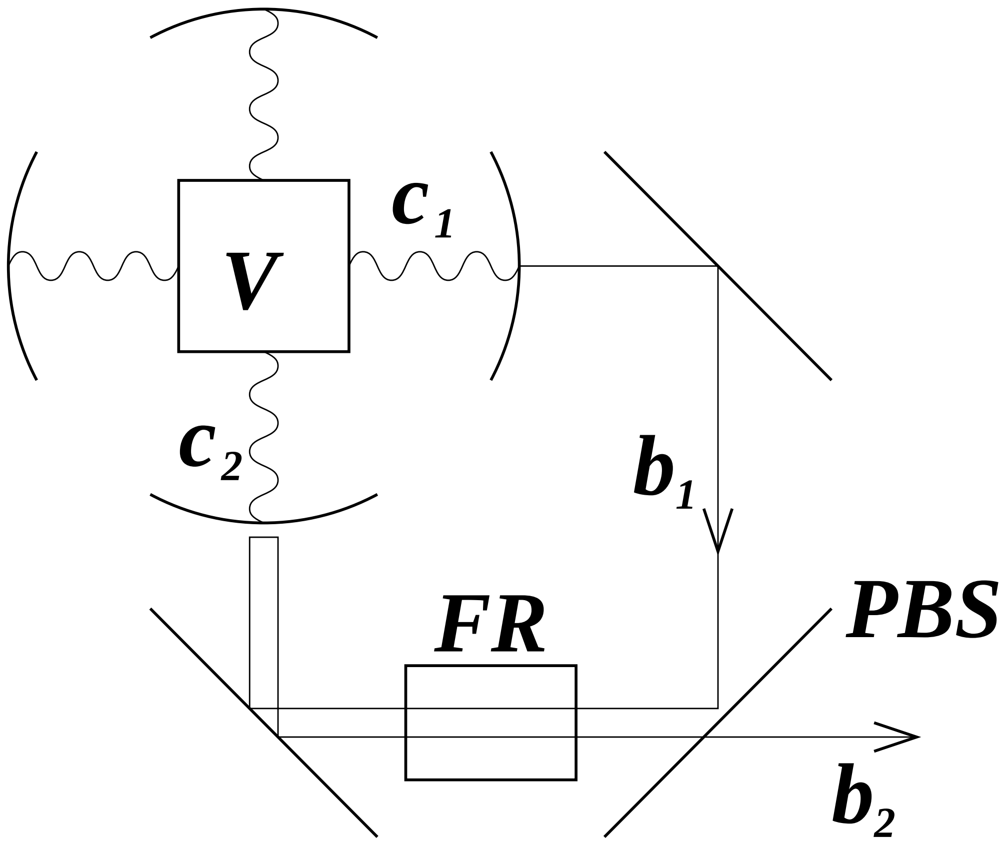

Figure 7.1: Diagram of the the general experimental scheme for all-optical feedback. The annihilation operators for the source and driven cavities are denoted c_1 and c_2 respectively, while b_1 and b_2 represent traveling waves. The nonlinear coupling between the two cavity modes is indicated by V. FR denotes a Faraday rotator, and PBS a polarization-sensitive beam splitter.

cavity. Specifically, let

$$V = c_2^\dagger c_2 K, \tag{7.32}$$

where K is an Hermitian operator on the source cavity. In this, and following sections, I am measuring time in inverse units of the decay constant γ_1 for the source cavity. The decay constant γ_2 for the driven cavity will be denoted simply γ. Now because of this interaction term, the dynamics of the source cavity is not independent of that of the driven cavity. Thus one must (at least initially) consider the master equation for the density operator of both modes R. According to Eq. (3.147), this obeys the master equation

$$\dot{R} = -\mathrm{i}[H + c_2^\dagger c_2 K, R] + \mathcal{D}[c_1]R + \gamma\mathcal{D}[c_2]R + \sqrt{\gamma}\left([c_1 R, c_2^\dagger] + [c_2, R c_1^\dagger]\right), \tag{7.33}$$

where H generates the internal dynamics of mode c_1. Note that here I have assumed that the bath is in the vacuum state ($N = M = 0$). The reason for this is basically the same as in Sec. 4.1 on direct detection, as will become apparent later.

In deriving Eq. (7.33), it is necessary to assume that the time τ of propagation between the cavities is equal to zero. However, this is no longer a formal mathematical limit as it was in Sec. 3.7. Rather, it is a physical condition that the time delay in the feedback loop should be negligible. This will be the case provided that the intercavity separation is of the same order as a length of the cavity (with highly reflective end mirrors as usual). A negligible delay is usually desirable in feedback loops, as a substantial time delay may lead to instabilities and chaos, rather than control. Similarly, for the feedback to be effective, the second cavity should respond much faster than the first. This ensures that the state of c_2 is effectively slaved to that of c_1, and the interaction effects instantaneous feedback as far as the source is concerned. Thus, in the limit $\gamma \gg 1$, it should be possible to derive a master equation including feedback for the source density operator ρ alone. This can be done by noting that, since $\gamma \gg 1$ and the bath is in the vacuum state, the driven cavity will be very close to being in the vacuum state also. This enables the adiabatic elimination procedure used previously in Sec. 6.6 to be employed again. For variety, I will give a slightly modified derivation.

As in Sec. 6.6, I expand R in powers of $1/\sqrt{\gamma}$ as

$$R = \rho_0 \otimes |0\rangle\langle 0| + (\rho_1 \otimes |1\rangle\langle 0| + \text{H.c.}) + \rho_2 \otimes |1\rangle\langle 1| + (\rho_{2'} \otimes |2\rangle\langle 0| + \text{H.c,}), \tag{7.34}$$

where the ρs exist in the source subspace and the Fock states in the driven subspace. Substituting the above expansion into the master equation (7.33) and ignoring $\rho_{2'}$ gives the following coupled equations

$$\dot{\rho}_0 = \gamma\rho_2 + \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1\rho_1^\dagger) + \mathcal{L}_0\rho_0, \tag{7.35}$$
$$\dot{\rho}_1 = -\tfrac{1}{2}\gamma\rho_1 - \mathrm{i}K\rho_1 - \sqrt{\gamma}[c_1\rho_0 + O(1/\gamma)] + \mathcal{L}_0\rho_1, \tag{7.36}$$
$$\dot{\rho}_2 = -\gamma\rho_2 - \mathrm{i}[K, \rho_2] - \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1\rho_1^\dagger) + \mathcal{L}_0\rho_2, \tag{7.37}$$

Here $\mathcal{L}_0\rho \equiv \mathcal{D}[c_1]\rho - \mathrm{i}[H, \rho]$. In this approximation, the source density operator is $\rho = \rho_0 + \rho_2$ which evidently obeys

$$\dot{\rho} = -\mathrm{i}[K, \rho_2] + \mathcal{L}_0\rho. \tag{7.38}$$

To turn this into a master equation, one requires an expression for ρ_2 in terms of $\rho \simeq \rho_0$. It is now obvious why I assumed a zero temperature bath. For N finite, ρ_2 would have a finite size irrespective of γ. Thus the signal due to the driving from the source, which is of order $1/\gamma$ would be swamped by the noise, and the feedback would not work.

To obtain ρ_2 it is first necessary to obtain ρ_1. Since almost all of the probability is in ρ_0, it is evident from Eq. (7.36) that ρ_1 relaxes much more rapidly than $\rho_0 \simeq \rho$. It is thus permissible to set ρ_1 equal to its steady state value of

$$\rho_1 = \left(1 + \mathrm{i}\frac{2K}{\gamma}\right)^{-1} \frac{-2}{\sqrt{\gamma}} c_1 \rho. \tag{7.39}$$

The expression on the right hand side will be a well defined operator if $K \ll \gamma$ in some sense. This assumption will be valid in practice, as single-photon nonlinearities are typically much smaller than damping rates. It allows the denominator to be expanded to first order in K/γ. Since ρ_1 is now slaved to ρ_0, it will evolve at a rate much smaller than γ. Thus, from Eq. (7.37), ρ_2 will relax to a steady state determined by the slaved value of ρ_1:

$$\dot\rho_2 = -\gamma\rho_2 - \mathrm{i}[K, \rho_2] + 4c_1\rho c_1^\dagger - 4\mathrm{i}\gamma^{-1}[K, c_1\rho c_1^\dagger] + \mathcal{L}_0\rho_2. \tag{7.40}$$

The slaved value of ρ_2, again to first order in K/γ, is

$$\rho_2 = \frac{4c_1\rho c_1^\dagger}{\gamma} - \frac{4\mathrm{i}}{2\gamma}\left[\frac{4K}{\gamma}, c_1\rho c_1^\dagger\right]. \tag{7.41}$$

Finally, substituting this into Eq. (7.38) gives the master equation

$$\dot\rho = -\mathrm{i}[Z, c_1\rho c_1^\dagger] - \tfrac{1}{2}[Z,[Z, c_1\rho c_1^\dagger]] + \mathcal{D}[c_1]\rho - \mathrm{i}[H, \rho], \tag{7.42}$$

where I have defined

$$Z = 4K/\gamma. \tag{7.43}$$

This master equation is the general equation for Markovian, intensity-dependent all-optical feedback in the small Z limit. Unfortunately, it is not of the Lindblad form (2.32). However, it is not difficult to see that, to second order in Z, it is equivalent to the direct detection feedback master equation

$$\dot\rho = -\mathrm{i}[H, \rho] + \mathcal{D}[e^{-\mathrm{i}Z} c_1]\rho, \tag{7.44}$$

which is of the appropriate form. This is the sense in which all-optical feedback can reproduce the dynamics of electro-optical feedback based on direct-detection. The adiabatic elimination only works in the small Z limit so it is perhaps not surprising that the correspondence is not exact for all orders. From the arguments of Sec. 6.1, it follows that the all-optical feedback considered in this section cannot produce a nonclassical state in the source cavity if K is a 'classical' operator, in the meaning of Sec. 6.1. Given that the nonlinear interaction Hamiltonian (7.32) is already of second order in the field of the driven (c_2) cavity, any coupling with a nonclassical K would require at least a five-wave mixing interaction (including an auxiliary pump field). This seems exceedingly impractical. Thus it can be concluded that intensity-dependent all-optical feedback is not a practical way to produce a nonclassical source state.

Quantum Langevin Equation

Just as electro-optical feedback can be formulated as a master equation or QLE, so can all-optical feedback. Furthermore, this yields some extra information in the case of all-optical feedback. The quantum Langevin equation equivalent to the master equation (7.33) is

$$\begin{aligned} dr \;=\;& \frac{1}{2}(2c_1^\dagger r c_1 - r c_1^\dagger c_1 - c_1^\dagger c_1 r)dt + \frac{\gamma}{2}(2c_2^\dagger r c_2 - r c_2^\dagger c_2 - c_2^\dagger c_2 r)dt \\ & -[dB_0^\dagger c_1 - c_1^\dagger dB_0, r] - \sqrt{\gamma}[dB_1^\dagger c_2 - c_2^\dagger dB_1, r] + \mathrm{i}[H + c_2^\dagger c_2 K, r]dt. \end{aligned} \tag{7.45}$$

Here $dB_0 = b_0 dt$ is the input vacuum field, and $dB_1 = dB_0 + c_1 dt$ is the output field from the first cavity. This field which is fed back is to be understood to be at a slightly earlier time than all of the other operators in the above equation. For this reason, it commutes with all other operators. If dB_1 is moved to the rear (far right) of any operator expression, the vacuum noise of dB_1 will not contribute to any average, as the annihilation operator will act directly on the bath in the vacuum state. Similar remarks hold for moving $dB_1^\dagger$ to the front of any expression. Thus, if one always put the bath operators in normal order as described here, then the fact that it is at a slightly earlier time can be ignored. This is the same technique as used in Sec. 7.1.

As above, the aim is to adiabatically eliminate mode c_2. This obeys

$$\dot{c}_2 = -\frac{\gamma}{2}c_2 - \sqrt{\gamma}b_1 - \mathrm{i}Kc_2. \tag{7.46}$$

Now the relaxation rate γ of mode c_2 cannot be much greater than the bandwidth of the vacuum fluctuations, which are assumed infinite. Hence, it is not strictly possible to slave c_2 to the vacuum fluctuations in b_1. However, as far as mode c_1 is concerned, vacuum fluctuations restricted to a bandwidth of γ are still effectively white, and so there is no harm in pretending that c_2 is slaved to the original vacuum fluctuations. This allows one to write the slaved value of c_2 as

$$c_2 = \left(1 - \frac{2\mathrm{i}K}{\gamma}\right)\frac{-2}{\sqrt{\gamma}}b_1. \tag{7.47}$$

Substituting this into the equation for a source cavity operator s gives

$$\begin{aligned} ds &= \frac{1}{2}(2c_1^\dagger s c_1 - s c_1^\dagger c_1 - c_1^\dagger c_1 s)dt - dB_0^\dagger[c_1, s] + [c_1^\dagger, s]dB_0 \\ &\quad + \mathrm{i}\frac{4}{\gamma}b_1^\dagger[K, s]b_1 dt - \frac{8}{\gamma^2}b_1^\dagger[K,[K,s]]b_1 dt + \mathrm{i}[H,s]dt. \end{aligned} \tag{7.48}$$

Substituting in the expression for b_1 gives

$$\begin{aligned} ds &= \left(c_1^\dagger s c_1 - \tfrac{1}{2}s c_1^\dagger c_1 - \tfrac{1}{2}c_1^\dagger c_1 s\right)dt - dB_0^\dagger[c_1, s] + [c_1^\dagger, s]dB_0 \\ &\quad + (c_1^\dagger + b_0^\dagger)\left(\mathrm{i}[Z,s] - \tfrac{1}{2}[Z,[Z,s]]\right)(c_1 + b_0)dt + \mathrm{i}[H,s]dt, \end{aligned} \tag{7.49}$$

where Z is as in Eq. (7.43). It is easy to verify by tracing over the bath that this equation is equivalent to the master equation (7.42). Also, Eq.(7.49) is equal to the direct detection feedback QLE derived above (7.12), expanded to second order in Z.

One property that a Langevin equation has which the master equation lacks is that it simply gives an expression for the field reflected from the driven cavity (see Fig. 1). Calling this field b_2,

$$b_2 = b_1 + \sqrt{\gamma}c_2. \tag{7.50}$$

From the adiabatic expression for c_2,

$$b_2 = -\left(1 - \frac{4\mathrm{i}K}{\gamma}\right)b_1. \tag{7.51}$$

Apart from a change of sign (due to reflection), this is equal to what would be obtained from the idealized Hamiltonian (7.3) of Sec. 7.1,

$$b_2 = e^{-\mathrm{i}Z}b_1, \tag{7.52}$$

to first order in Z. The statements made above regarding the inability of a classical feedback operator Z to produce a nonclassical source state do not necessarily apply to the output operator b_2. It may exhibit nonclassical features even if the source state is classical. I will not pursue the properties of this output field further in this section, but will in Sec. 7.4.

7.3.3 Quadrature Feedback

All-optical feedback as described in Sec. 7.3.1 can mimic electro-optical feedback of a homodyne photocurrent as well as a direct detection current. The required interaction Hamiltonian between the two modes is

$$V = (c_2 + c_2^\dagger)J, \tag{7.53}$$

which is linear in the real quadrature of the driven cavity. Here J is an Hermitian operator in the source cavity. In practice, conservation of energy would require at least one other auxiliary field which may be treated classically. With the input to the source cavity in the vacuum state as before, and with the damping rate γ of the second cavity much greater than that of the first, it is again possible to expand the combined density operator in powers of $1/\sqrt{\gamma}$ as in Eq. (7.34). In this case, the source cavity operators obey

$$\begin{aligned}
\dot\rho_0 &= \gamma\rho_2 + \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1\rho_1^\dagger) - \mathrm{i}[J\rho_1 - \rho_1^\dagger J] + \mathcal{L}_0\rho_0, &(7.54)\\
\dot\rho_1 &= -\tfrac{1}{2}\gamma\rho_1 - \mathrm{i}[J\rho_0 + O(1/\gamma)] - \sqrt{\gamma}[c_1\rho_0 + O(1/\gamma)] + \mathcal{L}_0\rho_1, &(7.55)\\
\dot\rho_2 &= -\gamma\rho_2 - \mathrm{i}[J\rho_1^\dagger - \rho_1 J] - \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1\rho_1^\dagger) + \mathcal{L}_0\rho_2. &(7.56)
\end{aligned}$$

The approximate source density operator $\rho = \rho_0 + \rho_2$ obeys

$$\dot\rho = -\mathrm{i}[J, \rho_1 + \rho_1^\dagger] + \mathcal{L}_0\rho. \tag{7.57}$$

Evidently, to derive a master equation for ρ in this case it is only necessary for ρ_1 to be slaved to $\rho = \rho_0 + O(1/\gamma)$. From Eq. (7.55), this is obviously true for large γ, with the slaved value

$$\rho_1 = -\frac{2}{\gamma}\left(\sqrt{\gamma}c_1 + \mathrm{i}J\right)\rho. \tag{7.58}$$

Here I am keeping only the lowest order in $1/\gamma$, and assuming that J is at most of order $\sqrt{\gamma}$. Substituting this expression into Eq. (7.57) gives the master equation

$$\dot\rho = -\mathrm{i}[F, c_1\rho + \rho c_1^\dagger] - \tfrac{1}{2}[F,[F,\rho]] + \mathcal{D}[c_1]\rho - \mathrm{i}[H,\rho], \tag{7.59}$$

where I have defined

$$F = \frac{-2J}{\sqrt{\gamma}}, \tag{7.60}$$

which is of order unity. This is precisely equal to the homodyne feedback master equation (6.54) derived in Sec. 6.2.

Quantum Langevin Equation

If the input to the source cavity is not a vacuum, then the master equation technique of adiabatic elimination cannot be used. Unlike the intensity dependent feedback of the preceding subsection, it is nevertheless possible to obtain a sensible result for non-vacuum input using a Langevin equation approach. An arbitrary operator obeys the explicit quantum equation (3.141) of Sec. 7.3, with Eq. (7.53) added to the Hamiltonian term. From this equation, an operator s for the source cavity will obey the equation

$$\begin{aligned}
ds &= \tfrac{1}{2}\Big\{(N+1)(2c_1^\dagger s c_1 - s c_1^\dagger c_1 - c_1^\dagger c_1 s) + N(2c_1 s c_1^\dagger - s c_1 c_1^\dagger - c_1 c_1^\dagger s) + M[c_1^\dagger,[c_1^\dagger,s]]\\
&\quad + M^*[c_1,[c_1,s]]\Big\}\,dt - [dB_0^\dagger c_1 - c_1^\dagger dB_0, s] + \mathrm{i}[H + (c_2 + c_2^\dagger)J, s]dt. &(7.61)
\end{aligned}$$

Evidently, to obtain a Langevin equation for a source cavity operator involving no driven cavity operators, it is necessary only to adiabatically eliminate c_2. From Eq. (3.141) with the Hamiltonian V added, this obeys

$$\dot{c_2}(t) = -\frac{\gamma}{2}c_2(t) - \sqrt{\gamma}b_1(t-\tau) - \mathrm{i}J(t). \tag{7.62}$$

As before, if $\gamma \gg 1$, then there is no harm in replacing c_2 by the slaved value

$$c_2(t) = \frac{-2}{\sqrt{\gamma}}\left[b_1(t-\tau) + \frac{\mathrm{i}J(t)}{\sqrt{\gamma}}\right], \tag{7.63}$$

providing that the resulting term in the Langevin equation is treated in the Stratonovich sense. That is to say, the effective feedback Hamiltonian

$$H_{\rm fb}(t) = (c_2 + c_2^\dagger)J = \mathrm{i}\left[b_1^\dagger(t-\tau)\left(\frac{2\mathrm{i}J(t)}{\sqrt{\gamma}}\right) - \left(\frac{-2\mathrm{i}J(t)}{\sqrt{\gamma}}\right)b_1(t-\tau)\right] \tag{7.64}$$

is to be treated in the same manner as the coupling Hamiltonian (3.137). This feedback Hamiltonian is of course the same one used in Sec. 2.2, namely

$$H_{\rm fb}(t) = b_1^\dagger(t-\tau)F(t) + F(t)b_1(t-\tau), \tag{7.65}$$

where F is as defined in Eq. (7.60). Thus, the present scheme of all-optical quadrature feedback will give the same QLE (7.31) as does homodyne-mediated electro-optical feedback.

7.4 In-Loop and Output Squeezing

As with intensity-dependent feedback, the all-optical quadrature-dependent feedback scheme described above is not a good way to try to produce nonclassical light in the source cavity. This is evident from the master equation (7.59). Ignoring the free Hamiltonian H of the source cavity, this will only produce squeezing if F is a nonclassical operator (see preceding chapter). Given the origin of F in the original coupling (7.53), it is evident that at least a $\chi^{(3)}$ nonlinearity would be required to produce nonclassical source light. However, as noted in Sec. 7.3.2, this requirement does not hold for the production of squeezing in the output light from the second cavity. Of course, this beam of light does not exist in the electro-optical case; it is only present for an all-optical feedback system. It turns out that this light can exhibit perfect squeezing, even though the source cavity remains in a classical state. The operator b_2 for this output beam is related to the in-loop field (denoted b_1 as above) by

$$b_2 = b_1 + \sqrt{\gamma}c_2. \tag{7.66}$$

Using the slaved value of c_2 (7.63), this is

$$b_2 = -(b_1 - \mathrm{i}F) = -(b_0 + c_1 - \mathrm{i}F), \tag{7.67}$$

which, apart from the unimportant sign change, is just what would be expected from the effective Hamiltonian (7.65).

Consider the simplest possible case, where b_0 is in a vacuum state and F is proportional to the y quadrature of the source cavity

$$F = -\frac{\lambda}{2}(-\mathrm{i}c_1 + \mathrm{i}c_1^\dagger), \tag{7.68}$$

as in Ch. 6. This is obtained from the interaction Hamiltonian

$$V = -\mathrm{i}\frac{g}{4}(c_1c_2^\dagger - c_1^\dagger c_2) - \mathrm{i}\frac{g}{4}(c_2c_1 - c_2^\dagger c_1^\dagger), \tag{7.69}$$

where $g = \sqrt{\gamma}\lambda$. Note that this is the same as coupling as considered in Sec. 6.6, used there to achieve a QND measurement. There, it was suggested that the two terms could be achieved by frequency conversion and nondegenerate parametric amplification respectively. Here, the two modes c_1 and c_2 are necessarily of the same frequency. However, they could be supported in the same cavity if the c_2 mode was polarized orthogonally to the c_1 mode. That is to say, the second cavity is physically the same as the first, unlike the diagrammatic representation in Fig. 7.1. Then the first term in the Hamiltonian (7.69) could describe mode conversion, via a polarization rotator. The coupling constant g would be proportional to the (small) proportion of light converted at each pass, divided by the round trip time of the cavity. The second term, also with strength g, could only be produced by a nonlinear medium, such as a $\chi^{(2)}$ crystal. The two polarization modes would be the signal and idler, and the second harmonic would have to be strongly driven and heavily damped so that it could be adiabatically eliminated. There are no obvious bars to setting up this scheme experimentally.

The feedback operator (7.68) gives linear equations of motion for the quadratures of the source cavity. Because they are linear, they can be obtained directly from the feedback Hamiltonian (7.65) without using stochastic calculus or worrying about the effect of a time delay. As explained in Sec. 7.1, the previous treatments of feedback [73, 155, 126] have only ever considered such linear equations. In this case, the equations for $x = c_1 + c_1^\dagger$ and $y = -\mathrm{i}c_1 + \mathrm{i}c_1^\dagger$ obtained from Eq. (7.31) are

$$\dot{x} = -\left(\tfrac{1}{2}x + \xi\right) - \lambda(x + \xi), \tag{7.70}$$

$$\dot{y} = -\left(\tfrac{1}{2}y + \upsilon\right). \tag{7.71}$$

Here ξ and υ are the vacuum quadrature noise operators introduced in Sec. 4.2.3, which obey

$$\langle\xi(t)\xi(t')\rangle = \langle\upsilon(t)\upsilon(t')\rangle = \delta(t - t'), \tag{7.72}$$

$$[\xi(t), \upsilon(t')] = 2\mathrm{i}\delta(t - t'). \tag{7.73}$$

Since the equations for the two quadratures are uncoupled, the operator nature of ξ and υ [as evidenced by Eq. (7.73)] is mostly unimportant. It is easy to derive the steady-state variance in x to be

$$V^x = \frac{(1+\lambda)^2}{1+2\lambda}. \tag{7.74}$$

This is always greater than one for any non-zero λ, which is as expected since this sort of feedback cannot produce a squeezed intracavity state. Eq. (7.74), agrees with the result (6.83) obtained in Sec. 6.4 for linear feedback when $l = \chi = 0$ and $\eta = 1$.

In-Loop Squeezing

To examine the extracavity fields, it is necessary to consider the noise at different frequencies. Denoting the Fourier transform by a tilde as usual, Eqs. (7.70,7.71) become

$$\tilde{x}(\omega) = -\frac{1+\lambda}{\frac{1}{2} + \lambda - \mathrm{i}\omega}\tilde{\xi}(\omega), \tag{7.75}$$

$$\tilde{y}(\omega) = -\frac{1}{\frac{1}{2} - \mathrm{i}\omega}\tilde{\upsilon}(\omega). \tag{7.76}$$

The frequency domain counterparts to the time domain relationships (7.72,7.73) are identical but for the replacement of $\delta(t-t')$ by $2\pi\delta(\omega+\omega')$. The in-loop quadratures of b_1 are

$$\tilde{x}_1(\omega) = -\frac{\frac{1}{2}+\mathrm{i}\omega}{\frac{1}{2}+\lambda-\mathrm{i}\omega}\tilde{\xi}(\omega), \tag{7.77}$$

$$\tilde{y}_1(\omega) = -\frac{\frac{1}{2}+\mathrm{i}\omega}{\frac{1}{2}-\mathrm{i}\omega}\tilde{\upsilon}(\omega). \tag{7.78}$$

These obey the commutation relations

$$[\tilde{x}_1(\omega),\tilde{y}_1(\omega')] = 4\pi\mathrm{i}\delta(\omega+\omega')\times\frac{\frac{1}{4}+\omega^2}{\frac{1}{4}+\omega^2+\lambda\left(\frac{1}{2}+\mathrm{i}\omega\right)}. \tag{7.79}$$

At first sight, the presence of the second factor here would seem to be a flaw in the theory, as it shows that the in-loop field does not obey the usual commutation relations for a free field. The commutator in Eq. (7.79) vanishes for low frequencies in the limit $\lambda\to\infty$. However, it must be remembered that the canonical commutation relations for the electromagnetic field, as introduced in Sec. 3.4, are defined between the field at different points in space, but at the same time. If they are defined in terms of frequency components, as here, then these are strictly speaking spatial, not temporal, frequencies. It is only for a field which is free to propagate over an infinite space that the distinction vanishes. In the present case, the in-loop field is spatially confined to a length $c\tau$, and I have let τ go to 0. For τ finite, it is not difficult to see that the only modification is to replace λ in the frequency domain by $\lambda e^{\mathrm{i}\omega\tau}$, as in Sec. 6.5. With this replacement in the above frequency commutation relations, it is possible to show as in Ref. [126] that the time commutation relations are only changed for times greater than τ. That is to say, the canonical (spatial) commutation relations are never violated. The field with operator $b_1(t)$ does not persist for longer than the time τ as it travels from the first to the second cavity, and so there is never in existence two values of the field which violate the canonical commutation relations.

For the Langevin approach to electro-optical feedback, these properties also apply to the field between the source and the detector in the feedback loop. This gives an alternative explanation as to why the in-loop photocurrent may be below the classical limit even for a classical feedback process. Recall that the original explanation in Sec. 6.3.1 was given in terms of the source dynamics. Regardless of the explanation, it is inappropriate to call the in-loop beam squeezed on the basis of a sub-shot noise homodyne photocurrent spectrum. This is a point of some contention, because Shapiro *et al* [126] make the statement that "Quantum mechanically, the feedback loop reduces the in-loop spectrum by squeezing". While not disputing their 'quantum mechanical' explanation, I believe that it is misleading to use the term 'squeezing' in this context. Squeezing is a quantum phenomenon whereby the noise in one quadrature is reduced at the expense of that in the other, due to Heisenberg's relations. For the in-loop light under consideration, the modified two-time commutation relations allow the frequency noise in one quadrature to be lowered while the other quadrature is unchanged. Without the standard (free field) two-time commutation relations, it is meaningless to talk of classical or nonclassical (squeezed) light. Also, the fact that there is an alternative pseudo-classical explanation for the sub-shot noise in-loop photocurrent, which Shapiro *et al* give, strongly suggests that the term squeezing should not be used.

Output Squeezing

Returning to the output field, one finds from Eq. (7.67) that the quadratures of the output field b_2 are (including the time delay τ),

$$\tilde{x}_2(\omega) = \frac{e^{\mathrm{i}\omega\tau}\left(\frac{1}{2}+\mathrm{i}\omega\right)}{\frac{1}{2}+\lambda e^{\mathrm{i}\omega\tau}-\mathrm{i}\omega}\tilde{\xi}(\omega), \tag{7.80}$$

$$\tilde{y}_2(\omega) = \frac{e^{\mathrm{i}\omega\tau}\left(\frac{1}{2}+\mathrm{i}\omega\right)+\lambda}{\frac{1}{2}-\mathrm{i}\omega}\tilde{\upsilon}(\omega). \tag{7.81}$$

That is to say, the x quadrature of the output is unchanged from the b_1 value (apart from the phase change due to the time delay), but the y quadrature has picked up an extra term. This extra term ensures that

$$[\tilde{x}_2(\omega), \tilde{y}_2(\omega')] = 4\pi\mathrm{i}\delta(\omega+\omega'), \tag{7.82}$$

as required because these fields may propagate to infinity. The spectrum for the x quadrature (which is equal to the spectrum of the photocurrent from a homodyne measurement) is defined by

$$S^x(\omega) = \int_{-\infty}^{\infty} \langle \tilde{x}(\omega)\tilde{x}(-\omega')\rangle d\omega', \tag{7.83}$$

and similarly for y. For the output field b_2,

$$S_2^x(\omega) = \frac{\frac{1}{4}+\omega^2}{|\frac{1}{2}+\lambda e^{\mathrm{i}\omega\tau}-\mathrm{i}\omega|^2}, \tag{7.84}$$

$$S_2^y(\omega) = 1/S_2^x(\omega). \tag{7.85}$$

Note that for $\lambda \to \infty$, there is perfect squeezing at low frequencies for the x quadrature, and infinite noise in the y quadrature, as required by Heisenberg's uncertainty relations.

Is there a simple way to understand this result, that the output may be perfectly squeezed, even though the cavity variance in x is unbounded [see Eq. (7.74)]? One may look for an answer by an analogy with electro-optical feedback. The equivalence of the intracavity dynamics with those of electro-optical feedback mediated by homodyne detection is of no use, because homodyne detection destroys the output beam. What would be necessary would be a QND (quantum non-demolition) measurement of the output field. Feedback based on a QND measurement of the output quadrature would have the same effect on the intracavity field as homodyne detection, provided it was efficient. It is not able to produce intracavity squeezing because an extracavity measurement is a poor measurement of the intracavity quadrature. However, if the output field x quadrature emerges from the QND device unchanged (or relatively little changed), then this quantity (the output field quadrature) can be well controlled by feedback. The fact that the feedback acts on the source cavity is relevant only so far as it affects the output. Such QND feedback schemes have been considered by Yamamoto *et al* [155, 73] and Shapiro *et al* [126]. By comparison, a QND measurement of the intracavity quadrature (as considered in Sec. 6.6) is good for controlling the intracavity variance, but of limited use in controlling the output noise.

In the all-optical quadrature feedback scheme, the second cavity can be considered as a QND apparatus for the x quadrature output of the source cavity. As shown above, it does not alter the statistics of the x quadrature which reflects from it. However, it increases the variance of the output y quadrature, as required for a QND apparatus. The nonlinearity used in this case (coupling the quadrature of one mode to that of another) is precisely what has been used to model ideal QND quadrature measurements as in Sec. 6.6. Rather than giving a current out as its measurement result, the output is directly coupled into the dynamics of the source cavity via that interaction Hamiltonian (7.53). This enables fluctuations in the quantity it is monitoring (the output x quadrature) to be suppressed to an arbitrary degree. The absence of the feedback parameter from the numerator of Eq. (7.84) implies that this would be the case even if there were additional processes which add noise to x. This is in contrast to the QND measurement of Sec. 6.6, where the output noise was limited by the intracavity noise because it was the intracavity state which was being monitored [2]. In summary, QND measurement of the output beam enables the correspondence between all-optical and electro-optical feedback to be extended to that output beam, which can be arbitrarily squeezed by that feedback.

It would seem that here is one potential application for all-optical feedback: producing squeezed light. However, as explained above, the coupling giving Eq. (7.68) would have to use a frequency- (but not polarization-) degenerate $\chi^{(2)}$ crystal, as well as a polarization converter. It would seem easier to use the traditional squeezer, a $\chi^{(2)}$ crystal acting as a degenerate parametric oscillator. Intensity-dependent all-optical feedback is even less practical, requiring a low loss $\chi^{(3)}$ nonlinearity to operate. The smallness of higher order nonlinearities is sufficient justification as to why I have not considered all-optical feedback with a coupling dependent on higher order field moments of the driven cavity. In fact, such higher order feedback does not produce any new results. At least in the regime where the second cavity can be adiabatically eliminated, the higher order terms either give a vanishing contribution, or reproduce the results of amplitude or intensity feedback. Thus one can conclude that all-optical feedback is probably not a practical way of controlling quantum noise, although there may be other applications. Nevertheless, the predicted results are interesting, and some experiments should be feasible with current technology.

7.5 Feedback Precluding Measurement

The two all-optical feedback schemes analyzed so far have both had an equivalent electro-optical scheme. The reason for this is that the couplings between driven and source cavity were QND couplings for the driven cavity. Since the driven cavity is slaved to the output of the source cavity, that means that the coupling effectively acts as a measurement of the output field of the source cavity, the result of which directly acts on the source cavity. In this section, I consider a Hamiltonian coupling between the two cavities which does not factorize as the direct product of a Hermitian operator in each cavity. This is a relaxation of the first item listed in Sec. 7.2.2 necessary to turn cascaded systems theory into feedback by measurement. With this relaxation, the feedback master equation obtained in this section cannot be derived from any electro-optical scheme. It is thus a matter of definition as to whether it is considered feedback at all. With the definition I have used so far in this thesis, a feedback scheme which could not have an interpretation in terms of measurement theory would not be possible. For this section, that definition is suspended.

7.5.1 Complex Amplitude Feedback

Consider the following Hamiltonian coupling between the two cavities

$$V = B^\dagger c_2 + c_2^\dagger B, \tag{7.86}$$

[2]However, the same device investigated in Sec. 6.6 could be used as a QND scheme for the output beam, in the same manner as for the QND device which will be investigated in the following chapter, Sec. 8.6.

where B is an operator on the source cavity. If B is Hermitian (up to a phase factor), then this coupling is equivalent to the quadrature-dependent feedback of Sec. 7.3.3. In general, however, this feedback is sensitive to both quadratures simultaneously and hence I have dubbed it complex amplitude feedback. The analysis is identical to that used in the case of quadrature feedback. I will quote the main results.

The master equation arising from complex amplitude feedback, including a squeezed or thermal input bath to the source cavity, is

$$\begin{aligned}\dot{\rho} &= (N+1)\left(\mathcal{D}[c_1+A]\rho - \mathrm{i}\left[\tfrac{\mathrm{i}}{2}(c_1^\dagger A - A^\dagger c_1),\rho\right]\right) + N\left(\mathcal{D}[c_1^\dagger + A^\dagger]\rho - \mathrm{i}\left[\tfrac{\mathrm{i}}{2}(c_1 A^\dagger - A c_1^\dagger),\rho\right]\right)\\ &\quad + M\left(\tfrac{1}{2}[c_1^\dagger + A^\dagger,[c_1^\dagger + A^\dagger,\rho]] + \mathrm{i}\left[\tfrac{\mathrm{i}}{2}[c_1^\dagger, A^\dagger],\rho\right]\right)\\ &\quad + M^*\left(\tfrac{1}{2}[c_1+A,[c_1+A,\rho]] + \mathrm{i}\left[\tfrac{\mathrm{i}}{2}[c_1,A],\rho\right]\right) - \mathrm{i}[H,\rho], \end{aligned} \tag{7.87}$$

where

$$A = \frac{2\mathrm{i}B}{\sqrt{\gamma}}, \tag{7.88}$$

where γ is the large damping rate of the driven cavity as previously. If $A = -\mathrm{i}F$, where F is Hermitian, then this equation is equal to Eq. (6.62) for quadrature feedback.

The QLE equivalent to the master equation (7.87) is

$$\begin{aligned} ds &= \frac{(N+1)}{2}\left(2\bar{c}_1^\dagger s \bar{c}_1 - s\bar{c}_1^\dagger \bar{c}_1 - \bar{c}_1^\dagger \bar{c}_1 s - [c_1^\dagger A - A^\dagger c_1, s]\right) dt\\ &\quad + \frac{N}{2}\left(2\bar{c}_1 s \bar{c}_1^\dagger - s\bar{c}_1\bar{c}_1^\dagger - \bar{c}_1\bar{c}_1^\dagger s - [c_1 A^\dagger - A c_1^\dagger, s]\right) dt\\ &\quad + \frac{M}{2}\left([\bar{c}_1^\dagger,[\bar{c}_1^\dagger,s]] + [[c_1^\dagger, A^\dagger],s]\right) dt + \frac{M^*}{2}\left([\bar{c}_1,[\bar{c}_1,s]] + [[c_1,A],s]\right) dt\\ &\quad - [dB_0^\dagger \bar{c}_1 - \bar{c}_1^\dagger dB_0, s] + \mathrm{i}[H,s]dt, \end{aligned} \tag{7.89}$$

where

$$\bar{c}_1 = c_1 + A. \tag{7.90}$$

This equation can be derived from the effective feedback Hamiltonian

$$H_{\mathrm{fb}} = \mathrm{i}(b_1^\dagger A - A^\dagger b_1). \tag{7.91}$$

The output field, reflected off the mirror of the second cavity, is

$$b_2 = -(b_1 + A) = -(b_0 + c_1 + A). \tag{7.92}$$

Again, these equations are equivalent to the quadrature feedback equations when $A = -\mathrm{i}F$.

Producing Non-Classical Light

One feature which distinguishes Eq. (7.87) from the quadrature feedback equation (6.62) is that for the optical case ($c_1 = a$), it can produce a nonclassical state in the source cavity even if A is linear in a and $a^\dagger$. To see this, consider the case $N = M = 0$, $H = 0$ to prevent any obscuring effects. The feedback master equation can then be rewritten as

$$\dot{\rho} = \mathcal{D}[a]\rho + \mathcal{D}[A]\rho + (A\rho a^\dagger + a\rho A^\dagger - A^\dagger a\rho - \rho a^\dagger A). \tag{7.93}$$

For A linear, this equation can be converted into a Fokker-Planck equation for the Glauber-Sudarshan P function representation of the density operator [52]. The condition for an initially positive P

function (representing a classical state) to remain so is that the diffusion matrix be positive semi-definite. The first term (damping) and third term (enclosed in round brackets above) will only give first order derivatives. Thus one needs to consider only the second term. Let

$$A = \frac{\lambda}{2}(a + \mu a^\dagger), \tag{7.94}$$

which can be achieved physically by the same means as Eq. (7.68) in Sec. 7.4. It can be readily shown that the eigenvalues of the diffusion matrix are proportional to $|\mu|^2 \pm |\mu|$. That is to say, if $0 < |\mu| < 1$, then this complex amplitude feedback will produce a nonclassical state in the cavity. The $|\mu| = 1$ limit gives the case of quadrature feedback, which, as shown in Sec. 7.3.3, cannot produce a nonclassical intracavity state. The other limit at $|\mu| = 0$ corresponds to simply shining the in-loop beam into a second mirror of the source cavity, not even requiring a nonlinear crystal (it is a classical geometrical optics problem). The property of nonclassicality from a linear feedback operator distinguishes this all-optical feedback from any form of electro-optical feedback.

Consider the case where λ is real and positive and μ real. This gives independent linear equations for the quadratures x, y of the field. Specifically, from Eq. (7.89), one obtains

$$\dot{x} = -\frac{1}{2}\left[1 + \lambda(1-\mu) + \left(\frac{\lambda}{2}\right)^2 (1-\mu^2)\right] x - \left[1 + \frac{\lambda}{2}(1-\mu)\right]\xi, \tag{7.95}$$

where ξ is as above. The equation for the y quadrature is identical, but for the replacement of x by y, ξ by υ, and μ by $-\mu$. Note that for $\mu = -1$, these equations agree with Eqs. (7.70,7.71), as required. The intracavity steady-state variance for x is

$$V^x = \frac{1 + \lambda(1-\mu) + \left(\frac{\lambda}{2}\right)^2 (1-\mu)^2}{1 + \lambda(1-\mu) + \left(\frac{\lambda}{2}\right)^2 (1-\mu^2)}. \tag{7.96}$$

Note that for $0 < \mu < 1$, this implies a variance in x less than the unit variance of a coherent state. For $0 < -\mu < 1$, the variance in x will be greater than one, but that for y [obtained by replacing μ by $-\mu$ in Eq. (7.96)] will be less than one. This is in accord with the result stated above, that for $0 < |\mu| < 1$, this linear all-optical feedback will produce a non-classical intracavity state. Furthermore, in the limit where λ is very large, and $\epsilon = 1 - \mu$ is very small (but not as small as λ^{-1}), then

$$V^x \to \epsilon/2. \tag{7.97}$$

That is to say, the intracavity state can be arbitrarily squeezed.

Now consider the output squeezing. From the method of Sec. 7.4, the output x spectrum can be calculated to be

$$S_2^x(\omega) = \frac{\frac{1}{4}(\sigma + \lambda\mu)^2 + \omega^2}{\frac{1}{4}(\sigma - \lambda\mu)^2 + \omega^2}, \tag{7.98}$$

where

$$\sigma = 1 + \lambda + \left(\frac{\lambda}{2}\right)^2 (1-\mu^2). \tag{7.99}$$

The spectrum $S_2^y(\omega)$ can be found by replacing μ by $-\mu$, as before. As shown in Sec. 7.3, the output of the feedback loop can show perfect squeezing even for 'classical' feedback with $|\mu| = 1$. Consider $0 < |\mu| < 1$, so that σ is always positive. For low frequencies, the output x quadrature is squeezed, with $S_2^x(0) < 1$, for $\mu < 0$. Note that this is the sign of μ which produces intracavity squeezing in the y quadrature. For $\mu > 0$, the output y quadrature is squeezed, while inside the cavity the x

quadrature exhibits the nonclassical statistics. In understanding these counter-intuitive results, it must be remembered that the output beam b_2 is not simply the output of the source cavity; the statistics of both quadratures are changed by its action as the feedback control beam.

7.5.2 Electro-Optic Analog

As stated above, all-optical complex amplitude feedback has no electro-optical counterpart in general. This is because it is not possible to measure both the x and y quadratures of the output field simultaneously with unit efficiency. Even a QND measurement of one quadrature would introduce noise into the other and so prevent a measurement of both. However, it is possible to do two inefficient measurements of both quadrature with the two efficiencies adding to one (or less than one in practice). It is simplest to consider heterodyne detection, which is equivalent to a homodyne measurement of each quadrature, each with efficiency of one half. This has been considered in Sec. 6.3.2.

In order to compare this all-optical complex amplitude feedback to feedback from heterodyne detection, it is convenient to rewrite Eq. (7.87) in terms of the Hermitian operators F, G defined by

$$A = G - \mathrm{i}F. \tag{7.100}$$

One obtains

$$\begin{aligned}
\dot{\rho} \;=\;& -\mathrm{i}[H,\rho] + (N+1)\left(\mathcal{D}[c_1]\rho - \mathrm{i}[F, c_1\rho + \rho c_1^\dagger] - \mathrm{i}[G, -\mathrm{i}c_1\rho + \mathrm{i}\rho c_1^\dagger] + \mathcal{D}[G - \mathrm{i}F]\rho\right) \\
& + N\left(\mathcal{D}[c_1^\dagger]\rho + \mathrm{i}[F, c_1^\dagger\rho + \rho c_1] + \mathrm{i}[G, \mathrm{i}c_1^\dagger\rho - \mathrm{i}\rho c_1] + \mathcal{D}[G + \mathrm{i}F]\rho\right) \\
& + M\left(\tfrac{1}{2}[c_1^\dagger, [c_1^\dagger, \rho]] + \mathrm{i}[F, [c_1^\dagger, \rho]] - \mathrm{i}[G, [\mathrm{i}c_1^\dagger, \rho]] + \tfrac{1}{2}[G + \mathrm{i}F, [G + \mathrm{i}F, \rho]]\right) \\
& + M^*\left(\tfrac{1}{2}[c_1, [c_1, \rho]] - \mathrm{i}[F, [c_1, \rho]] + \mathrm{i}[G, [-\mathrm{i}c_1, \rho]] + \tfrac{1}{2}[G - \mathrm{i}F, [G - \mathrm{i}F, \rho]]\right).
\end{aligned} \tag{7.101}$$

If A is linear in the field amplitude, then G and F are proportional to two orthogonal quadratures of the field. For controlling noise, it would be sensible for these to be proportional to the x and y quadratures respectively, as in the preceding section.

For heterodyne detection, the feedback Hamiltonian analogous to Eq. (7.91) with $A = G - \mathrm{i}F$ is

$$H_{\mathrm{fb}}(t) = I_{\mathrm{c}}^x(t)F + I_{\mathrm{c}}^y(t)G, \tag{7.102}$$

as considered in Sec. 6.3.2. The master equation derived there can be rewritten

$$\begin{aligned}
\dot{\rho} \;=\;& -\mathrm{i}[H,\rho] + (N+1)\left(\mathcal{D}[c_1]\rho - \mathrm{i}[F, c_1\rho + \rho c_1^\dagger] - \mathrm{i}[G, -\mathrm{i}c_1\rho + \mathrm{i}\rho c_1^\dagger] + 2\mathcal{D}[F] + 2\mathcal{D}[G]\right) \\
& + N\left(\mathcal{D}[c_1^\dagger]\rho + \mathrm{i}[F, c_1^\dagger\rho + \rho c_1] + \mathrm{i}[G, \mathrm{i}c_1^\dagger\rho - \mathrm{i}\rho c_1] + 2\mathcal{D}[F] + 2\mathcal{D}[G]\right) \\
& + M\left(\tfrac{1}{2}[c_1^\dagger, [c_1^\dagger, \rho]] + \mathrm{i}[F, [c_1^\dagger, \rho]] - \mathrm{i}[G, [\mathrm{i}c_1^\dagger, \rho]] + 2\mathcal{D}[F] - 2\mathcal{D}[G]\right) \\
& + M^*\left(\tfrac{1}{2}[c_1, [c_1, \rho]] - \mathrm{i}[F, [c_1, \rho]] + \mathrm{i}[G, [-\mathrm{i}c_1, \rho]] + 2\mathcal{D}[F] - 2\mathcal{D}[G]\right).
\end{aligned} \tag{7.103}$$

Note that the desired feedback terms (linear in G and F) are the same in this equation as in the all-optical Eq. (7.101), but the diffusion terms are different.

To elucidate this difference, consider the most basic example of all-optical feedback, alluded to above. The output of the source cavity is simply fed back into another mirror of that cavity. Let the loss rate of both mirrors be κ, and let the bath input be the vacuum. This is simply a geometrical optics problem, with the annihilation operator for the cavity obeying

$$\dot{a}(t) = -\frac{\kappa}{2}a(t) - \sqrt{\kappa}\nu(t) - \frac{\kappa}{2}a(t) - \sqrt{\kappa}[\sqrt{\kappa}a(t-\tau) + \nu(t-\tau)]. \tag{7.104}$$

In the Markovian approximation, the only effect of the time delay τ is to cause a phase shift ϕ in the field enclosed in square brackets above. Such feedback is covered by the master equation derived in this section, with

$$A = \sqrt{\kappa} e^{\mathrm{i}\phi} a. \tag{7.105}$$

This is one case where a second mode is evidently not necessary. From Eq. (7.93), the master equation for the cavity is

$$\dot{\rho} = 2\kappa(1 + \cos\phi)\mathcal{D}[a]\rho - \mathrm{i}\kappa \sin\phi [a^\dagger a, \rho], \tag{7.106}$$

which describes a new cavity with modified frequency and linewidth.

Now consider a cavity with one output mirror of loss rate κ subject to heterodyne detection. Attempting to replicate the all-optical feedback just described by feeding back the heterodyne photocurrent would require

$$H_{\mathrm{fb}}(t) = \tfrac{1}{2}\left(I_{\mathrm{c}}^{x}(t)[e^{\mathrm{i}\phi} a + e^{-\mathrm{i}\phi} a^\dagger] + \mathrm{i} I_{\mathrm{c}}^{y}(t)[e^{\mathrm{i}\phi} a - e^{-\mathrm{i}\phi} a^\dagger] \right). \tag{7.107}$$

This describes driving of the cavity (which would require a second mirror with negligible loss rate) with a coherent field of variable complex amplitude. Equation (7.103) then yields the master equation

$$\dot{\rho} = 2\kappa(1 + \cos\phi)\mathcal{D}[a]\rho - \mathrm{i}\kappa \sin\phi [a^\dagger a, \rho] + \kappa(\mathcal{D}[a] + \mathcal{D}[a^\dagger])\rho. \tag{7.108}$$

The equation of motion for the mean field from this equation is identical to that of Eq. (7.106). However, the presence of the extra term introduces noise into both quadratures equally. If $\phi = \pi$, so that the deterministic dynamics are eliminated [3], then the variance in each quadrature will simply grow linearly. This clearly shows the effect of the noise introduced by attempting to measure both quadratures in electro-optical feedback, as opposed to the coherent back-coupling of both quadratures in all-optical feedback. In general, the similarities and differences between all-optical and electro-optical feedback yield important insights about the nature of feedback.

[3] Of course, this is an approximation only. In reality, the lifetime of the cavity is enhanced by a factor of order $1/\kappa\tau \gg 1$.

Chapter 8

Using Feedback to Eliminate Back-Action

A quantum nondemolition (QND) or back-action evading (BAE) measurement is one which does not disturb the quantity it measures. It is possible to consider any measurement as a QND measurement followed by an additional back-action which may disturb the measured quantity. This chapter is concerned with the use of feedback to eliminate that unnecessary back-action, particularly for continuous measurements. I show that it is impractical to exactly eliminate the back action of direct photodetection. However, it is quite possible to do so for homodyne measurements. I propose a scheme which uses a $\chi^{(2)}$ crystal in a cavity, acting as a degenerate parametric oscillator at threshold. The feedback controls the coherent driving of the cavity, producing positive feedback of unit gain. As well as enabling a QND measurement of the x quadrature of the cavity, the device can be used to measure the quadrature of a traveling wave. I review the criteria for evaluating such measurements and show that in the limit of high nonlinearity the QND correlations are all close to one over some bandwidth.

8.1 First and Second Kind Measurements

The most general theory of quantum measurements was given in Sec. 2.1, in terms of operations and effects. There are many ways of dividing such measurements into two classes. For instance, in that section, I classified measurements as efficient or inefficient. In this chapter, I am concerned with a different classification of quantum measurements: those which do not disturb the quantity they measure, and those which do. This is in fact the oldest binary classification, dating to the 1933 book by Pauli [108]. He called these measurements of the first and second kind respectively. There are various definitions which make this distinction precise, appropriate for different applications. In this chapter, at least initially, I will use a definition which is appropriate only for efficient measurements.

First kind measurements are obviously better measurements in some sense. They are also known as quantum non-demolition (QND) or back-action evading (BAE) measurements. The interest in such measurements was revived in 1980 by the realization that planned gravitational wave detectors would only work if a QND measurement of position could be made [27]. Since then, the application of QND measurements in quantum optical communication systems has been of considerable interest. Several theoretical schemes have been proposed [103, 81, 1, 18, 79, 38], and some convincing experiments performed [87, 67]. For such experiments, a more practical definition is needed which handles inefficiencies without trouble. This will be investigated in Sec. 8.5.

For efficient measurements, the most elegant (in my opinion) definition for first and second kind measurements is that given in Ref. [17]. It follows directly from the operations and effects formalism of Sec. 2.1. Recall that for efficient measurements, the general theory can be formulated solely in terms of the set of what I have called measurement operators $\Omega_\alpha(T)$, where α denotes a measurement result and T the duration of the measurement. The operators $\Omega_\alpha(T)$ are arbitrary, apart from the condition that

$$\sum_\alpha \Omega_\alpha(T)^\dagger \Omega_\alpha(T) = 1. \tag{8.1}$$

The state of the system conditioned on the result α is simply given by

$$\rho_\alpha(t+T) = \Omega_\alpha(T)\rho(t)\Omega_\alpha^\dagger(T)/\mathrm{Pr}[\alpha], \tag{8.2}$$

where

$$\mathrm{Pr}[\alpha] = \mathrm{Tr}[W_\alpha(T)\rho(t)] \tag{8.3}$$

is the probability for that result. Recall that $W_\alpha(T)$ is called the effect and is given by

$$W_\alpha(T) = \Omega_\alpha^\dagger(T)\Omega_\alpha(T). \tag{8.4}$$

Note that many different sets of measurement operators Ω_α may have the same set of probability generating operators W_α. That is to say, a measurement is not completely specified by the probabilities of obtaining the results. What is missing is a further specification of the *back-action* of the apparatus on the system. This can be seen specifically in the case where all of the W_α are bounded operators [78, 17]. Then the operators $\Omega_\alpha(T)$ can be written

$$\Omega_\alpha(T) = U_\alpha(T)V_\alpha(T), \tag{8.5}$$

where

$$V_\alpha(T) = \sqrt{W_\alpha(T)} = V_\alpha^\dagger(T) \tag{8.6}$$

and

$$U_\alpha^\dagger(T) = U_\alpha^{-1}(T). \tag{8.7}$$

That is, $\Omega_\alpha(T)$ can be written as the product of a unitary and an Hermitian operator. Assuming a pure initial state, the unnormalized conditioned state vector can thus be written

$$|\tilde{\psi}_\alpha(t+T)\rangle = U_\alpha(T)V_\alpha(T)|\psi(t)\rangle. \tag{8.8}$$

The action of $V_\alpha(T)$ produces the minimum change in the system, required by Heisenberg's relation, to be consistent with a measurement giving the information about the state specified by the probabilities (8.3). The action of $U_\alpha(T)$ represents additional back-action, an unnecessary perturbation of the system.

A back-action evading measurement is reasonably defined by the requirement that, for all α, $U_\alpha(T)$ equals unity (up to a phase factor which can be ignored without loss of generality). This is equivalent to the requirement that all $\Omega_\alpha(T)$ be Hermitian. One criticism of this definition is that it disallows any Hamiltonian evolution of the system during the measurement. Such evolution would contribute a unitary $\exp[-\mathrm{i}HT]$ to *all* measurement operators $\Omega_\alpha(T)$. This evolution can thus be removed from all of the $U_\alpha(T)$ by making the unitary transformation into the interaction picture.

Thus, a better requirement for back-action evasion is that all $U_\alpha(T)$ be unity in the interaction picture. This is the definition which I will use to distinguish first kind measurements from all other (second kind) measurements.

For the case of continuous measurements (considered in Sec. 2.3), the measurement operators appropriate for the nonselective evolution $\dot{\rho} = \mathcal{D}[c]\rho$ are

$$\Omega_1(dt) = \sqrt{dt}\, c, \tag{8.9}$$
$$\Omega_0(dt) = 1 - \tfrac{1}{2}c^\dagger c dt. \tag{8.10}$$

The Hamiltonian evolution is absent in the interaction picture as explained above. The operator $\Omega_0(dt)$ defined in Eq. (8.10) is evidently Hermitian. Thus the classification of the measurement as first or second kind depends on whether c is Hermitian. In the case of simple damping of a cavity, c is proportional to the annihilation operator a of the cavity mode, which is not Hermitian. Hence the measurements permitted by damping are necessarily quantum demolition (second kind) measurements. The non-Hermitiancy of a in this case can be seen explicitly using the factorization (8.5)

$$a = e^{\mathrm{i}\Phi}\sqrt{a^\dagger a}, \tag{8.11}$$

where Φ is a phase operator for the single mode field. The difficulty in defining this operator [132, 110] is due to the fact that $a^\dagger a$ is not a bounded operator, which violates the assumptions made in writing down Eq. (8.5).

8.2 Back-Action Elimination by Feedback

In this section I will show that feedback can, in principle, turn any second kind measurement into a first kind measurement. In practice, the effectiveness of feedback is limited by the way that the system dynamics can be controlled. That is why I spent most of Ch. 6 discussing feedback controlling only the linear dynamics of a cavity mode. Here, I wish to ignore this limitation, and assume that the dynamics of the system can be arbitrarily well controlled. In that case, it is evident from the preceding section that feedback can eliminate the back-action of any measurement. The back-action of a particular measurement result α is produced by the unitary operator $U_\alpha(T)$ of Eq. (8.5). Hence, if the result α is obtained, then the Hamiltonian of the system should be changed by a large amount for a short time in order to induce the evolution $U_\alpha^{-1}(T)$. This turns the quantum demolition measurement with measurement operators $\Omega_\alpha(T)$ into a QND measurement with Hermitian measurement operators $V_\alpha(T)$.

For the case of a Markovian system with a single output, the requirement is for the feedback to act immediately following a detection. The induced evolution should undo the unitary component of the quantum jump caused by the operator c of Eq. (8.9). Say c is factorized as

$$c = e^{\mathrm{i}Z}\sqrt{c^\dagger c}, \tag{8.12}$$

so that $U_1 = e^{\mathrm{i}Z}$. Then it is evident from Ch. 6 that the necessary feedback Hamiltonian is

$$H_{\mathrm{fb}}(t) = I_{\mathrm{c}}(t)Z. \tag{8.13}$$

This gives the selective evolution of Eq. (6.12)

$$d\rho_{\mathrm{c}}(t) = \left\{dN_{\mathrm{c}}(t)\mathcal{G}[e^{-\mathrm{i}Z}c] - dt\mathcal{H}[-\tfrac{1}{2}c^\dagger c]\right\}\rho_{\mathrm{c}}(t), \tag{8.14}$$

which is equivalent to

$$d\rho_c(t) = \left\{ dN_c(t)\mathcal{G}[\sqrt{c^\dagger c}] - dt\mathcal{H}[-\tfrac{1}{2}c^\dagger c] \right\} \rho_c(t), \tag{8.15}$$

with $\mathrm{E}[dN_c(t)] = \mathrm{Tr}[c^\dagger c \rho_c(t)]$ as before. Evidently the new measurement with instantaneous feedback is indistinguishable from a continuous QND measurement with collapse operator $\sqrt{c^\dagger c}$. The new master equation is of course

$$\dot{\rho} = \mathcal{D}[\sqrt{c^\dagger c}]\rho. \tag{8.16}$$

As I stated above, this result assumes that the Hamiltonian of the system may be modified arbitrarily. In practice, this may not be so easy. For the case of direct detection of the photons emitted by a cavity, Eq. (8.11) shows that it would be necessary to have

$$Z = \Phi. \tag{8.17}$$

This could be produced by a nonlinear crystal which has an effective Hamiltonian proportional to the phase of the field, and which in addition can be turned on and off somehow by a current. It is extremely unlikely that such a crystal could be found. However, if one is only concerned with states which are close to some large coherent amplitude α then it is possible to obtain an approximate expression for Φ which is at most quadratic in the field operators and so could be produced by a simple $\chi^{(2)}$ nonlinearity. The procedure is as follows. Write

$$a = \alpha + \tfrac{1}{2}(x + \mathrm{i}y), \tag{8.18}$$

where x and y are operators of order unity. Let α be real so that x represents amplitude and y phase fluctuations. The quasi-unitary operator $e^{\mathrm{i}\Phi}$ is then defined by

$$U_1 = a(\sqrt{a^\dagger a})^{-1} \tag{8.19}$$

Expanding this to second order in $1/\alpha$ using Eq. (8.18) gives

$$e^{\mathrm{i}\Phi} \simeq 1 + \frac{\mathrm{i}y}{2\alpha} - \frac{y^2}{8\alpha^2} - \frac{\mathrm{i}yx}{4\alpha^2}. \tag{8.20}$$

Taking the log of both sides yields

$$\Phi \simeq \frac{y}{2\alpha} - \frac{yx}{4\alpha^2}. \tag{8.21}$$

This Hamiltonian is quite practical, as will be explained in Sec. 8.4. In this analysis, the large coherent amplitude of the intracavity field effectively acts as a local oscillator, so that the end result is an approximate QND measurement of the amplitude quadrature x. In Sec. 8.4 I show that the same form of feedback can be used to turn homodyne measurement into an exact QND measurement of one quadrature, for an arbitrary cavity state.

8.3 Putting the Photon Back

The effect of the operator $e^{-\mathrm{i}\Phi}$ is to translate the state in the photon number representation upward by one. Thus the action of the idealized feedback considered in the preceding section could be thought of as putting back the photon which was demolished by the direct detection. Note that the feedback does not undo the effect of the detection; it simply undoes the effect on the measured quantity (here photon number) while leaving the phase diffusion. From this explanation one might think that the photon could be 'put back' simply by sending excited atoms quickly through the cavity until one is detected to have given up a quantum of energy to the field. In this section, I show that the resultant master equation does not affect photon number statistics, as required for a QND measurement. However, it does affect the phase distribution to a greater extent than required by the information gained in the measurement. Specifically, in the limit of large photon numbers, the rate of phase diffusion is twice what it should be. In that sense, the measurement is not a QND measurement in the strict sense which I am using, because it is not an efficient measurement. The measurement scheme cannot be written in terms of measurement operators, but rather requires measurement operations.

Consider the passage of a two level atom through a cavity, where the interaction Hamiltonian is

$$H = \mathrm{i}\frac{g}{2}(\sigma a^\dagger - \sigma^\dagger a), \tag{8.22}$$

where a is the annihilation operator for the cavity and $\sigma = |1\rangle\langle 2|$ is the lowering operator for the atom. Let the atom initially be in the excited state $|2\rangle$, and let the interaction time 2τ be such that $g\tau\sqrt{\mu} \ll 1$, where μ is the mean photon number. Then the unitary operator $\exp(-\mathrm{i}2H\tau)$ acting on the initially factorized state $\rho \otimes |2\rangle\langle 2|$ can be expanded to second order to give the entangled state

$$R = \rho \otimes |2\rangle\langle 2| + g\tau\left(a^\dagger \rho \otimes |1\rangle\langle 2| + \text{H.c.}\right) + (g\tau)^2\left(a^\dagger \rho a \otimes |1\rangle\langle 1| - \tfrac{1}{2}\{aa^\dagger, \rho\} \otimes |2\rangle\langle 2|\right). \tag{8.23}$$

Now if the outgoing atom is detected in the excited state, the unnormalized conditioned state of the field is

$$\tilde{\rho}_e = \langle 2|R|2\rangle = \left(1 - (g\tau)^2 \mathcal{A}[a^\dagger]\right)\rho = \exp[-(g\tau)^2 aa^\dagger/2]\rho\exp[-(g\tau)^2 aa^\dagger/2], \tag{8.24}$$

where the superoperator $\mathcal{A}$ is as defined in Eq. (2.34). If the atom is detected in the ground state (which happens rarely), the state is

$$\tilde{\rho}_g = \langle 1|R|1\rangle = (g\tau)^2 \mathcal{J}[a^\dagger]\rho, \tag{8.25}$$

where the jump superoperator is defined in Eq. (2.8).

If the first atom is detected in the ground state, then the field has gained a photon and the process can stop. If it is detected in the upper state, one must try again with another (or the same) atom. Say K atoms are required before the $(K+1)$th is detected in the ground state. The unnormalized state after the $(K+1)$th atom is

$$\tilde{\rho}_K = (g\tau)^2 \mathcal{J}[a^\dagger]\exp[-K(g\tau)^2 aa^\dagger/2]\rho\exp[-K(g\tau)^2 aa^\dagger/2], \tag{8.26}$$

The norm of this density operator is equal to the probability that this many atoms are needed. Thus, the average density operator, given that an atom is finally detected in the ground state, is

$$\rho' = \sum_{K=0}^{\infty} \tilde{\rho}_K. \tag{8.27}$$

Using the fact that $g\tau$ is small, the sum can be converted to an integral by setting $u = (g\tau)^2 K$:

$$\rho' = \mathcal{J}[a^\dagger]\int_0^\infty \exp(-uaa^\dagger/2)\rho\exp(-uaa^\dagger/2)du. \tag{8.28}$$

This can be formally evaluated as

$$\rho' = \mathcal{J}[a^\dagger]\left(\mathcal{A}[a^\dagger]\right)^{-1}\rho. \tag{8.29}$$

This superoperator previously turned up in the investigation of the dynamics of an ideal laser in Sec. 5.6. It is easy to verify that it is well-defined and has unit norm.

Now, the purpose of sending these atoms through the cavity is to replace the photon lost when it is detected at the photodetector. The stochastic evolution equation for the conditional master equation is thus

$$d\rho_{\rm c}(t) = dN_{\rm c}(t)\left\{\mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1}\left(\mathcal{G}[a]+1\right)-1\right\}\rho_{\rm c}(t) + dt\mathcal{H}\left[-\tfrac{1}{2}a^\dagger a\right]\rho_{\rm c}(t), \tag{8.30}$$

where $\mathrm{E}[dN_{\rm c}(t)] = \mathrm{Tr}[\mathcal{J}[a]\rho_{\rm c}(t)]$ as before. Taking the ensemble average gives the master equation

$$\dot\rho = \left(\mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1}\mathcal{J}[a] - \mathcal{A}[a]\right)\rho. \tag{8.31}$$

In the photon number representation,

$$\dot\rho_{nm} = \left[\frac{2nm}{n+m} - \frac{n+m}{2}\right]\rho_{nm}. \tag{8.32}$$

Obviously, this leaves the photon number populations ($n=m$) unchanged. The off-diagonal elements will decay, which causes phase diffusion. This can be seen by considering the high photon number limit. Expanding in powers of the reciprocal of the photon number gives

$$\dot\rho_{nm} \simeq -\frac{(n-m)^2}{2(n+m)}\rho_{nm}. \tag{8.33}$$

These characteristics of the master equation are just what would be expected from a QND measurement of the photon number: it preserves photon populations but destroys coherences. However, the master equation for the ideal case of eliminating the back-action of photodetection by the feedback Hamiltonian $I_{\rm c}(t)\Phi$ of Sec. 8.2 is

$$\dot\rho = \mathcal{D}[\sqrt{a^\dagger a}]\rho. \tag{8.34}$$

In the photon number representation, this is

$$\dot\rho_{nm} = \left[\sqrt{nm} - \frac{n+m}{2}\right]\rho_{nm}. \tag{8.35}$$

Expanding as above yields

$$\dot\rho_{nm} \simeq -\frac{(n-m)^2}{4(n+m)}\rho_{nm}. \tag{8.36}$$

Thus, the rate of phase diffusion from the attempt to eliminate back-action using atomic feedback is twice that of the ideal case. That is to say, the back-action has not been completely eliminated; there is additional back-action in the form of excess phase diffusion above that required by the measurement. As stated above, this back-action is unimportant if one is only interested in photon populations. For simply determining the number of photons inside a cavity without changing that number, the scheme proposed in this section would work, at least in principle.

8.4 QND Homodyne Measurement by Feedback

8.4.1 From the Homodyne SME

Recall from Sec. 6.3 that feedback of the homodyne detection photocurrent by the Hamiltonian

$$H_{\rm fb}(t) = FI_{\rm c}^{\rm hom}(t), \tag{8.37}$$

gives the stochastic master equation for the cavity mode

$$\begin{aligned} d\rho_{\rm c}(t) &= dt\left\{-{\rm i}[H,\rho_{\rm c}(t)]+\mathcal{D}[a]\rho_{\rm c}(t)-{\rm i}[F,a\rho_{\rm c}(t)+\rho_{\rm c}(t)a^\dagger]+\mathcal{D}[F]\rho_{\rm c}(t)\right\} \\ &\quad +dW(t)\mathcal{H}[a-{\rm i}F]\rho_{\rm c}(t). \end{aligned} \tag{8.38}$$

If this is written in terms of the quadratures of the field, defined by $a=\frac{1}{2}(x+{\rm i}y)$, it has the suggestive form

$$\begin{aligned} d\rho_{\rm c}(t) &= dt\left\{-{\rm i}[H+\tfrac{1}{4}(x-{\rm i}y)F+\tfrac{1}{4}F(x+{\rm i}y),\rho_{\rm c}(t)]+\mathcal{D}[x/2+{\rm i}(y/2-F)]\rho_{\rm c}(t)\right\} \\ &\quad +dW(t)\mathcal{H}[x/2+{\rm i}(y/2-F)]\rho_{\rm c}(t). \end{aligned} \tag{8.39}$$

Choosing $F=y/2$, one obtains

$$d\rho_{\rm c}(t)=dt\left\{-{\rm i}\left[H+\tfrac{1}{8}(xy+yx),\rho_{\rm c}(t)\right]+\mathcal{D}[x/2]\rho_{\rm c}(t)\right\}+dW(t)\mathcal{H}[x/2]\rho_{\rm c}(t). \tag{8.40}$$

This is precisely the conditioning equation appropriate for a QND measurement of the x quadrature as in Sec. 6.6, apart from the extra term in the Hamiltonian. This term increases the variance in the x quadrature, and so the measurement with feedback does not leave the statistics of the x quadrature unchanged.

This unwanted effect of the feedback can be overcome by adding another parametric term of opposite sign, $H_0=-\frac{1}{8}(xy+yx)$. That is, the total Hamiltonian required is

$$H(t)=H_0+H_{\rm fb}(t)=-\tfrac{1}{8}(xy+yx)+I_{\rm c}^{\rm hom}(t)y/2, \tag{8.41}$$

where $H_{\rm fb}(t)$ is as defined in Eq. (8.37). Then the evolution of the system is exactly that required of a Markovian QND measurement of the x quadrature. In the nonselective case,

$$\dot\rho=\mathcal{D}[x/2]\rho\equiv\mathcal{L}\rho. \tag{8.42}$$

The expression for the current is unchanged as

$$I_{\rm c}^{\rm hom}(t)=\langle x\rangle_{\rm c}(t)+\xi(t) \tag{8.43}$$

However, the two-time correlation function is altered to become

$$E[I_{\rm c}^{\rm hom}(t+\tau)I_{\rm c}^{\rm hom}(t)]={\rm Tr}\left[xe^{\mathcal{L}\tau}\tfrac{1}{2}\{x,\rho(t)\}\right]+\delta(\tau). \tag{8.44}$$

Note that this will always give a super-shot noise spectrum because it measures symmetrically ordered moments for x, rather than normally ordered moments as from homodyne detection without feedback.

Unlike the phase operator Φ necessary for creating direct QND detection, the Hamiltonian (8.41) giving homodyne QND detection could be achieved relatively easily in the laboratory. The feedback Hamiltonian y simply corresponds to controlling the driving onto the cavity. This could be achieved by controlling the intensity of a strong coherent beam (the same source as the local oscillator could be used) incident on another mirror of the cavity. If the transmittivity of this mirror is sufficiently high, the extra damping it causes can be ignored. Alternatively, the driving could take place at the same mirror as the damping, but the modulated amplitude removed from the output beam before it is detected. This will be explained in detail in Sec. 8.6. The intensity modulator could be effected by using a electro-optic polarization modulator combined with a polarization-dependent beam splitter. The auxiliary Hamiltonian $H_0=-(xy+yx)/8$ is well approximated by the action of a degenerate parametric oscillator (DPO) below threshold. In fact, the magnitude of this DPO nonlinearity puts it at threshold in the cavity [1]. However, this difficulty can be avoided by assuming that there are other linear losses apart from that allowing the measurement to be made. This will be the case in practice, and is necessary if the device is to be used to monitor a traveling wave, as will be investigated in the following sections.

8.4.2 From the Homodyne Measurement Operators

It is natural to ask what is the connection between this elimination of back-action by feedback, and that derived from general principles in Sec. 8.2? To do this it is necessary to return to the measurement operators for homodyne measurement before the large amplitude approximation has been made. From Sec. 4.2.1 these are, with the local oscillator amplitude γ real,

$$\Omega_1(dt) = \sqrt{dt}(a+\gamma), \tag{8.45}$$

$$\Omega_0(dt) = 1 - \tfrac{1}{2}dt\left[(a\gamma - a^\dagger\gamma) + (a+\gamma)^\dagger(a+\gamma)\right]. \tag{8.46}$$

Consider first the measurement operator for a detection, $\Omega_1(dt)$, and write it in a form whereby its factorization (8.5) can be easily accomplished. Keeping terms up to second order in $1/\gamma$,

$$\Omega_1(dt) = \sqrt{dt}\,\gamma\exp\left(\frac{a}{\gamma} - \frac{a^2}{2\gamma^2}\right). \tag{8.47}$$

Using the Baker-Hausdorff theorem [92] for the first order terms, this can be factorized (to second order in $1/\gamma$) as

$$\Omega_1(dt) = U_1 V_1(dt), \tag{8.48}$$

where

$$U_1 = \exp\left(\frac{a-a^\dagger}{2\gamma} - \frac{a^2 - a^{\dagger 2}}{4\gamma^2}\right) = \exp(\mathrm{i}Z) \tag{8.49}$$

is unitary, and

$$V_1(dt) = \sqrt{dt}\,\gamma\exp\left(\frac{a+a^\dagger}{2\gamma} - \frac{a^2 + a^{\dagger 2} + [a^\dagger, a]}{4\gamma^2}\right) = \sqrt{dt\left[\gamma^2 + \gamma(a+a^\dagger) + a^\dagger a\right]} \tag{8.50}$$

is Hermitian.

From the expression for U_1 (8.49), it is evident that the Hermitian operator Z is

$$Z = \frac{-\mathrm{i}a + \mathrm{i}a^\dagger}{2\gamma} + \frac{\mathrm{i}a^2 - \mathrm{i}a^{\dagger 2}}{4\gamma^2} = \frac{y}{2\gamma} - \frac{xy+yx}{8\gamma^2}. \tag{8.51}$$

Thus, according to Sec. 8.2, the feedback Hamiltonian needed to undo the back-action of a detection is

$$H_{\rm fb}(t) = [dN_{\rm c}(t)/dt]\left(\frac{y}{2\gamma} - \frac{xy+yx}{8\gamma^2}\right). \tag{8.52}$$

This should be compared with Eq. (8.21). Now, keeping terms of two orders in $1/\gamma$ gives

$$H_{\rm fb}(t) = \gamma y/2 - \tfrac{1}{8}(xy+yx) + \frac{dN_{\rm c}(t) - \gamma^2 dt}{\gamma dt}y/2. \tag{8.53}$$

[1] It is necessary to operate near a critical point because the desired vanishing damping rate for the x quadrature is indicative of a critical slowing down of the dynamics.

Unlike with direct detection, the measurement operator for a null result in homodyne detection $\Omega_0(dt)$ is not Hermitian. It can be factorized as

$$\Omega_0(dt) = \left[1 - \tfrac{1}{2}(a\gamma - a^\dagger\gamma)dt\right]\left[1 - \tfrac{1}{2}dt(a+\gamma)^\dagger(a+\gamma)\right]. \tag{8.54}$$

The first factor in this expression represents the unitary back-action which can be undone by the Hamiltonian

$$H_0 = \mathrm{i}\tfrac{1}{2}(a\gamma - a^\dagger\gamma) = -\gamma y/2 \tag{8.55}$$

Thus the Hamiltonian which must be added to change the homodyne measurement into a QND measurement is

$$H(t) = H_0 + H_{\mathrm{fb}}(t) = -\tfrac{1}{8}(xy + yx) + I_{\mathrm{c}}^{\mathrm{hom}}(t)y/2, \tag{8.56}$$

where the definition (4.51) of $I_{\mathrm{c}}^{\mathrm{hom}}(t)$ has been used. This final expression is of course just what was suggested above (8.41).

8.5 QND Evaluation Criteria

It was shown in the preceding section that feedback of the homodyne photocurrent, combined with a $\chi^{(2)}$ nonlinearity, can turn damping into a perfect measurement of the first kind of the x quadrature of the intracavity field. In reality, imperfections would arise due to inefficient photodetectors, time delay in the feedback loop, and other losses. Also, it is usually more desirable to be able to measure the quadrature of a traveling wave, rather than that of a single mode cavity. The device I have proposed can be used to this effect. The traveling wave to be measured (called the signal) would reflect off a second mirror to the cavity (which is controlled as before). However, in order to evaluate the effectiveness of such a measurement, it is necessary to introduce means of discrimination other than simply inspecting master equations for ideal cases. The criteria I will use are those defined in Ref. [79]. This section summarizes that work.

The type of measurements to which the criteria apply can be modeled as a 'black box' with two inputs and two outputs. One input is the signal or system to be measured, labeled S_{in} and the other is the probe, labeled B_{in}. After interacting with the probe, the system leaves as the output S_{out}, while the probe output which contains the information of the measurement is B_{out}. A perfect measurement is characterized by three qualities: accuracy, conservativity, and predictivity. Accuracy means that the probe output is a good measure of the signal input, conservativity that the interaction conserves the state of the system (so that the signal output equals the signal input), and predictivity that the probe output is a good indicator of the system output. It is possible to define three correlation functions, A, C, and P which quantify these qualities. A value of 1 indicates perfect correlation, and a value of 0 no correlation. Of course, the three correlation functions are not independent (see Sec. I.3 of Ref. [137]) [2].

The accuracy A of the measurement is defined as

$$A = \frac{|\langle S_{\mathrm{in}}, B_{\mathrm{out}}\rangle|^2}{V(S_{\mathrm{in}})V(B_{\mathrm{out}})}. \tag{8.57}$$

[2] Consider the case when each of A, C, and P is either 0 or 1. Then, either all three are 0 (trivial), or all three are 1 (perfect), or one is 1 and two are 0. For the last case, $A = 1$ indicates that the input signal is channeled directly into the output probe, $C = 1$ indicates that the input signal is channeled directly into the output signal, and $P = 1$ indicates that the device produces correlated probe and signal outputs which have nothing to do with the signal input.

Here, angle brackets denote quantum expectation values, and

$$\langle a, b\rangle = \tfrac{1}{2}\langle ab + ba\rangle - \langle a\rangle\langle b\rangle \tag{8.58}$$

for arbitrary operators a and b, and

$$V(a) = \langle a, a\rangle. \tag{8.59}$$

The conservativity C is

$$C = \frac{|\langle S_{\text{in}}, S_{\text{out}}\rangle|^2}{V(S_{\text{in}})V(S_{\text{out}})}. \tag{8.60}$$

This measures the extent to which the measurement does not disturb the system. The predictivity P is

$$P = \frac{|\langle S_{\text{out}}, B_{\text{out}}\rangle|^2}{V(S_{\text{out}})V(B_{\text{out}})}. \tag{8.61}$$

This measures the ability of the measurement to prepare output states.

A simple example to illustrate the significance of these quantities is a beam splitter. The signal input is say one quadrature of a beam of light incident on the beam splitter with transmittance η, and the probe input would usually be a vacuum. The signal output is the transmitted light, and the probe output the reflected light. Consider the case where the signal input is a coherent state also. This serves as a point of reference. Then it is easy to show [79] that

$$A = 1 - \eta\ ; \quad C = \eta\ ; \quad P = 0. \tag{8.62}$$

Defining this example to be the classical reference point, the quantities A and C are evidently limited classically by the inequality

$$A + C \leq 1, \tag{8.63}$$

where equality applies for measurements with no losses. The more light used in the measurement, the more accurate it will be, but the greater the degradation of the signal. QND measurements can overcome this limitation of pseudoclassical optics. The degree to which $A + C$ is greater than one, and approaches the maximum value of 2, is an indicator of how well the measurement approximates a QND scheme. A non zero value of P also indicates this. If $P = 1$, then knowledge of the probe state allows the experimenter to infer completely the system output state. In general, this ability can be quantified by the conditioned normalized variance of the signal output, given the probe output. For the case of Gaussian statistics (which is common), this conditioned variance is related to P by [79]

$$V(S_{\text{out}}|B_{\text{out}}) = V(S_{\text{out}})(1 - P). \tag{8.64}$$

In quantum optics, it is common for the signal to be continuous in time. Then it is convenient to define the above correlation functions as a function of frequency. Assuming that the quantities of interest are the x quadratures of the field, one defines the spectral covariance of two such quantities a, b by

$$S(a, b; \omega) = \int_{-\infty}^{\infty} d\omega' \langle \tilde{a}(\omega), \tilde{b}(\omega')^*\rangle, \tag{8.65}$$

where $\tilde{a}(\omega)$ denotes the Fourier transform of $a(t)$. Then the accuracy of the measurement as a function of frequency is defined as

$$A(\omega) = \frac{|S(S_{\text{in}}, B_{\text{out}}; \omega)|^2}{S(S_{\text{in}}, S_{\text{in}}; \omega)S(B_{\text{out}}, B_{\text{out}}; \omega)}. \tag{8.66}$$

Analogous definitions apply to $C(\omega)$ and $P(\omega)$.

8.6 Evaluating the Homodyne QND Scheme

8.6.1 Inputs and Outputs for the Device

As explained in the preceding section, the scheme for eliminating back-action by feedback described in Sec. 8.4 can be used as a QND scheme with input and output beams. The QND variable is the x quadrature of the signal, reflected off one end of the cavity. The probe output is the beam which is detected by the homodyne apparatus whose current controls the feedback driving. This modulated driving can act at the probe mirror. An experimental configuration which removes the modulated reflections from the probe beam before detection is shown in Fig. 8.1. The modulated beam is added to the input probe beam (usually a vacuum) by a low reflectivity beam splitter. The transmitted part (almost all) of the modulated beam is reflected by a transverse mirror, and put through the low reflectivity beam splitter again. The reflected part of the modulated beam drives the cavity, and is reflected off it along with the probe output. If the optical path lengths are set correctly, complete destructive interference between the modulated reflection from the cavity transmitted by the beam splitter, and the small fraction of the returning modulated beam reflected at the beam splitter, will occur. Thus the light coming from the cavity through the beam splitter will be the same as if it had a vacuum input. This light is then detected by homodyne detection and the resultant photocurrent used in the feedback loop.

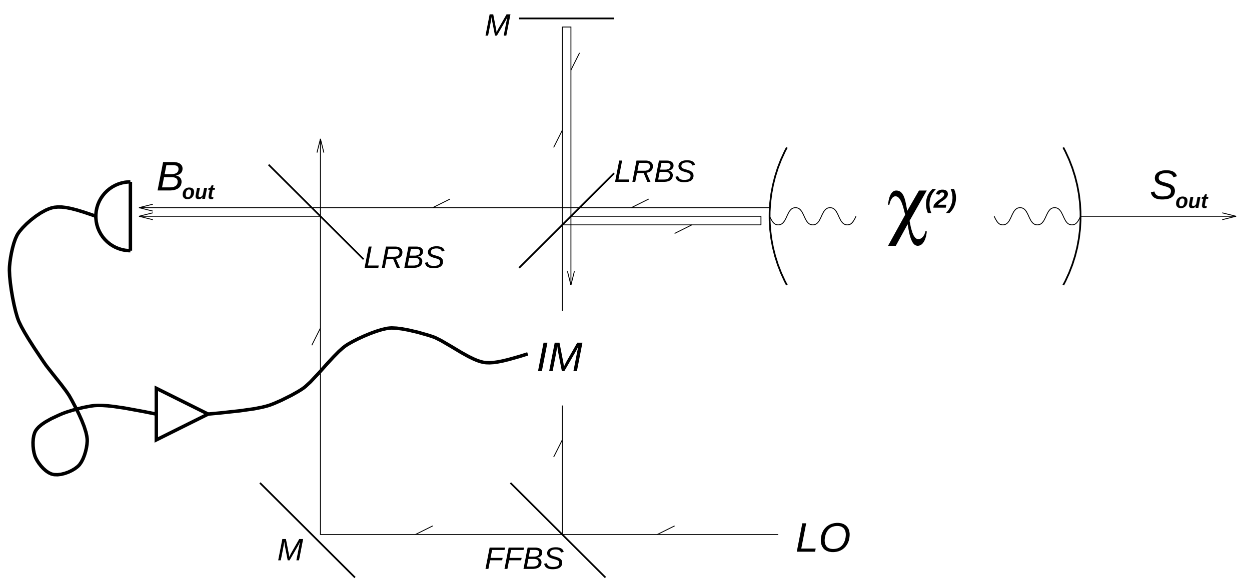

Figure 8.1: Schematic diagram of traveling wave QND device using feedback. The current-carrying wire is indicated by a thick curve; narrow lines represent light beams. $S_{\rm out}$ and $B_{\rm out}$ are the signal and probe output beams, as explained in the text. LO denotes the local oscillator source, IM a current-controlled intensity modulator, M a mirror, $LRBS$ a low reflectivity beam-splitter, and $FFBS$ a 50/50 beam splitter.

Let the damping rates at the signal and probe ends of the cavity be κ_1 and κ_2 respectively. Assume that the field inputs at both ends are in vacuum states, and so represented by the vacuum annihilation operators $\nu_1(t)$ and $\nu_2(t)$ respectively. These bath operators have zero mean, but obey the commutation relations

$$[\nu_i(t), \nu_j^\dagger(t')] = \delta_{ij}\delta(t - t'), \tag{8.67}$$

with all other commutators vanishing. The Langevin equation for an arbitrary cavity operator s is then

$$ds = \left\{ \mathrm{i}[H(t), s] - [s, a^\dagger] \left(\frac{\kappa_1 + \kappa_2}{2} a + \sqrt{\kappa_1}\nu_1 + \sqrt{\kappa_2}\nu_2 \right) + \left(\frac{\kappa_1 + \kappa_2}{2} a^\dagger + \sqrt{\kappa_1}\nu_1^\dagger + \sqrt{\kappa_2}\nu_2^\dagger \right) [s, a] \right\} dt \tag{8.68}$$

Here, H is the Hamiltonian (8.41)

$$H(t) = -\frac{\kappa_2}{8}(xy + yx) + \tfrac{1}{2} y \int_0^\infty I_c^{\text{hom}}(t - s) h(s) ds, \tag{8.69}$$

where $a = \frac{1}{2}(x + iy)$ as before, and where I have generalized the feedback by including a response function $h(s)$. The instantaneous feedback of Sec. 8.4 corresponds to $h(s) = \delta(s)$. The output fields, denoted by $b_1(t)$ and $b_2(t)$ for the signal and probe respectively, are given by

$$b_i(t) = \nu_i(t) + \sqrt{\kappa_i}\, a(t). \tag{8.70}$$

In order to treat Eq. (8.68) consistently, the current $I_c^{\text{hom}}(t)$ must be an operator. That is to say, the calculations in this section use the quantum Langevin approach to feedback of Ch. 7. The homodyne current operator is of course simply the x quadrature of the output probe field. Define commuting bath operators $\xi_i(t) = \nu_i(t) + \nu_i^\dagger(t)$ as usual. Then the homodyne current can be defined as

$$I_c^{\text{hom}}(t) = \kappa_2 x(t) + \sqrt{\kappa_2}\, \xi_2(t). \tag{8.71}$$

In order to consider detectors of efficiency η_i for the two output beams, it is necessary to modify this expression. The effect of non-unit efficiency is equivalent to that of placing a beam splitter of transmittance η_i in front of a perfectly efficient detector. Normalizing the transmitted field so that the deterministic part (proportional to x) remains the same, the result is

$$I_c^{\text{hom}}(t) = \kappa_2 x(t) + \sqrt{\kappa_2}\, \xi_2(t) + \sqrt{\kappa_2 \epsilon_2}\, \xi_4(t), \tag{8.72}$$

where

$$\epsilon_i = (1 - \eta_i)/\eta_i, \tag{8.73}$$

and $\xi_4(t)$ is an independent Gaussian white noise term.

Since the probe output beam has to be measured in order to carry out the feedback, it is only sensible to define the probe output to be that measured photocurrent. Choosing a convenient normalization,

$$B_{\text{out}}(t) = \sqrt{\kappa_2} x(t) + \xi_2(t) + \sqrt{\epsilon_2}\, \xi_4(t). \tag{8.74}$$

Since this expression includes the possibility of inefficient detectors, it would seem consistent to define the signal output in the same way. That is,

$$S_{\text{out}}(t) = \sqrt{\kappa_1}\, x(t) + \xi_1(t) + \sqrt{\epsilon_1}\, \xi_3(t), \tag{8.75}$$

where $\xi_3(t)$ is another noise term. It was assumed above that both probe and signal inputs were in the vacuum state. This is a convenient choice, as explained in the preceding section. Thus, with the same normalization,

$$B_{\text{in}}(t) = \xi_2(t) \text{ and } S_{\text{in}}(t) = \xi_1(t). \tag{8.76}$$

This completes the definitions.

The probe and signal outputs depend only on the noise operators and the intracavity quadrature operator $x(t)$. From Eq. (8.68), this obeys

$$\begin{aligned} \dot{x}(t) &= -\frac{\kappa_1+\kappa_2}{2}x(t) - \frac{\kappa_2}{2}x(t) - \sqrt{\kappa_1}\xi_1(t) - \sqrt{\kappa_2}\xi_2(t) \\ &\quad + \int_0^\infty \left[\kappa_2 x(t-s) + \sqrt{\kappa_2}\xi_2(t-s) + \sqrt{\kappa_2\epsilon_2}\xi_4(t-s)\right] h(s)ds, \end{aligned} \tag{8.77}$$

where the definition (8.72) has been used. Transforming to the frequency domain,

$$\tilde{x}(\omega) = \frac{-\sqrt{\kappa_1}\,\tilde{\xi}_1(\omega) - \sqrt{\kappa_2}[1-\tilde{h}(\omega)]\tilde{\xi}_2(\omega) + \sqrt{\kappa_2\epsilon_2}\,\tilde{h}(\omega)\tilde{\xi}_4(\omega)}{\kappa_1/2 + \kappa_2[1-\tilde{h}(\omega)] - \mathrm{i}\omega}, \tag{8.78}$$

where the noise terms in the frequency domain are complex and obey

$$\xi_i(\omega)^* = \xi_i(-\omega), \tag{8.79}$$

$$\langle \xi_i(\omega)\xi_j(\omega')\rangle = 2\pi\delta_{ij}\delta(\omega+\omega'). \tag{8.80}$$

The Fourier transformed probe and signal outputs are

$$\begin{aligned} \tilde{B}_{\text{out}} &= \frac{-\sqrt{\kappa_1\kappa_2}\tilde{\xi}_1 + (\kappa_1/2 - \mathrm{i}\omega)\tilde{\xi}_2 + \sqrt{\epsilon_2}(\kappa_1/2 + \kappa_2 - \mathrm{i}\omega)\tilde{\xi}_4}{\kappa_1/2 + \kappa_2(1-\tilde{h}) - \mathrm{i}\omega}, \\ \tilde{S}_{\text{out}} &= \frac{-\kappa_1\tilde{\xi}_1 + [\kappa_1/2 + \kappa_2(1-\tilde{h}) - \mathrm{i}\omega](\tilde{\xi}_1 + \sqrt{\epsilon_1}\,\tilde{\xi}_3) - \sqrt{\kappa_1\kappa_2}(1-\tilde{h})\tilde{\xi}_2 + \sqrt{\epsilon_2\kappa_1\kappa_2}\,\tilde{h}\tilde{\xi}_4}{\kappa_1/2 + \kappa_2(1-\tilde{h}) - \mathrm{i}\omega}, \end{aligned}$$

where the argument ω has been suppressed.

8.6.2 QND Correlation Coefficients

Using these results and the noise statistics (8.79,8.80), the correlation coefficients A, C, and P may be found. First, it is useful to define the dimensionless quantities

$$\Omega \equiv \omega/\kappa_1\ ; \quad G \equiv \kappa_2/\kappa_1. \tag{8.81}$$

The symbol G is used for the ratio of the damping rates of the end mirrors, because it is effectively the gain of the QND measurement. In terms of these parameters,

$$A(\Omega) = \left(1 + \frac{\frac{1}{4}+\Omega^2}{G} + \epsilon_2\frac{\left(G+\frac{1}{2}\right)^2+\Omega^2}{G}\right)^{-1}, \tag{8.82}$$

$$C(\Omega) = \left(1 + \frac{G|1-\tilde{h}|^2 + G\epsilon_2|\tilde{h}|^2 + \epsilon_1\left|\frac{1}{2}+G(1-\tilde{h})-\mathrm{i}\Omega\right|^2}{\left|-\frac{1}{2}+G(1-\tilde{h})-\mathrm{i}\Omega\right|^2}\right)^{-1}, \tag{8.83}$$

$$P(\Omega) = A(\Omega)\times C(\Omega)\times\left(1 + \frac{\left|\left(\frac{1}{2}-\mathrm{i}\Omega\right)\left(1-\tilde{h}\right)\right|^2 + (\epsilon_2)^2\left|\tilde{h}\left(\frac{1}{2}+G-\mathrm{i}\Omega\right)\right|^2}{\left|-\frac{1}{2}+G(1-\tilde{h})-\mathrm{i}\Omega\right|^2}\right). \tag{8.84}$$

To understand these formulae, first consider the ideal case $\epsilon_1 = \epsilon_2 = 0$, $\tilde{h} = 1$. Then it is easy to see that

$$C(\Omega) = 1. \tag{8.85}$$

$$A(\Omega) = P(\Omega) = \frac{G}{G+\frac{1}{4}+\Omega^2}. \tag{8.86}$$

That is to say, the system is unaffected by the interaction, as required for a true QND measurement. The accuracy and predictivity of the measurement depend on the gain G. For large gain, they both approach one also, indicating that the device is a perfect QND detector in this limit. The bandwidth is (in original units) equal to the geometric mean of the two decay rates. This behaviour is shown in Fig. 8.2(a), which plots A, C, and P versus Ω for $G=4$. Evidently this value of gain can be considered quite large. However, this apparently excellent rôle for back-action elimination by feedback is somewhat diminished when one considers the same case, but with the feedback turned off ($\tilde{h}=0$). The results shown in Fig. 8.2(b) is worse than that obtained with feedback, but it is still a QND measurement in the sense that P is non zero, and $A+C$ is greater than one over some bandwidth. Furthermore, in the limit $G\to\infty$, the three correlation coefficients all approach unity, just as in the feedback case. It thus appears that the central rôle in the QND device is being played by the $\chi^{(2)}$ medium, rather than by the feedback [3].

To assess the usefulness of this device, it is necessary to consider non-ideal conditions, in particular, imperfect photodetectors. Even with better than 95% efficient detectors ($\epsilon_1=\epsilon_2=0.05$), the effect on the quality of the measurement is dramatic, as shown in Fig. 8.3. Figure 8.3(a) the results with feedback, and 8.3(b) those without. The feedback still gives an improvement, but it is less dramatic than in the ideal case of Fig. 8.2. In Fig. 8.3(a), the feedback loop response function $\tilde{h}(\omega)$ is no longer unity. By analysis of the above formulae for the QND correlation coefficients, one finds that the results are better at $\Omega=0$ if $\tilde{h}(0)$ is greater than one. However, the feedback loop gain is limited by the stability requirement that $\frac{1}{2}\kappa_1+\kappa_2[1-\tilde{h}(0)]>0$ [see Eq. (8.77)]. Working at the limit of stability suggests $\tilde{h}(0)=1+\kappa_1/(2\kappa_2)$. In Fig. 8.3(a) I have also included a time delay and some exponential smoothing in the feedback loop, by taking the total response function to be

$$\tilde{h}=\left(1+\frac{\kappa_1}{2\kappa_2}\right)\frac{\exp(\mathrm{i}\omega/\kappa_2)}{1+\mathrm{i}\omega/\kappa_2}=\left(1+\frac{1}{2G}\right)\frac{\exp(\mathrm{i}\Omega/G)}{1+\mathrm{i}\Omega/G}. \tag{8.87}$$

The non-flat response has little effect at low frequencies, where the QND correlations are best.

A complete analysis of this QND apparatus would require one to relax the assumption that the strength of the nonlinearity χ equals the damping rate κ_2 of the second mirror. Then, for a fixed χ, κ_1, and ϵ_i, one would have to maximize some suitable combination of $A(0)$, $C(0)$, and $P(0)$, as a function of κ_2 and $\tilde{h}(0)$ (assuming one is most interested in the result at zero frequency). This would be necessary to find the optimal operating region, and to find out how much improvement the feedback can offer over the nonlinearity alone. In any case, for making a QND measurement of the intracavity x quadrature, as explored in Sec. 8.4, the feedback is essential. Without it, a homodyne detection would simply be measuring the squeezing produced by the DPO. If a detector of efficiency η is included in that model, the QND master equation (8.42) becomes

$$\dot{\rho}=\mathcal{D}[x/2]\rho+\frac{1-\eta}{\eta}\mathcal{D}[y/2]\rho\equiv\mathcal{L}\rho \tag{8.88}$$

The extra term causes the variance in x to increase linearly with time, which obviously violates the QND definition. However, for η close to one, an approximate first kind measurement could be carried out for a short time. As well as modifying $\mathcal{L}$, the loss increases the noise to signal ratio in the two-time correlation function (8.44)

$$E[I_{\rm c}^{\rm hom}(t+\tau)I_{\rm c}^{\rm hom}(t)]=\mathrm{Tr}\left[xe^{\mathcal{L}\tau}\tfrac{1}{2}\{x,\rho(t)\}\right]+\frac{1}{\eta}\delta(\tau). \tag{8.89}$$

[3] It has been brought to my attention that the scheme I am referring to here, with no feedback, has been analyzed recently in Ref. [127].

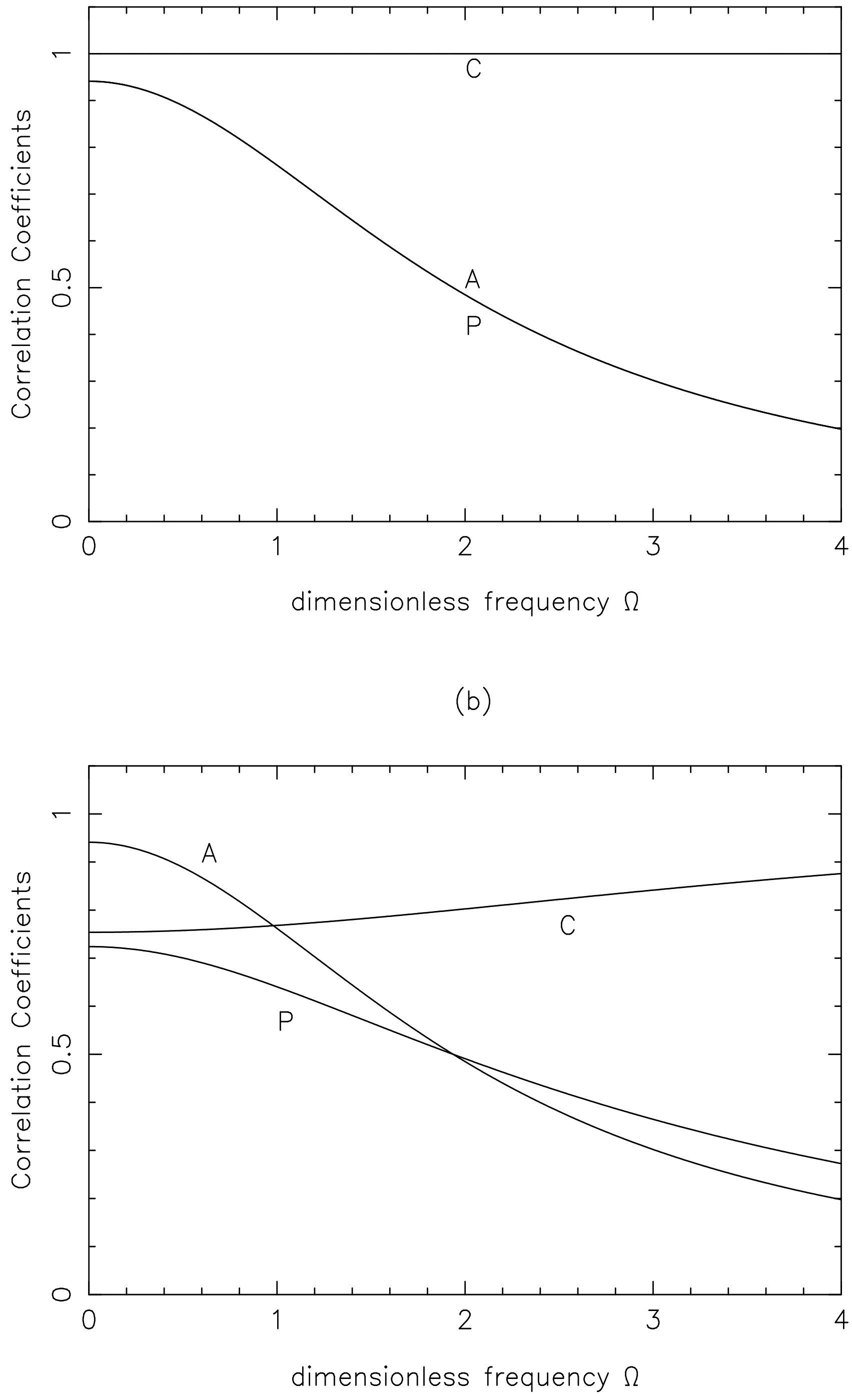

Figure 8.2: Plot of the correlation coefficients for accuracy A, conservativity C, and predictivity P of the QND device, versus dimensionless frequency Ω. In (a), the feedback is on, with a response function $\tilde{h} = 1$, and in (b) the feedback is off. For both cases, the efficiency of the photodetectors is 100%, and the QND gain $G = 4$.

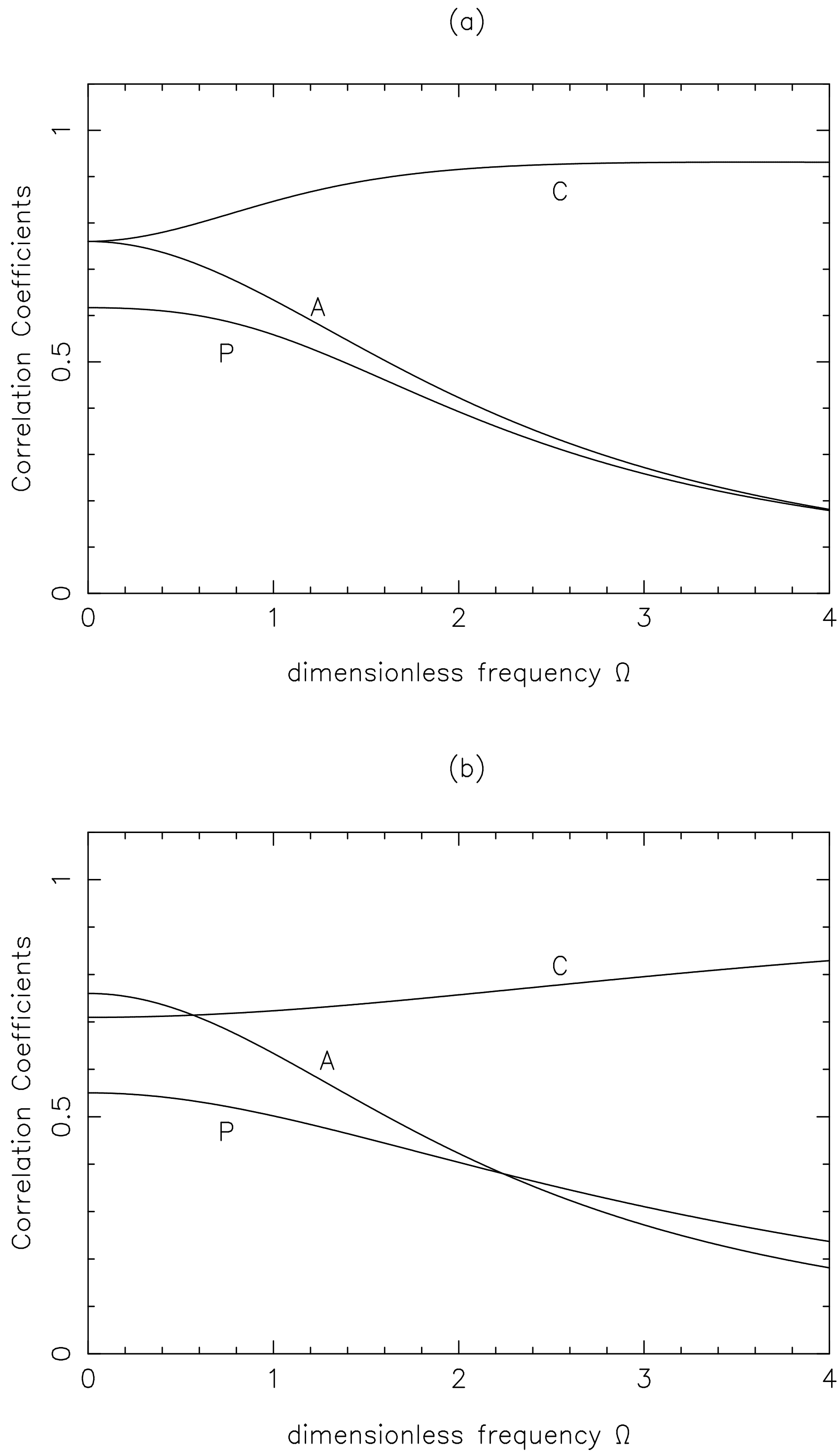

Figure 8.3: As in Fig. 2, but with 95% efficient detectors, and with $\tilde{h}$ given by Eq. (8.87)

Chapter 9

Conclusion

This conclusion has three sections. The first is a recapitulation of the logical structure of this thesis. The second discusses applications of the theory of quantum trajectories and feedback. This includes the examples given in this thesis (not covered in the summary), applications which have been worked out by myself and others based on the work in this thesis but not included in it, and applications which remain to be realized. The final section presents some unresolved questions, relating to non-Markovian systems and information theory.

9.1 Summary

If, having reached the conclusion, the reader is of the opinion that the subject of quantum trajectories and feedback is a mostly trivial application of some basic principles of quantum mechanics, then I have achieved my goal. I have tried to explain each step clearly (although see Bohr's caveat on p. ii), so that it should appear as an obvious extension of the developing theory. Needless to say, the actual course of development of the theory bears little resemblance to the way I have presented it. My hope is that my own tortuous, and often wrong, process of working has not muddied my presentation of what is, in hindsight, quite a straightforward area of the quantum theory of open systems. In this section I wish to review the logical structure of this thesis. I leave a discussion of the specific cases treated in the body of the thesis, which are essential for building intuition, to the following section.

The roots of almost all of this thesis can be found in Chapter 2, "Quantum Measurement and Feedback Theory", even though it is the shortest proper chapter. This is perhaps not surprising from the chapter title, because quantum trajectories are (as I have emphasized) simply an application of quantum measurement to continuous observation. Because they apply to continuous observation of real systems, quantum trajectories cannot be based on projective measurements, as it is well known that continuous projective measurements lead to the quantum Zeno paradox [104]. Rather, they are an example of a more general class of measurements described by operations and effects. As shown in Ch. 2, the theory of operations and effects can be derived by including a quantum description of the apparatus as well as the system, and then considering projective measurements on the apparatus. For the case of quantum optics, the quantum apparatus is the continuum of external field modes, weakly coupled to the object such as an atom or cavity. It is the infinite number of independent bath modes which allow each infinitesimal time interval to constitute a new measurement. This coupling was considered in detail in Ch. 3.

It was shown in Ch. 2 from the general theory of operations and effects that the quantum trajectories for a continuously monitored system will be jump-like. The measurement record consists of a series of events which correspond to a sudden change in the system state. Between these events, the system changes smoothly but not unitarily. In Ch. 4 I showed that, for a quantum optical system, these events correspond to photodetections. A stochastic master equation generating the quantum trajectories conditioned on the stochastic photocurrent was derived. It was generalized for homodyne and heterodyne detection, for which it is possible to deal with a bath input contaminated with white noise. Because of the origin of the operations and effects for photodetection in the entanglement between the system and the apparatus, the measured photocurrents have two complementary descriptions. The first is as a stochastic c-number, the classical photocurrent which conditions the system state. The second is as an operator in the Hilbert space of the bath which has interacted with the system. Observable quantities, such as two-time photocurrent correlation functions or distributions for integrated photocurrents, can be determined from either method, as shown in Ch. 4.

The next topic of Ch. 2 was feedback, which I defined as the use of a measurement result to influence the later dynamics of the system. In Ch. 6 I investigated continuous feedback using quantum trajectories. As well as the stochastic term describing the conditioning of the state on the photocurrent, another stochastic term describing the feedback (controlled by that photocurrent) is added to the quantum trajectory. In the limit of Markovian feedback (zero time delay), it is possible to take an ensemble average over all possible photocurrents, and thus derive a new master equation incorporating the effects of the feedback. In this derivation it is necessary to take into account two facts: that a physically fed-back photocurrent will not have an infinite bandwidth (unlike its mathematical idealization), and that the feedback must act after the measurement even in the Markovian limit. It was to deal with the idealized mathematical photocurrent that I developed (non-rigorously) a general stochastic calculus in Ch. 3. In the non-Markovian case, I showed that it is possible to solve the quantum trajectories exactly for linear systems where the feedback is mediated by the homodyne photocurrent.

As with the photocurrent, there are two complementary descriptions of quantum feedback, of which I have just described the first. As shown in Ch. 2, if the apparatus (bath) is allowed to interact with the system again (after the measurement interaction) then the information it stores about the system can be fed-back without ever being realized as a classically measured result. In Ch. 7 I derived this quantum mechanical treatment of feedback, with the (direct or homodyne) photocurrent represented by an operator. For convenience, this was done in the Heisenberg picture, and the result was a non-Markovian quantum Langevin equation for the source. In the Markovian limit, the same master equation could be derived as from the quantum trajectory approach. Furthermore, I showed in Ch. 7 that feedback without measurement can be more than a mathematical nicety. With all-optical feedback, the essential elements of the feedback loop (light beams) can be treated fully quantum-mechanically. By this means it is possible to reproduce the effects of direct and homodyne feedback, without photodetection. Also, all-optical (as opposed to electro-optical) feedback produces an output beam from the feedback loop, which has interesting properties.

The one structural element of this thesis which diverged from the plan of Ch. 2 was the final section of Ch. 7 entitled "Feedback Precluding Measurements". From the definition in Ch. 2, this is a contradiction in terms. Nevertheless feedback which is incompatible with a measurement interpretation arises naturally in an all-optical scheme if the feedback interaction between the apparatus (the in-loop light beam) and the system is not assumed to factorize as the product of two Hermitian operators. This allows both quadratures of the in-loop beam to be fed back simultaneously. There is no equivalent formulation using quantum trajectories because it is impossible to measure both quadratures simultaneously, except inaccurately, such as by heterodyne detection. I included this section because the comparison of all-optical feedback (if it is permitted to be called such) and electro-optical feedback is a valuable guide to the nature of quantum-limited feedback. I do not expect that all-optical feedback will find many applications, whereas electro-optical feedback is likely to become increasingly important as practical optical devices evolve towards the quantum-limited noise regime.

9.2 Present and Future Applications

9.2.1 Applications for Quantum Trajectories

In this thesis, the main structural rôle of quantum trajectories was as a path to deriving a quantum theory of feedback. However, I also placed considerable emphasis on quantum trajectories as a subject in itself. I even devoted an entire chapter to the interpretation of quantum trajectories, which was not mentioned at all in the above summary. In this chapter I mentioned what is almost certainly the main application for quantum trajectories in the physics community at present, numerically solving master equations in Hilbert spaces of necessarily large dimension. I used little space discussing this application not because it is unimportant but because it does not use the property of quantum trajectories by which I have defined them in this thesis. That property is of course that they give the stochastic evolution of a system conditioned on the measurement record. This interpretation in terms of quantum measurement theory was emphasized in Ch. 5. It is a useful interpretation for several reasons.

Aid to Understanding

Firstly, quantum trajectories are a new way of looking at the dynamics of open quantum systems and as such are an aid to understanding such dynamics. In Ch. 5 I considered fairly simple systems: a two-level atom, an empty cavity, and an ideal laser. In each of these cases, quantum trajectories provided new insights into the behaviour of the system, especially in noting the importance of the measurement scheme on the way the nonselective evolution is unraveled. Quantum trajectories also illuminate the way classical concepts, such as the phase of a laser, emerge from the mists of quantum uncertainty. This aspect may be relevant with more complicated systems, whose classical analog has limit cycles or even strange attractors. The phase space distribution of the steady state solution to the master equation would presumably bear some relation to the classical attractor in the limit of large quantum numbers. However, this stationary solution does not have the dynamical property of a classical point moving on the attractor. With quantum trajectories, it may be possible to see such motion. The different behaviour under different measurement schemes could be a particularly interesting aspect of this re-examination of the classical limit.

Chaos and Control

A more specific application of quantum trajectories to systems with complicated dynamics would be to search for sensitive dependence on measurement results. It is well known that the classical definition of chaos as sensitive dependence on initial conditions cannot apply to closed quantum systems because the overlap of two state vectors remains constant under unitary evolution [71]. With nonunitary evolution as generated by a master equation, the system may exhibit such sensitive dependence. Quantum trajectories would allow the investigation to be broadened to examine how sensitive the final state of the system is to the measurement record (the photocurrent). One might expect that a quantum state starting in a classically chaotic area of phase space would evolve so that nearly identical photocurrents would give quite different final states. This phenomenon has been investigated by Caves and co-workers [125, 6] and has been called by them hypersensitivity to perturbations (a random perturbation can always be re-interpreted as a measurement). The evolution of nonclassical initial conditions (superpositions of widely separated areas of phase space) is known to be extremely sensitive to the measurement record even for systems with trivial dynamics [22]. Again, the choice of the measurement scheme is crucial.

The issue of the sensitivity of the quantum state to the photocurrent may turn out to be a practical one also, if the feedback control of chaotic systems is ever extended to the quantum limit. It is intriguing to imagine a supercomputer hooked up to the output of a photodetector, calculating the conditioned quantum state in real time so that the appropriate forces can be applied to the system to keep it near the desired orbit. If the system is too sensitive to the photocurrent noise, then it may be impossible to track its state because there will inevitably be some spurious electronic noise added to the signal in the detection process. At a more mundane level, quantum trajectories may also be the most practical way of calculating the effect of a standard electronic feedback loop when quantum fluctuations in the system are comparatively large. This is the limit in which the system cannot be linearized, and so the photocurrent must be treated as it is, a series of discrete detections convoluted with the response function of the feedback loop. It is this limit where the Markovian approximation (which would enable the feedback to be treated as a term in a master equation) would be likely to be inadequate also.

Calculation of Observables

Aside from feedback, there are numerous other numerical applications of quantum trajectories for which the measurement interpretation is central. The traditional way of calculating observable quantities is to represent these as operators and calculate the expectation values. This method is well suited for simple quantities such as averages and variances. However, experimentalists often have access to much more information. They calculate averages and variances from the entire distribution function. For example, theoreticians often describe the photon statistics of the output of a quantum optical system in terms of the second order coherence function $g^{(2)}(\tau)$ which is proportional to the expectation value of the product of the field intensity at two times separated by τ. This is proportional to the probability to detect a photon a time τ later than a photodetection. Experimentalists, on the other hand, find it easier to measure the waiting time distribution $w(\tau)$, which is the probability density for the time between *successive* photodetections to be τ. Actually, experimentalists measure something distinct from $w(\tau)$ also; Ref. [55] discusses this issue in detail.

It is possible to write an operator expression for $w(\tau)$, but it is much more complicated than the expression for $g^{(2)}(\tau)$. Furthermore, it cannot be derived from $g^{(2)}(\tau)$, although the converse derivation is possible. The waiting time distribution $w(\tau)$ is in many ways therefore a better description of the output field. With the quantum trajectory treatment of photodetection, the waiting time distribution is, if anything, easier to calculate numerically than the second order coherence function. In fact it was the reinterpretation of the complicated operator expressions for $w(\tau)$ in terms of the source dynamics which led Carmichael *et al* [23] towards the theory of quantum trajectories.

Quantum trajectories are the natural way to calculate experimentally observable quantities, because they are in essence numerical simulations of experiments, including the experimental data. I believe that in future more theoreticians in quantum optics will use quantum trajectories for this purpose, rather than relying on experimentalists to massage their data so that the result appears to be a measurement of what the theoreticians have calculated.

Another similar application of quantum trajectories, on which I have spent a little time, is for calculating integrated photocurrents for freely decaying systems. As shown in Ch. 4, this is a trivial exercise for a harmonic oscillator (an optical cavity). The integrated direct detection current measures photon number, while the integrated homodyne and heterodyne photocurrents measure the marginal Wigner distribution and the Husimi distribution respectively. Consideration of the problem for other systems shows that this is a special property of harmonic oscillators; in general, the distribution measured by an integrated heterodyne measurement, for example, would be soluble most easily by a numerical calculation using quantum trajectories. An example of such a general system would be a spin-j system. This could be realized physically as superfluorescence [70]: an ensemble of $2j$ identical two-level atoms strongly coupled to a decaying cavity mode (which is measured) and weakly coupled to other reservoirs. An alternative realization would be a two-mode (pump and signal) lossless cavity [120] coupled through a $\chi^{(2)}$ medium to a third, heavily damped mode (idler) which is measured. The special case of $j = \frac{1}{2}$ (a single atom or a single photon for the two realizations respectively) is soluble because it is isomorphic to a harmonic oscillator with only the lowest two energy levels populated. It can also be solved using the Schrödinger - Langevin equation described in App. B. The answer for arbitrary j, and the relation to atomic distribution functions as defined using atomic coherent states [112, 138], is a problem which remains to be solved, and quantum trajectories seem the only tractable approach.

9.2.2 Applications for Quantum Feedback

I have already mentioned applications for quantum-limited feedback, treated using quantum trajectories, in the preceding section. In this section I wish to concentrate on applications of the feedback theory itself, independent of quantum trajectories. If the time delay in the feedback loop is negligible, then the appropriate feedback master equation (for direct or homodyne detection) can be used. Recall that the feedback master equation can be derived from either a quantum trajectory approach or a quantum Langevin approach. If the response function of the feedback loop is important, then linearization would probably be necessary to solve the problem (unless it is a truly linear system as I have generally considered in this thesis). This will be valid for systems with a large mean excitation. It is most easily undertaken using the quantum Langevin description of feedback, although it is perfectly possible to use the quantum trajectory approach as I showed in Ch. 6. In this thesis I considered two applications for feedback: reducing noise, and creating QND measurements. I will review the former first.

Squeezing

In Ch. 6 I considered a special case of quantum feedback where the quadrature dynamics was linear, the feedback is controlled by the homodyne photocurrent, and the feedback itself was linear (driving the cavity). Some of these assumptions are not essential. For a system with large photon number, the amplitude and phase dynamics are often approximately independent and linear, and direct detection of the output is equivalent to homodyne measurement of the amplitude quadrature. The essential points are that the information used in the feedback loop is derived from detection of the cavity output beam, and that only classical dynamics is controlled by the feedback. These limitations are important because they correspond to classical feedback, which in turn corresponds to what is (relatively) easy to achieve in the laboratory. That is to say, in order to obtain information any other way, or to control nonclassical dynamics, it would be necessary to use nonclassical optical elements. Such nonclassical elements are defined as those which are necessary to produce nonclassical light (irrespective of feedback). It is thus not surprising that, as I showed in Ch. 6, classical feedback cannot produce nonclassical light. The simple explanation of this given in Ch. 6 is that measuring the output of a cavity cannot produce a nonclassical conditioned state, so that shifting the mean (which is what driving achieves) by feedback cannot reduce the variance.

Although classical feedback cannot produce nonclassical light, it can enhance the nonclassicality of light which is already squeezed. Again, this has an explanation in terms of conditioning of the states by the measurement. The stationary conditioned variance is always less than the unconditioned variance, except if the latter is equal to that of a coherent state. The nonselective stationary variance is larger because the conditioned mean fluctuates due to the fluctuations in the homodyne photocurrent. The effect of the feedback is to counter these fluctuations, enabling the lower conditioned variance to be seen. It is interesting that in this case, the feedback must be positive (that is, destabilizing), which has the same explanation as the sub-shot noise spectrum of squeezed light. What the feedback cannot do is to enhance the squeezing in the output of the system. That is because some of the cavity output must be used in the feedback loop, and the squeezing in the remainder is degraded so as to more than counteract the increase in squeezing due to the feedback. Of course, if the cavity is producing classically noisy light, then classical feedback can always reduce this, but not without limitations. For classical noise reduction, the feedback is negative as expected from intuition.

The limited noise reduction capabilities of feedback based on external photodetection suggest that an intracavity measurement may be better. In Ch. 6 I modeled feedback of an ideal continuous QND (quantum non-demolition) measurement of one quadrature. This was shown to allow an arbitrarily squeezed state to be produced by feedback inside the cavity. The cavity output can also be squeezed, but the degree of squeezing is determined by the nature of the intracavity dynamics. The reason for this limitation is that an intracavity QND measurement is only an indirect measurement of the output statistics. To achieve arbitrary squeezing of the output of a system it would be necessary to make a QND measurement on that output. This would require a traveling wave QND device, which could be built from a single mode QND device in a lossy cavity. Such a feedback scheme was considered in Ch. 7, where it was designed as an all-optical device. This detail is not important; what is important is that the QND measurement of the output can be used in a feedback loop to influence the source in such a way that the output becomes arbitrarily squeezed. This feedback will not necessarily cause the source itself to be squeezed; in fact it was shown in Ch. 7 to do the opposite.

Second Harmonic Generation

Quantum non-demolition measurements are difficult to do. For this reason, it is sensible to consider other measurement schemes which are easier to achieve, but which nevertheless are nonclassical and so will allow the classical limits to noise reduction by feedback to be overcome. One such feedback scheme was suggested by Taubman and Bachor[1]. I worked out the theory, and they are planning to undertake the experiment. The proposal uses a $\chi^{(2)}$ crystal acting as a second harmonic generator. The first harmonic (red) resides in a good cavity and is driven by a laser at the end with very high reflectivity. The second harmonic (green) effectively has no cavity, but is reflected so as to form a single output beam. By itself (in the absence of feedback) this system produces amplitude squeezing in both the first and second harmonic. The amplitude squeezing in the red can be understood to be due to two-photon absorption, which is the effect of the adiabatically eliminated green mode. Such nonlinear absorption preferentially damps large fluctuations and so reduces the variance. The antibunching in the green mode can be attributed to the fact that the creation of a green photon requires the loss of two red photons, which reduces the chances of this event re-occurring.

Without feedback, the optimum low frequency squeezing in the red mode output is

$$R_r = -\frac{1}{3} \quad \text{for} \quad \chi = \frac{1}{3}, \tag{9.1}$$

and that in the green mode is

$$R_g = -\frac{8}{9} \quad \text{for} \quad \chi \to \infty. \tag{9.2}$$

Here I am using the notation of Ch. 6 for noise reduction R (zero equals shot noise, -1 equals perfect squeezing), while χ is the ratio of the nonlinear loss rate to the linear loss rate. Now there are two possible ways to feedback onto the driving of the system. The first is to use the green photocurrent to control the amplitude of the red driving, in order to try to reduce the noise in the red output light. A linearized treatment gives a new minimum of

$$R_r = -\frac{1}{2} \quad \text{for} \quad \chi = 1, \quad T = \frac{1}{2}, \tag{9.3}$$

which is significantly less than the no-feedback value (9.1). Here, T is the low frequency loop transfer function, equal to the round-loop gain of the feedback [129]. Note that it is positive, again corresponding to destabilizing feedback. The behaviour of R_r as a function of χ without feedback and with optimal feedback is shown in Fig. 9.1(a), where the required feedback loop gain is plotted also.

The other sort of feedback is to control the driving of the red mode using the detected red photocurrent, looking to enhance the squeezing in the green output. It turns out that the minimum R_g (9.2) as $\chi \to \infty$ cannot be lowered by feedback. However, for finite values of χ, the feedback can give a definite improvement. For example, with no feedback

$$R_g = -\frac{1}{2} \quad \text{for} \quad \chi = 1, \tag{9.4}$$

whereas the optimal feedback gives

$$R_g = -\frac{2}{3} \quad \text{for} \quad \chi = 1, \quad T = \frac{1}{4}. \tag{9.5}$$

Figure 9.1(b) plots the variation of R_g with χ for the cases of no feedback and optimal feedback, and also the optimal feedback transfer function T. All of these results are calculated for the case of unit efficiency photodetectors. The effect of non-unit efficiency is to reduce the effectiveness of the feedback, and to alter the conditions of optimality. I have determined the general solution, including the full spectrum with an arbitrary transfer function $T(\omega)$, but this is not the place to present such details.

[1] as part of an Australian Research Council co-operative research project between the Australian National University and the University of Queensland

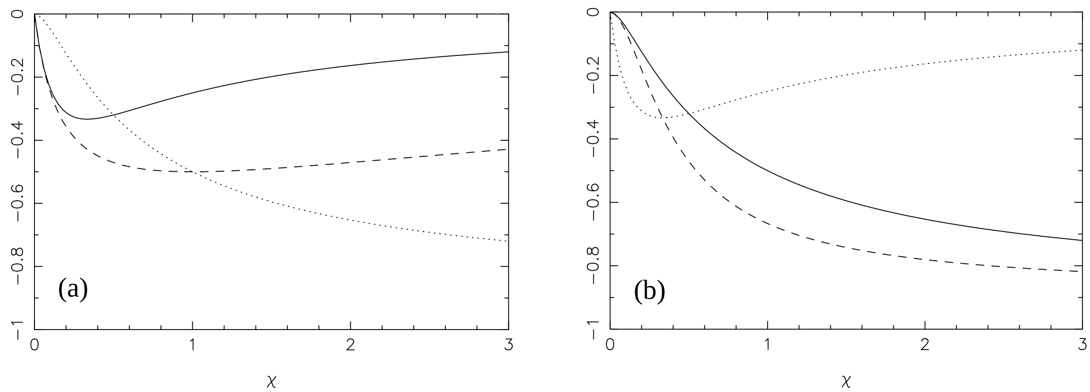

Figure 9.1: Plot of low-frequency squeezing (minus shot noise) in second harmonic generation with feedback, as a function of χ, the ratio of nonlinear loss to linear loss. Fig. 9.1(a) shows squeezing in the red, with the green used in the feedback loop (which controls the driving of the red), while 9.1(b) shows squeezing in the green, with the red used in the feedback loop. The solid line is the squeezing without feedback, the dashed line the optimal squeezing with feedback, and the dotted line the negative of the low-frequency transfer function (round-loop gain) producing the optimal results.

The Micromaser

Another example of a nonlinear but not strictly QND measurement which can be used in conjunction with feedback to give nonclassical noise reduction is that of the micromaser [47]. This consists of a small microwave cavity through which a monochromatic beam of resonant two-level atoms is passed. The atomic state upon exit can be measured and the result used in feedback. The case of modifying the cavity quality factor has been considered by Liebman and Milburn [89]. Because of the nature of the Jaynes-Cummings Hamiltonian, the micromaser dynamics are complicated without feedback, and even more complicated with. However, one result is easy to explain. In the limit of short transit time the atoms (assumed all to enter in the upper state) act simply as a linear amplifier of the cavity mode. In the absence of feedback the stationary state is thermal, with a photon number variance much greater than the mean. With weak feedback, increasing the cavity damping rate whenever an outgoing atom is detected in the lower state, the photon distribution can be made sub-Poissonian, with a variance equal to half the mean. For longer transit times, the no-feedback dynamics show the effects of trapping states (where the atom undergoes an integer number of Rabi cycles in transit [47]), and the minimum stationary variance is typically as low as one quarter of the mean. Feedback can produce an arbitrary small minimum variance near a trapping state. However, this result is very sensitive to the transit time and so may be washed out by a realistic atomic velocity profile.

Back-Action Elimination

The final application for quantum feedback is quite distinct from noise reduction. It is the subject of Ch. 8: turning quantum demolition measurements into quantum non-demolition (QND) measurements. A distinguishing feature of this idea is that it is a purely quantum application for feedback. As far as I know, it is a completely new idea, and is in some ways the most novel contribution of this thesis. The basic idea is that any measurement can be considered a QND measurement (which disturbs only the quantity conjugate to that being measured, according to Heisenberg's uncertainty principle), followed by additional back-action which may disturb the measured quantity. By feeding back the measurement result it is possible to undo the unnecessary back-action and so turn a quantum demolition into a quantum non-demolition measurement. In Ch. 8 I proposed a practical scheme for turning homodyne measurement into a QND measurement of a field quadrature. The feedback required controlling only the amplitude of a driving beam, and the scheme also requires a $\chi^{(2)}$ nonlinear crystal inside a cavity. The device could be used to measure either an intracavity quadrature, or that of a light beam (over some bandwidth). With perfect detection, the feedback produces a measurement which is significantly closer to an ideal QND measurement than would be obtained without the feedback (but with the nonlinearity). With imperfect detectors the advantage of using feedback is not as clear, so it remains to be seen as to whether the scheme is practical. Nevertheless, it is an interesting idea and there may be other applications for it in as-yet undeveloped technologies.

9.3 Remaining Questions

On the basis of my thesis so far, the reader could be forgiven for thinking that the fundamental theory of quantum trajectories and feedback was a closed book and that only applications (however interesting and significant they may be) are left to find. In fact I am aware of at least two important theoretical questions which remain to be addressed. The first is of how to deal with continuous measurements (and hence possibly continuous feedback) for non-Markovian systems. The second is whether quantum trajectories can be related to more fundamental areas of physics, such as the physics of information and quantum cosmology. These are the issues with which I will close my thesis.

9.3.1 Non-Markovian Systems

All of the examples of master equations, continuous quantum measurements and feedback considered in this thesis have been in quantum optical systems. This is no coincidence; quantum optics is the area for which master equations have been developed to the greatest extent, and quantum trajectories and continuous quantum feedback have not, to my knowledge, been used outside this area. The reason for this is simple: optical frequencies are large. A crucial assumption in deriving a master equation for a system is that the interaction between the system and its environment is small compared to the system evolution. This is almost always the case for optical systems because the dipole oscillation frequency is much greater than other time scales. Specifically, this allows one to make the rotating wave approximation in the coupling between the system dipole and the electric field in Eq. (3.74), giving Eq. (3.75). With this approximation, the evolution is relatively easy to solve, as I showed in Ch. 3 (ignoring any subtle issues such as the origin of energy shifts *et cetera*).

With systems for which this approximation cannot be made, the derivation of the master equation can go wrong. An example of this is quantum Brownian motion (QBM). This is meant to model a particle coupled to a thermal bath which damps its momentum as well as giving it random momentum kicks (as originally observed by Brown). As shown in Ref. [52], it is still possible to derive a Markovian equation of motion for the density operator, assuming that the effect of the bath is small in some sense. However, contrary to the claim in that reference, the equation of motion is not of the Lindblad form (2.32). It is worth briefly examining the QBM master equation to see how it fails:

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar}\left[\frac{p^2}{2m} + V(x), \rho\right] - \frac{\mathrm{i}\gamma}{2\hbar}[x, p\rho + \rho p] + \frac{2\gamma m k_B T}{\hbar^2}\mathcal{D}[x]\rho, \tag{9.6}$$

where γ is the damping rate for momentum and T is the temperature of the bath. The second term here looks precisely like the sort of desired feedback resulting from a QND measurement of p, as considered in Sec. 6.6. The third term looks like the noise introduced by this feedback. What is missing is the term representing the QND measurement, which would be

$$\dot{\rho} = \frac{\gamma}{8\eta m k_B T}\mathcal{D}[p]\rho, \tag{9.7}$$

where $\eta \leq 1$ is the efficiency of the measurement. It is easy to verify that this term vanishes from the Langevin equations for x and p in the limit

$$\frac{\gamma\hbar}{4\eta k_B T} \ll 1. \tag{9.8}$$

The presence of $\hbar$ in the numerator ensures that this is satisfied in the classical limit. However, this classical limit could equally well be viewed as the high temperature, or lightly damped limit.

With the failure to obtain a valid master equation, the whole structure of this thesis will not apply; the QBM master equation generates negative probabilities. There are two possible resolutions for this problem. One is that there is a correct Markovian master equation to describe QBM, but that it has not yet been derived. The simplest possible modification to the QBM master equation which makes it of the Lindblad form is to add the extra term (9.7). Although this term vanishes in the classical limit, as necessary, there is no physical (as opposed to mathematical) justification for it. If a modified QBM master equation of the Lindblad form is derived, then one could examine quantum trajectories for the particle *et cetera*, which would be an interesting exercise. The other possibility is that quantum Brownian motion is inherently non-Markovian, unlike classical Brownian motion. If this is the case, then one is lead to the issue of how measurement theory can be applied to non-Markovian systems. There are many questions which can be asked relating to this issue. Is it possible to make measurements on the bath, revealing information about the system, without affecting the non-Markovian nonselective evolution of the system? If it is, then would these measurements give a non-Markovian quantum trajectory for the conditioned state vector of the system?

Some light may be thrown on these questions by considering examples from quantum optics. For some systems, it is not possible to make measurements on the bath without affecting the system. For example, if the output beam of a cavity is reflected off a distant mirror into the cavity again, then a measurement on the output beam would obviously interfere with the evolution of the system. On the other hand, with the all-optical feedback considered in Ch. 7, where there is a Faraday isolator in the path of the output beam, then the output beam may be measured without affecting the system. Yet if the feedback loop delay is significant, the evolution of the cavity will be non-Markovian. In this case, the conditioned system state would not be pure in general. This is because the measurement of the output beam does not reveal the state of the in-loop beam. Because the in-loop beam is entangled with the system, the latter would have to be described by a state matrix even with perfect measurement of the output. However, if the all-optical feedback interaction is a QND interaction for a particular observable of the output beam (as considered in Secs. 7.3 and 7.4), and if that observable is also measured at the output, then this measurement will reveal the state of the in-loop beam also because that observable is unchanged by the feedback. In that case, the quantum trajectory for the system would be a non-Markovian evolution equation for the state vector. Evidently the answers to the above questions depend on the exact system and measurement scheme under consideration.

Unfortunately, it is not obvious how these examples from quantum optics relate to the problem of a particle undergoing QBM. There are further questions also. Under some conditions the non-Markovian evolution of a system may be described by a non-Markovian deterministic master equation for the system state matrix alone, such as in Ref. [36]. What are these conditions? Are they satisfied for quantum Brownian motion? If they are satisfied, then does this mean that QBM could be unraveled into quantum trajectories? With a Markovian master equation, the only valid form is that which can be unraveled as a quantum trajectory, but it is not obvious whether there would be a corresponding theorem for non-Markovian master equations. A more general question is, how can the validity of a non-Markovian master equation be checked? It is known that non-Markovian master equations which have been considered in the derivation of Markovian master equations do not preserve positivity [4]. Some or all of these questions may have answers in the literature; I have not made a thorough search. In any case, they are not addressed in standard books on quantum stochastic processes such as Refs. [52, 137]. This suggests that there is much to be learned in investigating quantum trajectories for non-Markovian systems.

9.3.2 Information Theory

It has been somewhat of an *idée fixe* of this thesis that there is no unique unraveling of a master equation as a quantum trajectory. From a formal standpoint, as shown in Ch. 2, the master equation is invariant under a continuous group of transformations parameterized by γ. Physically, this γ corresponds to the strength of the local oscillator, allowing different measurement schemes (such as direct detection or homodyne detection) to be used. The reader will therefore probably be surprised as I now claim that there is a unique optimal unraveling. Here, I am referring to optimality in the context of information theory [28]. An optimal measurement is that for which the thermodynamic cost of maintaining a record of the system evolution is minimized. According to Landauer, the thermodynamic cost is the energy required to erase the measurement record in an environment at thermal equilibrium of temperature T. This is equal to $k_B T I$, where I is the average information in the record measured in nats (1 bit equals $\log 2$ nats), and k_B is Boltzmann's constant. If the measurement has N outcomes of probability p_i, $i = 1 \ldots N$, then the average information cost is given by Shannon's formula

$$I \simeq H = -\sum_{i=1}^{N} p_i \log p_i, \tag{9.9}$$

where the approximate equality indicates a constant uncertainty of order 1 [28], with I always greater than H.

Consider this in the context of quantum measurement theory. For simplicity, I will restrict the discussion to efficient measurements (as defined in Sec. 2.2) which are specified by the set of measurement operators Ω_i, and an initially pure state $|\psi\rangle$. The nonselective state of the system following the measurement is

$$\rho = \sum_{i=1}^{N} \Omega_i |\psi\rangle\langle\psi| \Omega_i^\dagger. \tag{9.10}$$

The entropy cost of storing this measurement is

$$H(\{\Omega_i\}, \psi) = -\sum_{i=1}^{N} \langle W_i \rangle \log \langle W_i \rangle \tag{9.11}$$

where the averages are with respect to the initial state and $W_i = \Omega_i^\dagger \Omega_i$ is the effect for the result i. Now there are many possible measurement schemes $\{\Omega_i\}$ which produce the same nonselective evolution. It is easy to see that the entropy cost of all such schemes is bounded below by

$$k_B H(\{\Omega_i\}, \psi) \geq S(\rho) = -k_B \mathrm{Tr}[\rho \log \rho], \tag{9.12}$$

the von-Neumann entropy [139] for the post-measurement state matrix.

Furthermore, it is easy to construct the measurement scheme for which this lower bound is attained. This scheme is what I am defining to be the optimal scheme. First solve the eigenvalue equation for ρ,

$$\rho|\tilde{\phi}_k\rangle = \langle\tilde{\phi}_k|\tilde{\phi}_k\rangle|\tilde{\phi}_k\rangle \tag{9.13}$$

The states $|\tilde{\phi}_k\rangle$ are orthogonal, with normalization defined by this equation. The subscript k ranges from 1 to $R = \mathrm{Rank}[\rho] \leq N$ [80]. Then construct the new measurement operators

$$\Omega_i^{\mathrm{opt}} = \sum_{j=1}^{N} U_{ij}\Omega_j, \tag{9.14}$$

where U_{ij} is the c-number square matrix

$$U_{ij} = \frac{\langle\tilde{\phi}_i|\Omega_j|\psi\rangle}{\langle\tilde{\phi}_i|\tilde{\phi}_i\rangle}. \tag{9.15}$$

Here, the undefined kets $|\tilde{\phi}_i\rangle$ $i = R+1, \ldots, N$ have been defined to be the null state. This procedure guarantees that U_{ij} is a unitary matrix [80], so that the new measurement operators Ω_j^{opt} generate the same nonselective state matrix, as explained in Ch. 2. Now however, the conditional states following the measurement $|\tilde{\phi}_i\rangle$ are orthogonal by construction in Eq. (9.13). Thus the entropy for the optimal scheme is the minimum possible, given $|\psi\rangle$ and ρ,

$$k_B H(\{\Omega_i^{\mathrm{opt}}\}, \psi) = S(\rho). \tag{9.16}$$

Now consider the case of continuous measurements, and let there be a single loss source so the nonselective evolution is governed by the master equation

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \mathcal{D}[c]\rho. \tag{9.17}$$

It was shown in Sec. 2.3 that this master equation can be unraveled continuously with two operators for measurements of infinitesimal duration,

$$\begin{aligned} \Omega_1(dt) &= \sqrt{dt}(c + \gamma), && (9.18) \\ \Omega_0(dt) &= 1 - \left[\mathrm{i}H + \tfrac{1}{2}\left(\gamma\gamma^* + 2\gamma^* c + c^\dagger c\right)\right] dt. && (9.19) \end{aligned}$$

Here γ is a c-number which parameterizes all possible measurement schemes on the (single) output. Let the conditioned state of the system at time t be $|\psi_{\mathrm{c}}(t)\rangle$. The optimal measurement for the following infinitesimal time interval is simply defined by requiring that the two conditional states $|\tilde{\psi}_0(t+dt)\rangle$, $|\tilde{\psi}_1(t+dt)\rangle$ be orthogonal. That is to say,

$$\langle\psi_{\mathrm{c}}(t)|\Omega_0^{\mathrm{opt}\,\dagger}(dt)\Omega_1^{\mathrm{opt}}(dt)|\psi_{\mathrm{c}}(t)\rangle = 0. \tag{9.20}$$

To lowest order in dt this gives the very simple condition on γ

$$\gamma^{\mathrm{opt}} \equiv \gamma_{\mathrm{c}}(t) = -\langle c\rangle_{\mathrm{c}}(t). \tag{9.21}$$

Note that this is conditioned on the state. In practical terms, this would require a computer to compute the conditioned state of the system in order to control the local oscillator amplitude. This is feedback of a sort quite unlike what I have considered before. It does not affect the system evolution at all, only the measurement record.

The optimal unraveling of the master equation (9.17) can be written simply as the stochastic Schrödinger equation

$$\begin{aligned} d|\psi_{\rm c}\rangle &= \left[dN_{\rm c}(t)\left(\frac{c-\langle c\rangle_{\rm c}}{\sqrt{\langle c^\dagger c\rangle_{\rm c} - \langle c^\dagger\rangle_{\rm c}\langle c\rangle_{\rm c}}} - 1 \right) \right. \\ &\left. + dt\left(\frac{\langle c^\dagger c\rangle_{\rm c}}{2} - \frac{c^\dagger c}{2} - \langle c^\dagger\rangle_{\rm c}\langle c\rangle_{\rm c} + \langle c^\dagger\rangle_{\rm c} c - {\rm i}H \right) \right] |\psi_{\rm c}(t)\rangle, \end{aligned} \tag{9.22}$$

where the time arguments in the quantum averages have been omitted. This equation could be viewed as a unique quantum jump SSE, in the same way that the equation of Gisin and Percival (5.2) is a unique quantum diffusion SSE. Both of these equations are invariant under the transformation

$$c \to e^{{\rm i}\phi}(c+\gamma)\ , \quad H \to H - {\rm i}\tfrac{1}{2}(c\gamma^* - c^\dagger\gamma). \tag{9.23}$$

However, it is only the quantum jump SSE (9.22) which is optimal in the information theoretic sense. This optimality can be seen from the rate of jumps,

$${\rm E}[dN_{\rm c}(t)] = \left(\langle c^\dagger c\rangle_{\rm c} - \langle c^\dagger\rangle_{\rm c}\langle c\rangle_{\rm c} \right) dt. \tag{9.24}$$

This is the lowest possible rate which can be achieved by changing the measurement on the bath. Thus the measurement record for the optimal SSE will consist of the minimum number of photodetections, and so is the most efficient way of keeping track of the conditioned system evolution. The actual record could be stored as the time between detections, so that the record increases roughly at the rate of an extra real number per characteristic time scale of the system dissipation. By contrast, the measurement record for a heterodyne measurement (corresponding to the Gisin and Percival SSE) consists of two real numbers (from the complex photocurrent) for each infinitesimal time interval, and would be a very inefficient way of keeping track of the system state.

Although the SSE (9.22) is certainly unique, it could be disputed that it is in fact the optimal unraveling. My argument was based on the equivalence between the average information I in the record and the entropy H of the ensemble following the measurement expressed in Eq. (9.9). However, this is only an approximate equality, with I possibly larger than H by a constant of order unity [28]. For continuous measurements of infinitesimal duration, the entropy cost is also infinitesimal, with the lowest order term

$$dH_{\rm c}(t) = -{\rm E}[dN_{\rm c}(t)] \log dt. \tag{9.25}$$

Thus, the apparatus-dependent constant term in I will likely overwhelm the contribution due to actually making the measurement, and so it is questionable whether any claims about optimality can be made based on the size of ${\rm E}[dN_{\rm c}(t)]$. Certainly it is unlikely that anyone would ever use my suggested measurement in an attempt to save on memory, as the computing power needed to keep track of the system state would far outweigh any advantage. Theoretically, the computing could be done reversibly [12], which is why it need not be included in the analysis here.

There is another questionable aspect to continuous measurements. Although $dH_{\rm c}(t)$ goes to zero with dt, the integral $\int dH_{\rm c}(t) \sim \log dt$ is not well defined as $dt \to 0$. This is not inconsistent with the actual measurement record as described above, because there it takes an arbitrarily large amount of information to store real numbers (the time intervals between detections). It is an unsolved question as to what is actually the optimal way to keep track of the state of an open quantum system. For example, with the driven, damped and detuned two-level atom considered in Sec. 5.2, the optimal measurement scheme for continuous measurements over a long time would be direct detection, rather than the 'optimal' measurements I have defined. The reason for this is that whenever a direct photodetection occurs the atom jumps to the ground state. Thus the state of the atom can be stored simply by one number, the time since the last detection. By contrast, all of the detection times under the 'optimal' SSE (9.22) would have to be recorded to deduce the system state. This is because the quantum trajectory will in general wander over the entire Bloch sphere, jumping to a diametrically opposite point whenever a detection occurs. Recall that in the Teich and Mahler model [135] discussed in Ch. 5, the quantum jumps were also between orthogonal states, but that these states were stationary under the deterministic dynamics. If the Teich and Mahler scheme could be realized, it would be ideal because at steady state the system would be known to be in one of the fixed points, so that one need only store which state that was. Unfortunately the Teich and Mahler model can never be realized; it is a fiction which is not even self-consistent, as argued in Ch. 5.

Irrespective of the doubts about the actual optimality of Eq. (9.22), it is nevertheless an unraveling with unique properties. It is surprising that it has not, to my knowledge, been suggested before in any context. The remaining question for this section is, what use is it? I will hazard a few guesses. Firstly, the property of having sparse (in time) jumps might be useful for some numerical solutions of master equations using quantum trajectories. Secondly, the optimal SSE and the Gisin and Percival SSE represent the natural extremities in the continuum of possible unravelings of a master equation. Both have the desirable property of being invariant under the transformation (9.23). The optimal SSE has jumps which are maximally infrequent and large, in the sense that they take a state to an orthogonal state (which is maximally distant in Hilbert space [154]). The Gisin and Percival SSE has infinitely frequent, infinitesimally small jumps, which are represented by diffusion. Thus, if one wishes to investigate the stochastic quantum dynamics of an open system with no particular measurement scheme in mind, then these two limits suggest themselves as starting points for such an investigation. One area where it would be hard to specify the relevant measurement scheme might be quantum cosmology. While the universe is a closed system by definition, it has been suggested that the observable universe could have an environment, such as 'baby universes' which are attached to it by wormholes [68]. If it were possible to model this dissipation as a Markovian process, then the quantum trajectories suggested above could help in conceptualizing the stochastic evolution of an individual 'universe', as seen by a hypothetical external observer.

Appendix A

The Quantum Metaphysics of London and Bauer

This appendix contains a large part of the section "Measurement and Observation. The Act of Objectification" from the short book *The Theory of Observation in Quantum Mechanics* by Fritz London and Edmond Bauer [91]. It explains well my own attitude to the "problem" of measurement in quantum theory. London and Bauer begin this section by considering the interaction of a quantum object x with a quantum apparatus y by unitary evolution, giving an entangled state described by the wavefunction

$$\Psi(x, y) = \sum_k \psi_k u_k(x) v_k(y). \tag{A.1}$$

In their notation, ψ_k is a complex coefficient, and u_k and v_k are orthogonal wavefunctions in the system and apparatus space respectively. Both the apparatus and the system are mixtures (impure states). They go on to say

> So far we have only coupled one apparatus with one object. But a coupling, even with a measuring device, is not yet a measurement. A measurement is only achieved when the position of the pointer has been *observed*. It is precisely this increase in knowledge, acquired by observation, that gives the observer the right to chose among the different components of the mixture predicted by theory, to reject those which are not observed, and to attribute thenceforth to the object a new wave function, that of the pure case which he has found.
>
> We note the essential role played by the consciousness of the observer in this transition from the mixture to the pure case. Without his effective intervention, one would never obtain a new ψ function. In order to see this point clearly, let us consider the ensemble of three systems, (*object* x) + (*apparatus* y) + (*observer* z), as a combined and unique system. We will describe it by the global wavefunction with a form analogous to (A.1),
>
> $$\Psi(x, y, z) = \sum_k \psi_k u_k(x) v_k(y) w_k(z), \tag{A.2}$$
>
> where the w_k represent different states of the observer.
>
> "Objectively" — that is to say, *for us* who consider as "object" the combined system x, y, z — the situation seems little changed compared to what we had just met when

> we were considering only apparatus and object. We now have three mixtures, one for each system, with those statistical correlations between them that are tied to a pure case for the combined system. Thus the function $\Psi(x, y, z)$ represents a maximal description of the combined "object", consisting of the actual object x, the apparatus y, and the observer z; and nevertheless we do not know in what state the object is.
>
> The observer has a completely different impression. For him it is only the object x and apparatus y that belong to the external world, to what he calls "objectivity". By contrast, he has *with himself* relations of a very special character. He possesses a characteristic and quite familiar faculty which we can call the "faculty of introspection." He can keep track from moment to moment of his own state. By virtue of this "immanent knowledge" he attributes to himself the right to create his own objectivity — that is, to cut the chain of statistical correlations summarized in Eq. (A.2) by declaring "I am in state w_k" or more simply, "I see [the state of the apparatus is v_k]," or even directly, "[the state of the object is u_k]."
>
> Accordingly, we will label this creative action as "making objective." By it the observer establishes his own framework of objectivity and acquires a new piece of information about the object in question.[1]
>
> Thus it is not a mysterious interaction between the apparatus and the object that produces a new ψ for the system during the measurement. It is only the consciousness of an "I" who can separate himself from the former function $\Psi(x, y, z)$ and, by virtue of his observation, *set up a new objectivity* in attributing to the object henceforward a new function $\psi(x) = u_k(x)$.

Although London and Bauer's argument is the clearest approach to quantum theory which I know, it is incomplete in one way. It does not explain why the observer should, over a series of experiments, find himself in the state w_k with a probability roughly equal to $|\psi_k|^2$. This is the extra postulate, of equating Hilbert space measure with perceived probabilities, which was referred to in Sec. 1.2 . One can make a convincing case for why the appropriate measure must be the standard Hilbert space measure [46], but this does not answer the question as to how probabilities enter the argument at all. On reflection, it is difficult to conceive how else a measure could be perceived. This seems to make probability a very nebulous concept. However, any theory which hopes to explain quantum experiments must introduce a notion of probability at some point. This holds even in deterministic theories such as Bohm's [16], where the probability enters in the initial conditions for the universe. Wherever probability enters in such theories, one could ask, "What does it mean? What distinguishes it as a probability rather than a measure in some space?" The answer may ultimately be no more convincing than the one given above.

One could argue that probability did not enter into the initial conditions in Bohm's model, but rather that they were chosen by God [2] and only appear random to us. Leaving aside the implications that this may have for the sudden appearance of non-random behaviour, violating the usual probability rules of quantum theory, I wish to raise another question. In Bohm's theory, the wavefunction for the universe still exists, as in the theories of London and Bauer [91] and Everett [46], and indeed as it must if the theory is to reproduce the predictions of standard quantum theory. However, in Bohm's theory, it is only the positions of the particles in 3-space which are regarded as

[1] This paragraph is a typed addition by Prof. Fritz London in his own copy of the printed book, translated by the editors of Ref. [142].

[2] Generator of data

"real". The ultimate end of any physical theory is where it describes perceptions, so one could say rather that it is only these positions which are perceived. My question is, why should perception be attached to these points? There is a universal wavefunction describing multitudes of superpositions of virtual brains interacting with virtual particles, but the only ones which are really conscious are those which are picked out by Bohm's hidden variables. It seems incongruous that the location of the spirit should be determined by something as mundane as a point in configuration space. In London and Bauer's theory, the universal wavefunction contains all physics. The hypothetical objective positions of particles are perhaps a useful aid to understanding in bridging the gap from classical to quantum physics, but ultimately should be discarded to fully appreciate the latter. An analogous rôle is played by the electromagnetic ether in bridging the gap from Newtonian to relativistic physics. In relativity, simultaneity and other concepts are determined by the observer's frame. In quantum physics, the observer is even more central, creating one world out of the infinite virtual possibilities.

Bell [9] has pointed out that Bohm's theory is in fact a special case of a more general non-local hidden variable theory which reproduces the predictions of standard quantum theory. In Bell's theory, the hidden variables are completely arbitrary. They are associated with projection operators in Hilbert space, and so could perhaps more aptly be called "hidden states". The universal state vector evolves as usual, but "in reality" the universe is always in one of the hidden states, and swaps stochastically between them. Bohm's theory is the special case where the hidden states are the joint eigenstates of the position operators of all the particles. In this continuum limit, the stochastic jumping is replaced by smooth evolution along a path in configuration space. Viewed as a generalization of Bohm's theory, Bell's theory shows the arbitrariness of Bohm's (or anyone's) choice of hidden states. Again the question arises, why associate eigenstates of position with consciousness? Even if one were to choose the hidden states as being those which appear to describe conscious observers, it still seems inelegant to have consciousness determined by a label. Why shouldn't those unlucky brains, in branches of the universal state vector outside the hidden state, feel conscious? After all, their physical state changes in precisely the same manner as those of the lucky brains inside the hidden state. (If the hidden state were to cause different physical behaviour, then this could be distinguished experimentally. If this were discovered, then the arguments of this appendix would be worthless. However, as established in Sec. 1.2, I am not considering that possibility here.) In the end, the most natural position to take is to admit, as do London and Bauer, that it is consciousness that determines reality, not the other way round.

Appendix B

The Schrödinger-Langevin Equation

This appendix shows that the relationship between quantum stochastic differential equations in the Schrödinger picture, and quantum trajectories as stochastic Schrödinger equations, can be made much closer than has been presented in the body of this thesis. The relegation of this demonstration to an appendix has two reasons. Firstly, I believe that the derivations in this appendix obscure the conceptual differences between a realized measurement and a system-bath entanglement. Secondly, I only became aware of the work of Goetsch and Graham [65] after having written the chapters on quantum trajectories. I have included this appendix in order to complete my coverage of the topic.

B.1 Homodyne Detection

In the new edition of the book *Stochastic Processes in Physics and Chemistry*, van Kampen writes down a "Schrödinger-Langevin equation"

$$d|\tilde{\psi}_0\rangle = dt\left[-\mathrm{i}H - \tfrac{1}{2}c^\dagger c + \ell_0(t)c\right]|\tilde{\psi}_0\rangle, \tag{B.1}$$

where I am using my own notation, the reason for which will become evident later. Here, $\ell_0(t)$ represents white noise satisfying

$$\mathrm{E}[\ell_0(t)] = 0\ ;\ \ \mathrm{E}[\ell_0^*(t)\ell_0(t')] = \delta(t-t'). \tag{B.2}$$

As noted by van Kampen, this noise need not be Gaussian. However, in this section I will be concerned with the case where $\ell_0(t) = \xi_0(t)$, representing real Gaussian white noise. Note that $|\tilde{\psi}_0\rangle$ is not normalized, and that it should not be normalized when determining the ensemble average state matrix

$$\rho = \mathrm{E}\left[|\tilde{\psi}_0\rangle\langle\tilde{\psi}_0|\right]. \tag{B.3}$$

It is easy to see that this obeys

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \mathcal{D}[c]\rho. \tag{B.4}$$

Thus, Eq. (B.1) represents an unraveling of the master equation in terms of (unnormalized) state vectors. However, it is different from the stochastic Schrödinger equations with Gaussian white noise (appropriate for homodyne or heterodyne detection), in that it is a linear equation. This appendix aims to show how it is related to those SSEs.

Goetsch and Graham [65] also consider Eq. (B.1), but they derive it rather than postulating it. They consider the quantum optical system-bath coupling I described in Sec. 3.4. In my notation, the interaction Hamiltonian for the system and bath at time t is

$$V(t) = \mathrm{i}[b^\dagger(t)c - c^\dagger b(t)]. \tag{B.5}$$

Assume that the bath is in the vacuum state, appropriate for the master equation (B.4). Further assume that the system at time t can be specified by a state vector $|\psi(t)\rangle$. Then the initial state of the system and bath at time t is

$$|\Psi(t)\rangle = |\psi(t)\rangle|0\rangle, \tag{B.6}$$

where the ket $|0\rangle$ denotes the vacuum state about to interact with the system, with

$$b(t)|0\rangle = 0. \tag{B.7}$$

Now the entangled state a time dt later is, ignoring the Hamiltonian evolution via H,

$$|\Psi(t+dt)\rangle = \exp[dB^\dagger(t)c - c^\dagger dB(t)]|\Psi(t)\rangle, \tag{B.8}$$

where $dB(t) = b(t)dt$ as usual. Expanding the exponential using the rules of quantum Itô calculus as explained in Sec. 3.4 yields

$$|\Psi(t+dt)\rangle = \left[1 - \tfrac{1}{2}c^\dagger c dt + c dB^\dagger(t)\right]|\psi(t)\rangle|0\rangle. \tag{B.9}$$

Now because of Eq. (B.7) it is possible to replace $dB^\dagger$ in Eq. (B.9) by $dB^\dagger + dB$, giving

$$|\Psi(t+dt)\rangle = \left\{1 - \tfrac{1}{2}c^\dagger c dt + c[dB^\dagger(t) + dB(t)]\right\}|\psi(t)\rangle|0\rangle. \tag{B.10}$$

This is useful if one now wishes to measure the x quadrature of the bath after it has interacted with the system. This measurement is modeled by projecting the field onto the eigenstates $|\xi_0(t)\rangle$, where

$$[b(t) + b^\dagger(t)]|\xi_0(t)\rangle = \xi_0(t)|\xi_0(t)\rangle. \tag{B.11}$$

The system state conditioned on the measurement is thus

$$|\tilde{\psi}(t+dt)\rangle = \langle\xi_0(t)|\Psi(t+dt)\rangle = \left[1 - \tfrac{1}{2}c^\dagger c dt + c\xi_0(t)dt\right]|\psi(t)\rangle\sqrt{\Upsilon_\nu[\xi_0(t)]}, \tag{B.12}$$

where

$$\Upsilon_\nu[\xi_0(t)] = |\langle\xi_0(t)|0\rangle|^2 = \sqrt{\frac{dt}{2\pi}}\exp\left[-\tfrac{1}{2}dt\xi_0(t)^2\right]. \tag{B.13}$$

The ν subscript in Υ_ν refers to the vacuum input. The norm of the state $|\tilde{\psi}(t+dt)\rangle$ gives the probability for the result $\xi_0(t)$.

Taking the trace over the bath is the same as averaging over the measurement result, yielding

$$\rho(t+dt) = \int d\xi_0(t)|\tilde{\psi}(t+dt)\rangle\langle\tilde{\psi}(t+dt)|. \tag{B.14}$$

It is easy to verify that this $\rho(t+dt) = \rho(t) + dt\dot{\rho}(t)$, where $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ and $\dot{\rho}$ is given by Eq. (B.4). The factor $\sqrt{\Upsilon_\nu[\xi_0(t)]}$ can be removed from the state vector by changing the integration measure for the measurement result. That is,

$$\rho(t+dt) = \int d\mu_0(\xi_0(t))|\tilde{\psi}_0(t+dt)\rangle\langle\tilde{\psi}_0(t+dt)|, \tag{B.15}$$

where

$$|\tilde{\psi}_0(t+dt)\rangle = \left[1 - \tfrac{1}{2}c^\dagger c dt + c\xi_0(t)dt\right]|\psi(t)\rangle \tag{B.16}$$

and

$$d\mu_0\big(\xi_0(t)\big) = \Upsilon_\nu[\xi_0(t)]d\xi_0(t). \tag{B.17}$$

Here, the 0 subscript refers to the use in Eq. (B.16) of $\xi_0(t)$, which, according to the integration measure $d\mu_0\big(\xi_0(t)\big)$, is to be interpreted as real Gaussian white noise. Thus it is evident that $|\tilde{\psi}_0\rangle$ evolves according to the Schrödinger-Langevin equation (B.1) with $\ell_0(t) = \xi_0(t)$.

In this formulation, the noise $\xi_0(t)$ is Gaussian in one sense, due to its origin in the vacuum fluctuations of the input bath, yet its actual statistics are weighted by the norm of the conditioned ket $|\tilde{\psi}_0(t+dt)\rangle$. The actual distribution for $\xi_0(t)$ can be seen by using ket $|\breve{\psi}_0(t+dt)\rangle$, defined to be $|\tilde{\psi}_0(t+dt)\rangle$ normalized to have unit norm. The reason for retaining an identifier above the state will become evident later. To ease the confusion of the reader, the kets defined so far, and those still to be defined, are set out in table B.1. In terms of the normalized $|\breve{\psi}_0(t+dt)\rangle$, the nonselective density operator is found to be

$$\rho(t+dt) = \int d\mu_1\big(\xi_0(t)\big)|\breve{\psi}_0(t+dt)\rangle\langle\breve{\psi}_0(t+dt)|. \tag{B.18}$$

where the new measure for the result $\xi_0(t)$ is

$$d\mu_1\big(\xi_0(t)\big) = d\mu_0\big(\xi_0(t)\big)\langle\tilde{\psi}_0(t+dt)|\tilde{\psi}_0(t+dt)\rangle. \tag{B.19}$$

From Eq. (B.16) it is easy to show that

$$\langle\tilde{\psi}_0(t+dt)|\tilde{\psi}_0(t+dt)\rangle = 1 + \langle x(t)\rangle\xi_0(t)dt, \tag{B.20}$$

where $x = c + c^\dagger$, and the average is with respect to $|\psi(t)\rangle$. Using Eq. (B.13), this allows the new measure to be related to the old by

$$d\mu_1\big(\xi_0(t)\big) = d\mu_0\big(\xi_0 - \langle x(t)\rangle\big). \tag{B.21}$$

Alternatively, the actual measurement result is $\xi_1(t)$, which is defined as

$$\xi_1(t) = \xi_0(t) + \langle x(t)\rangle, \tag{B.22}$$

where $\xi_0(t)$ is assigned the measure $d\mu_0\big(\xi_0(t)\big)$.

Eq. (B.22) is familiar from the homodyne detection theory of Ch. 4 as the homodyne photocurrent, equal to a deterministic term proportional to the quantum mean of x, plus a white noise term which there was derived from the shot noise of the local oscillator. Goetsch and Graham also claim that Eq. (B.22) can be regarded as the eigenvalue solution of the input-output relation of Gardiner and Collet [53], as derived in Sec. 3.4,

$$\xi_1(t) = \xi_0(t) + x(t). \tag{B.23}$$

Here $\xi_0 = b_0 + b_0^\dagger$ is the operator for the input field quadrature (in the vacuum state), and $\xi_1 = b_1 + b_1^\dagger$ is that for the output field. In any case, what Eq. (B.22) indicates is that it is more natural to replace the measurement result $\xi_0(t)$ by $\xi_1(t)$, and to retain the original probability distribution $\Upsilon_\nu[\xi_0(t)]$ for $\xi_0(t)$. To see what this means, replace $\xi_0(t)$ by $\xi_1(t)$ in Eq. (B.16), giving

$$|\bar{\psi}_0(t+dt)\rangle = |\tilde{\psi}_1(t+dt)\rangle = \left[1 - \tfrac{1}{2}c^\dagger c dt + c\xi_1(t)dt\right]|\psi(t)\rangle. \tag{B.24}$$

states	unnormalized	normalized
$I^{\rm hom} = \xi_0$	$\lvert\tilde\psi_0\rangle$; $\rho = {\rm E}\left[\lvert\tilde\psi_0\rangle\langle\tilde\psi_0\rvert\right]$	$\lvert\breve\psi_0\rangle = \lvert\tilde\psi_0\rangle/\sqrt{\langle\tilde\psi_0\lvert\tilde\psi_0\rangle}$
$I^{\rm hom} = \xi_1$	$\lvert\bar\psi_0\rangle = \lvert\tilde\psi_1\rangle$	$\lvert\psi_0\rangle = \lvert\bar\psi_0\rangle/\sqrt{\langle\bar\psi_0\lvert\bar\psi_0\rangle} = \lvert\breve\psi_1\rangle$; $\rho = {\rm E}\big[\lvert\psi_0\rangle\langle\psi_0\rvert\big]$

Table B.1: Definitions of conditioned state vectors used in the text. The subscript 0 or 1 indicates the use of ξ_0 or ξ_1 in the definition of the increment. $I^{\rm hom}(t)$ is the measured homodyne photocurrent. The starting point $|\tilde\psi_0(t+dt)\rangle$ is defined by Eq. (B.15)

Now this new ket is still unnormalized, but its norm is not required to calculate the output $\xi_1(t)$, as this is given by Eq. (B.22). Instead, the unnormalized ket

$$|\bar\psi_0(t+dt)\rangle = \left\{1 - \tfrac{1}{2}c^\dagger c dt + c[\xi_0(t) + \langle x(t)\rangle]dt\right\}|\psi(t)\rangle \tag{B.25}$$

should simply be normalized to give $|\psi_0(t+dt)\rangle$, representing the state conditioned on the homodyne photocurrent $\xi_1(t)$. But Eq. (B.25) is of course the same unnormalized equation (4.55) describing homodyne measurements derived in Sec. 4.2, from where the reader was first directed to this appendix. Obviously, the normalized state vector $|\psi_0(t)\rangle$ will obey the homodyne SSE derived there as Eq. (4.54),

$$d|\psi_0(t)\rangle = \left\{-{\rm i}H dt - \tfrac{1}{2}\left[c^\dagger c - 2\langle x/2\rangle(t)c + \langle x/2\rangle^2(t)\right]dt + [c - \langle x/2\rangle(t)]dW_0(t)\right\}|\psi(t)\rangle, \tag{B.26}$$

where $dW_0(t) = \xi_0(t)dt$. The ensemble average of this state vector, according to the measure (B.17), is the nonselective state matrix $\rho(t)$.

The Schrödinger-Langevin equation (B.1) is much simpler than the normalized stochastic Schrödinger equation (B.26), and also has the property of being linear. This would presumably make it easier to find analytic solutions, which Goetsch and Graham [65] have done for a damped harmonic oscillator, and a parametrically driven damped harmonic oscillator starting in the vacuum state. Although the latter result sounds impressive, it can also be derived readily from the normalized nonlinear equation (B.26), in the manner shown in Sec. 6.4. A system which is trivial to solve using the unnormalized linear equation (B.1) is the damped two-level atom. By contrast both the SSE (B.26) and the Heisenberg picture Langevin equations are not easily solved because of the nature of the nonlinearity. The solution using the Schrödinger-Langevin equation allows the operations and effects for the measurement of the integrated homodyne photocurrent to be determined analytically. (They can also be determined from noting that a two-level atom is isomorphic to the lowest two levels of a harmonic oscillator, as stated in Sec. 9.2.) The linearity of the Schrödinger-Langevin equation may also make it useful for some techniques of numerical simulation for more complicated systems. In such simulations, the norm of the final state vector must be taken into account as the weight given to that particular trajectory. This has the disadvantage that a particular trajectory cannot be regarded as typical, so the ensemble may not converge as fast as an ensemble generated by the nonlinear SSE (B.26). In summary, either the linear or nonlinear forms of quantum trajectories for homodyne detection can be used; the choice depends on the purpose, and the particular system.

B.2 Heterodyne Detection

It is not difficult to see that the Schrödinger-Langevin equation for heterodyne detection is obtained from Eq. (B.9) by considering measurements of the complex amplitude of the output field. This is modeled by forming the inner product with the state $|\zeta_0(t)\rangle$, where $\zeta_0(t)$ is the eigenvalue for the annihilation operator $b(t)$:

$$b(t)|\zeta_0(t)\rangle = \zeta_0(t)|\zeta_0(t)\rangle. \tag{B.27}$$

Because of this eigenvalue equation, it is not necessary to modify Eq. (B.9) before making the measurement. Taking into account the overcompleteness of the coherent states (see Sec. 4.5.3), the conditioned vector at time $t+dt$ is

$$|\tilde{\psi}(t+dt)\rangle = \frac{1}{\sqrt{\pi}}\langle\zeta_0(t)|\Psi(t+dt)\rangle = \left[1 - \tfrac{1}{2}c^\dagger c dt + c\zeta_0^*(t)dt\right]|\psi(t)\rangle\sqrt{\Upsilon_\nu[\zeta_0(t)]}, \tag{B.28}$$

where

$$\Upsilon_\nu[\zeta_0(t)] = \frac{1}{\pi}\left|\langle\zeta_0(t)|0\rangle\right|^2 = \frac{dt}{\pi}\exp\left[-dt|\zeta_0(t)|^2\right]. \tag{B.29}$$

This gives the heterodyne version of Eq. (B.1) as

$$d|\tilde{\psi}_0(t)\rangle = dt\left[-\tfrac{1}{2}c^\dagger c + \zeta_0^*(t)c\right]|\tilde{\psi}_0(t)\rangle. \tag{B.30}$$

The remainder of the analysis for heterodyne detection goes through in the same manner as for homodyne detection, with the final nonlinear version given as Eq. (4.80).

B.3 Detection with a Squeezed Input

The analysis given above for homodyne and heterodyne detection has been for an input in the vacuum state. The addition of a coherent driving field to the input is trivial to work out. What is more interesting is the case where the input is contaminated by white noise. This is defined by the quantum Itô rules

$$dB^\dagger(t)dB(t) = N dt\ ;\ \ dB(t)^2 = M dt. \tag{B.31}$$

If the input is in a mixed state (such as a thermal state with $N \neq 0$ and $M = 0$), then the conditioned system state will not remain pure even if the output is detected efficiently. This could be treated by using an unnormalized state matrix, rather than an unnormalized state vector. This is quite possible, but is a complication which I wish to avoid, if for no other reason than that the notation would become unbearably confusing. I will consider only a bath which is in a pure squeezed state, with $|M|^2 = N(N+1)$. Then Eq. (B.8) when expanded yields

$$|\Psi(t+dt)\rangle = \left\{1 - \tfrac{1}{2}dt\left[(N+1)c^\dagger c + Ncc^\dagger - Mc^{\dagger 2} - M^*c^2\right] + c\,dB^\dagger(t) - c^\dagger dB(t)\right\}|\psi(t)\rangle|M\rangle, \tag{B.32}$$

where I am denoting the input state by the ket $|M\rangle$. It is easy to verify that Eq. (B.7) is replaced by

$$[(N+M^*+1)b(t) - (N+M)b^\dagger(t)]|M\rangle = 0, \tag{B.33}$$

by showing that the mean of the square of this operator is zero.

Consider homodyne detection on the output. Any multiple of the operator in Eq. (B.33) can be added to Eq. (B.32) without affecting it. Thus it is possible to replace $dB^\dagger(t)$ by

$$dB^\dagger(t) + \frac{N+M^*+1}{L}dB(t) - \frac{N+M}{L}dB^\dagger(t) = \frac{N+M^*+1}{L}[dB^\dagger(t)+dB(t)], \tag{B.34}$$

and $dB(t)$ by

$$dB(t) - \frac{N+M^*+1}{L}dB(t) + \frac{N+M}{L}dB^\dagger(t) = \frac{N+M}{L}[dB^\dagger(t)+dB(t)], \tag{B.35}$$

where $L = 2N + M^* + M + 1$ as usual. This yields

$$\begin{aligned}|\Psi(t+dt)\rangle &= \Big\{1 - \tfrac{1}{2}dt\left[(N+1)c^\dagger c + Ncc^\dagger - Mc^{\dagger 2} - M^* c^2\right] \\ &+ \left(c\frac{N+M^*+1}{L} - c^\dagger\frac{N+M}{L}\right)[dB^\dagger(t)+dB(t)]\Big\}|\psi(t)\rangle|M\rangle, \end{aligned}\tag{B.36}$$

Projecting onto eigenstates of the output quadrature then gives the unnormalized conditioned state

$$\begin{aligned}|\tilde\psi(t+dt)\rangle &= \Big\{1 - \tfrac{1}{2}dt\left[(N+1)c^\dagger c + Ncc^\dagger - Mc^{\dagger 2} - M^* c^2\right] \\ &+ \xi_0(t)dt\left(c\frac{N+M^*+1}{L} - c^\dagger\frac{N+M}{L}\right)\Big\}|\psi(t)\rangle\sqrt{\Upsilon_M[\xi_0(t)]}, \end{aligned}\tag{B.37}$$

where

$$\Upsilon_M[\xi_0(t)] = |\langle\xi_0(t)|M\rangle|^2 = \sqrt{\frac{dt}{2\pi L}}\exp[-\tfrac{1}{2}\xi_0(t)^2 dt/L]. \tag{B.38}$$

Note that the variance of $\xi_0(t)^2$ is L/dt, which, depending on the modulus and argument of M, may be larger or smaller than its vacuum value of $1/dt$.

The Schrödinger-Langevin equation for the unnormalized state vector $|\tilde\psi_0(t)\rangle$, defined as above, is

$$\begin{aligned}d|\tilde\psi_0(t)\rangle &= dt\Big\{-\tfrac{1}{2}\left[(N+1)c^\dagger c + Ncc^\dagger - Mc^{\dagger 2} - M^* c^2\right] \\ &+ \xi_0(t)\left(c\frac{N+M^*+1}{L} - c^\dagger\frac{N+M}{L}\right)\Big\}|\psi(t)\rangle. \end{aligned}\tag{B.39}$$

From the norm of this state it is easy to find the true distribution for the measurement result $\xi_0(t)$. The upshot is that the measurement result, renamed as $\xi_1(t)$ is given by

$$\xi_1(t) = \langle x(t)\rangle + \xi_0(t) = \langle x(t)\rangle + \sqrt{L}\xi(t), \tag{B.40}$$

where $\xi(t)$ is normalized Gaussian white noise. Using this result in place of $\xi_0(t)$ in Eq. (B.39) yields the unnormalized SSE for $|\bar\psi_0(t)\rangle$ (as defined in table B.1)

$$\begin{aligned}d|\bar\psi_0(t)\rangle &= dt\Big\{-\tfrac{1}{2}\left[(N+1)c^\dagger c + Ncc^\dagger - Mc^{\dagger 2} - M^* c^2\right] \\ &+ [\langle x(t)\rangle + \sqrt{L}\xi(t)]\left(c\frac{N+M^*+1}{L} - c^\dagger\frac{N+M}{L}\right)\Big\}|\psi(t)\rangle, \end{aligned}\tag{B.41}$$

as quoted in Sec. 4.4. The normalized version of this equation is complicated for the state vector, but simple for the state matrix, which is how it was derived in Sec. 4.4. The modification of these formulae for the case of heterodyne detection in a squeezed bath is not difficult to derive.

B.4 Direct Detection

The above sections show how the quantum trajectories for homodyne and heterodyne detection can be derived directly from the quantum Langevin equation in the Schrödinger picture. In a sense, the Schrödinger-Langevin equation is a way of replacing quantum noise by classical noise while still retaining the correct quantum evolution. The Gaussian detection noise, which is attributed to local oscillator shot noise in the derivations of Ch. 4, arises directly from the quantum Gaussian noise of the field operators. What is less obvious is how to replace the quantum noise by classical point process noise. That is to ask, how can the quantum trajectories for direct detection be derived from the quantum Langevin equation? This question was not considered by Goetsch and Graham. The difficulty lies in the fact that, if the input bath is in the vacuum state, then it is an eigenstate of the photon flux operator, so that there is no input noise from which a point process can be derived. One way to treat direct detection is to consider an input bath with a coherent amplitude. This amplitude can then be taken to zero after the stochastic Schrödinger equation has been derived. Unlike the case of homodyne or heterodyne detection, the linear Schrödinger-Langevin equation is not particularly easy to solve, and indeed does not exist in the limit of a vacuum input. Nevertheless, it is good to understand how direct detection can be derived within the same framework.

Let the input bath at time t be denoted $|\beta\rangle$ where

$$[b(t)-\beta]|\beta\rangle = 0. \tag{B.42}$$

The Langevin equation

$$|\Psi(t+dt)\rangle = \left[1 - \tfrac{1}{2}dtc^\dagger c + cdB^\dagger(t) - c^\dagger dB(t)\right]|\psi(t)\rangle|\beta\rangle, \tag{B.43}$$

can thus be rewritten as

$$|\Psi(t+dt)\rangle = \left[1 - \tfrac{1}{2}dtc^\dagger c + c\frac{b(t)^\dagger b(t)}{\beta}dt - c^\dagger \beta dt\right]|\psi(t)\rangle|\beta\rangle. \tag{B.44}$$

Now project the output onto eigenstates $I_0(t)$ of the output defined by

$$b^\dagger(t)b(t)|I_0(t)\rangle = I_0(t)|I_0(t)\rangle, \tag{B.45}$$

giving the unnormalized state

$$|\tilde{\psi}(t+dt)\rangle = \left[1 - \tfrac{1}{2}dtc^\dagger c + c\frac{I_0(t)}{\beta}dt - c^\dagger \beta(t) dt\right]|\psi(t)\rangle\sqrt{\Upsilon_\beta[I_0(t)]}, \tag{B.46}$$

where

$$\Upsilon_\beta[I_0(t)] = |\langle I_0(t)|\beta\rangle|^2 = (1-|\beta|^2dt)\delta\big(I_0(t)\big) + |\beta|^2 dt\,\delta\big(I_0(t)-1/dt\big). \tag{B.47}$$

That is to say, $dN_0(t) = I_0(t)dt$ is a point process of mean $|\beta|^2dt$.

Define a new unnormalized state as usual

$$|\tilde{\psi}_0(t+dt)\rangle = \left[1 - \tfrac{1}{2}dtc^\dagger c + c\frac{I_0(t)}{\beta}dt - c^\dagger \beta dt\right]|\psi(t)\rangle. \tag{B.48}$$

Then the nonselective state matrix is given by

$$\rho(t+dt) = \int d\mu_0\big(I_0(t)\big)|\tilde{\psi}_0(t+dt)\rangle\langle\tilde{\psi}_0(t+dt)|, \tag{B.49}$$

where the measure for the output flux is

$$d\mu_0\big(I_0(t)\big) = dI_0(t)\Upsilon_\beta[I_0(t)]. \tag{B.50}$$

The nonselective evolution can be readily evaluated to be generated by the master equation

$$\dot{\rho} = [\beta^* c - \beta c^\dagger, \rho] + \mathcal{D}[c]\rho, \tag{B.51}$$

which is that appropriate for a system driven coherently.

Defining the normalized ket $|\breve{\psi}_0(t+dt)\rangle$ as in Sec. B.1, the evolved density operator can be re-expressed as

$$\rho(t+dt) = \int d\mu_1\big(I_0(t)\big)\,|\breve{\psi}_0(t+dt)\rangle\langle\breve{\psi}_0(t+dt)|, \tag{B.52}$$

where

$$\begin{aligned} d\mu_1\big(I_0(t)\big) &= d\mu_0\big(I_0(t)\big)\langle\tilde{\psi}_0(t+dt)|\tilde{\psi}_0(t+dt)\rangle \\ &= dI_0(t)\left\{\left[1-\langle(c^\dagger+\beta^*)(c+\beta)\rangle dt\right]\delta\big(I_0(t)\big) + \langle(c^\dagger+\beta^*)(c+\beta)\rangle dt\,\delta\big(I_0(t)-1/dt\big)\right\}, \end{aligned}$$

where the quantum means are again evaluated using $|\psi(t)\rangle$. The meaning of this new measure is that the actual statistics for the output are given by the point process $dN_1(t) = I_1(t)dt$, with

$$\mathrm{E}[I_1(t)] = \langle(c^\dagger+\beta^*)(c+\beta)\rangle. \tag{B.53}$$

Replacing I_0 by I_1 in Eq. (B.48) gives

$$|\bar{\psi}_0(t+dt)\rangle = |\tilde{\psi}_1(t+dt)\rangle = \left[1-\tfrac{1}{2}dt c^\dagger c + c\frac{dN_1(t)}{\beta} - c^\dagger\beta dt\right]|\psi(t)\rangle, \tag{B.54}$$

where the normalization of this state has no significance. Although this equation appears linear, like Eq. (B.48), it is not because the noise $dN_1(t)$ is generated according to Eq. (B.53) which uses the system state. Finally, one can normalize $|\bar{\psi}_0(t+dt)\rangle$ to arrive at the SSE

$$\begin{aligned} d|\psi_0(t)\rangle &= \Bigg[dN_1(t)\left(\frac{c+\beta}{\sqrt{\langle(c^\dagger+\beta^*)(c+\beta)\rangle}}-1\right) \\ &\quad + dt\left(-\tfrac{1}{2}c^\dagger c + \tfrac{1}{2}\langle c^\dagger c\rangle - \beta c^\dagger + \tfrac{1}{2}\langle\beta c^\dagger + \beta^* c\rangle\right)\Bigg]|\psi_0(t)\rangle. \end{aligned} \tag{B.55}$$

This is the same SSE as derived in Sec. 4.1.4. Furthermore, it is now possible to let the coherent amplitude β go to zero, as the equation is well-defined in this limit. The result is of course the standard SSE for direct detection in a vacuum,

$$d|\psi_0(t)\rangle = \left[dN_1(t)\left(\frac{c}{\sqrt{\langle c^\dagger c\rangle}}-1\right) + dt\left(-\tfrac{1}{2}c^\dagger c + \tfrac{1}{2}\langle c^\dagger c\rangle\right)\right]|\psi_0(t)\rangle. \tag{B.56}$$

Appendix C

Essential Model of a Laser

In this appendix I present a simple physical model for a laser. It reproduces the essential features of an ideal laser. The steady-state photon number distribution is Poissonian, as in a coherent state, but the phase diffuses, giving a finite linewidth. The laser consists of a single-mode optical cavity containing N_A four-level atoms. The atomic level structure is shown in Fig. C.1. The lowering operator from an upper to a lower level is denoted σ_{lu}. The cavity mode has annihilation operator a, freely rotating at frequency ω_0. This is resonant with an atomic transition between levels $|1\rangle$ and $|2\rangle$. The dipole coupling constant is g, which I take to be real for simplicity. The lower level $|1\rangle$ spontaneously decays to the atomic ground state $|0\rangle$ at rate γ_1. The atoms are incoherently pumped from $|0\rangle$ to $|3\rangle$. This is modeled as thermal excitation with damping rate γ_0 and thermal photon number $N \gg 1$. Level $|3\rangle$ decays to the upper level of the lasing transition $|2\rangle$ at rate γ_3. For completeness, dephasing of the atomic dipole (caused by elastic collisions) at rate γ_d, and decay of the upper level $|2\rangle$ to the lower level $|1\rangle$ at rate γ_2, are also included. Finally, the cavity mode is damped to the external continuum of electromagnetic modes at rate κ.

The aim is to adiabatically eliminate the atoms, so it is only necessary to consider the interaction between the field and a single atom, and subsequently scale the effect by N_A. The density operator R for the field plus one atom obeys the following master equation:

$$\begin{aligned}\dot{R} &= -\mathrm{i}\frac{g}{2}[a^\dagger\sigma_{12} + a\sigma_{12}^\dagger, R] + \gamma_1\,\mathcal{D}[\sigma_{01}]R + \gamma_0(N+1)\,\mathcal{D}[\sigma_{03}] + \gamma_0 N\,\mathcal{D}[\sigma_{03}^\dagger]R \\ &\quad + \gamma_3\,\mathcal{D}[\sigma_{23}] + \gamma_d\mathcal{D}[\tfrac{1}{2}\sigma_{12}^z] + \kappa\,\mathcal{D}[a]R. \end{aligned} \tag{C.1}$$

where, $\mathcal{D}[c]$ is as defined in Eq. (2.33). The atom may be adiabatically eliminated if it relaxes on a time scale much smaller than the cavity lifetime κ^{-1}. Under this approximation, the damping of the cavity can be temporarily ignored. Consider the following ansatz for the total state of atom plus field:

$$R = \sum_{l=0}^{3} \rho_l \otimes |l\rangle\langle l| + \rho_d \otimes |2\rangle\langle 1| + \rho_d^\dagger \otimes |1\rangle\langle 2|. \tag{C.2}$$

Substituting this into Eq. (C.1) with cavity damping ignored gives the following coupled equations for the field density operators

$$\dot{\rho}_0 = -\beta\rho_0 + \beta\rho_3 + \gamma_1\rho_1, \tag{C.3}$$

$$\dot{\rho}_1 = -\mathrm{i}\frac{g}{2}(a^\dagger\rho_d - \rho_d^\dagger a) - \gamma_1\rho_1 + \gamma_2\rho_2, \tag{C.4}$$

$$\dot{\rho}_2 = -\mathrm{i}\frac{g}{2}(a\rho_d^\dagger - \rho_d a^\dagger) - \gamma_2\rho_2 + \gamma_3\rho_3, \tag{C.5}$$

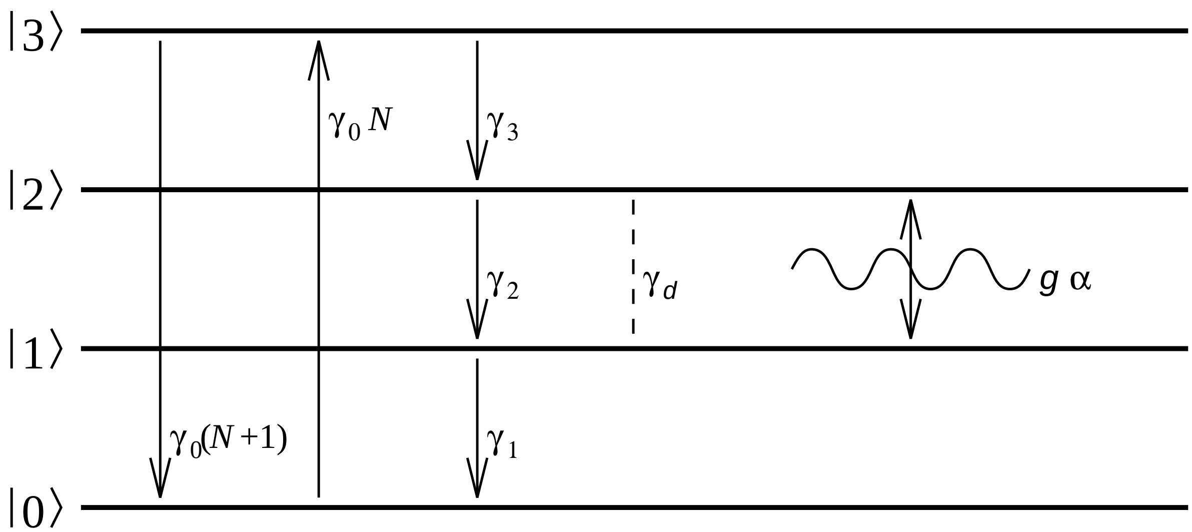

Figure C.1: Schematic diagram for atomic processes in the ideal laser model. The dipole coupling constant for the $|1\rangle - |2\rangle$ lasing transition is g and the field amplitude is denoted α. Atomic damping rates are denoted $\gamma_l(l = 0, 1, 2, 3)$ and the dipole dephasing rate is γ_d. The $|0\rangle - |3\rangle$ transition is the thermal pump, with $N \gg 1$ the number of thermal photons. All other baths are taken to be at zero temperature.

$$\dot{\rho}_3 = -\beta\rho_3 - \gamma_3\rho_3 + \beta\rho_0, \tag{C.6}$$

$$\dot{\rho}_d = -\mathrm{i}\frac{g}{2}(a\rho_1 - \rho_2 a) - \frac{\Gamma}{2}\rho_d, \tag{C.7}$$

where $\Gamma = \gamma_1 + \gamma_2 + \gamma_d$ and $\beta = N\gamma_0$, where 1 has been ignored compared to N. The reduced density operator for the field is given by

$$\rho = \sum_{l=0}^{3} \rho_l. \tag{C.8}$$

From Eqs. (C.3–C.6) this obeys

$$\dot{\rho} = \mathrm{i}\frac{g}{2}(a\rho_d^\dagger - \rho_d a^\dagger + a^\dagger\rho_d - \rho_d^\dagger a). \tag{C.9}$$

Now Eq. (C.9) is only a master equation if ρ_d can be written in terms of ρ. That is to say, the dipole of the atom must be slaved to the field. This can be achieved under the following conditions:

$$\gamma_1, \gamma_3 \gg \gamma_2, g\alpha \gg \beta, \tag{C.10}$$

where α is the magnitude of the field amplitude. The following argument is qualitative but can be rigorously justified. The requirement that γ_1 is very large ensures that level 1 becomes depopulated with

$$\rho_1 \simeq 0. \tag{C.11}$$

Substituting Eq. (C.11) into Eq. (C.7), and using the fact that $\Gamma \geq \gamma_1 \gg g\alpha$ allows the atomic dipole to be slaved to the atomic inversion:

$$\rho_d \simeq \mathrm{i}\frac{g}{\Gamma}\rho_2 a. \tag{C.12}$$

Substituting this into Eq. (C.9) gives

$$\dot{\rho} = \chi \mathcal{D}[a^\dagger]\rho_2, \tag{C.13}$$

where $\chi = g^2/\Gamma$.

This equation is still not a master equation unless ρ_2 can be slaved to ρ. Because the downward transition rates of the atom are so large, the atomic population will tend to accumulate in the ground state with

$$\rho_0 \simeq \rho. \tag{C.14}$$

Next, the requirement that γ_3 is very large gives, from Eq. (C.6),

$$\rho_3 = \frac{\beta}{\gamma_3}\rho. \tag{C.15}$$

Substituting this and Eq. (C.12) into Eq. (C.5) gives

$$\dot{\rho}_2 = -(\gamma_2 + \chi \mathcal{A}[a^\dagger])\rho_2 + \beta\rho. \tag{C.16}$$

Then the smallness condition on β allows ρ_2 to be slaved as

$$\rho_2 = \beta(\gamma_2 + \chi \mathcal{A}[a^\dagger])^{-1}\rho, \tag{C.17}$$

where this inverse superoperator is well defined, as will be shown later.

Substituting this into Eq. (C.13) gives the laser gain master equation

$$\dot{\rho} = G n_s \mathcal{D}[a^\dagger](n_s + \mathcal{A}[a^\dagger])^{-1}\rho, \tag{C.18}$$

where the laser gain G (scaled by N_A atoms) is defined by

$$G = N_A \beta / n_s = N_A \gamma_0 N / n_s, \tag{C.19}$$

where the saturation photon number is

$$n_s = \frac{\gamma_2}{\chi} = \frac{\gamma_2(\gamma_1 + \gamma_2 + \gamma_d)}{g^2}. \tag{C.20}$$

Eq. (C.18) can be put in the form required for a master equation (2.32) as

$$\dot{\rho} = G n_s \int_0^\infty dx \left\{ n_s \mathcal{D}\left[\exp\left(-x(n_s + aa^\dagger)/2\right)\right] + \mathcal{D}\left[a^\dagger \exp\left(-x(n_s + aa^\dagger)/2\right)\right] \right\} \rho. \tag{C.21}$$

This shows how the inverse superoperator in Eq. (C.18) is defined. The proof can be demonstrated easily in the Fock basis, in which

$$\dot{\rho}_{n,m} = G n_s \left(\frac{\sqrt{nm}}{n_s + \frac{1}{2}(n+m)} \rho_{n-1,m-1} - \frac{\frac{1}{2}(n+m+2)}{n_s + \frac{1}{2}(n+m+2)} \rho_{n,m} \right). \tag{C.22}$$

Eq. (C.18) describes incoherent excitation. In the limit $\bar{n} \ll n_s$ (where $\bar{n}$ is a characteristic photon number in the cavity), it reduces to thermal excitation

$$\dot{\rho} = G \mathcal{D}[a^\dagger]\rho. \tag{C.23}$$

Here the rate of addition of photons is proportional to the number of photons already present (plus one). In the other limit ($n_s \ll \bar{n}$), the rate of addition of photons is independent of the cavity state.

This limit is what I refer to as the ideal laser limit, in which the pump is Poissonian. Then the effective master equation is

$$\dot{\rho} = \kappa\mu\mathcal{E}[a^\dagger]\rho, \tag{C.24}$$

where $\kappa\mu = Gn_s$ and I have defined a new, incoherent Poissonian excitation superoperator

$$\mathcal{E}[a^\dagger]\rho = \mathcal{J}[a^\dagger]\left(\mathcal{A}[a^\dagger]\right)^{-1} - 1. \tag{C.25}$$

The inverse superoperator here is well defined because $aa^\dagger$ has eigenvalues bounded below by 1. The justification for defining $\mu = Gn_s/\kappa$ is that this is the stationary mean photon number when damping is reintroduced into the full master equation

$$\kappa^{-1}\dot{\rho} = \left(\mu\mathcal{E}[a^\dagger] + \mathcal{D}[a]\right)\rho. \tag{C.26}$$

The criterion for this to be a valid description of a laser near steady state is $\mu \gg n_s$.

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