An optimally convergent discontinuous-Galerkin-based extended finite element method for fracture mechanics

Yongxing Shen and Adrian Lew*

Department of Mechanical Engineering, Stanford University

SUMMARY

The extended finite element method (XFEM) enables the representation of cracks in arbitrary locations of a mesh. We introduce here a variant of the XFEM rendering an optimally convergent scheme. Its distinguishing features are: a) the introduction of singular asymptotic crack tip fields with support on only a small region around the crack tip, the enrichment region, b) only one and two enrichment functions are added for anti-plane shear and planar problems, respectively, and c) the relaxation of the continuity between the enrichment region and the rest of the domain, and the adoption of a discontinuous Galerkin (DG) method therein. The method is provably stable for any positive value of a stabilization parameter, and by weakly enforcing the continuity between the two regions it eliminates “blending elements” partly responsible for the suboptimal convergence of some early XFEMs. Moreover, the particular choice of enrichment functions results in a surprisingly sparse stiffness matrix that remains reasonably conditioned as the mesh is refined. More importantly, the stress intensity factors can be extracted with a satisfactory accuracy as primary unknowns. Quadrature strategies needed for the optimal convergence are also discussed. Finally, the DG method was modified to retain stability based on an inf-sup condition. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: solids, extended finite element method, discontinuous Galerkin, fracture, lifting operator, immersed boundary method

*Correspondence to: Mechanics and Computation, 496 Lomita Mall, Durand Building, Stanford University, Stanford, California 94305-4040, U. S. A.

Contract/grant sponsor: National Institutes of Health through the NIH Roadmap for Medical Research; contract/grant number: U54GM072970

Contract/grant sponsor: Department of the Army Research Grant; contract/grant number: W911NF-07-2-0027

Contract/grant sponsor: NSF Career Award; contract/grant number: CMMI-0747089

Contract/grant sponsor: ONR Young Investigator Award; contract/grant number: N000140810852

Received 4 July 2009
1. INTRODUCTION

Fracture mechanics plays an essential role in failure analysis of engineering structures. Accurately and efficiently computing the stress field is crucial for predicting the evolution of cracks. In fracture mechanics problems, cracks are usually idealized as geometries of one less dimension than that of the embedding solid, i.e., submanifolds of codimension 1. This idealization gives rise to a stress singularity of the type $1/\sqrt{r}$ for isotropic linear elastic materials, where $r$ is the distance to the crack tip, and leads to suboptimal convergence rates if standard finite element methods (FEMs) with piecewise polynomial spaces are employed.

There are virtually two families of strategies to achieve better convergence behavior: adaptive mesh refinement and extension of the finite element (FE) space with the a priori known singular functions. This introduction will focus on developments along the second family of the aforementioned methods.

With minimal alteration to existing FE programs, Henshell and Shaw [1] and Barsoum [2] independently proposed using isoparametric 8-noded quadrilateral elements while placing some of the mid-side nodes at the quarter positions so as to produce the $\sqrt{r}$ behavior of the displacement field in the physical domain. The quarter-point elements are straightforward to implement. However, the singularity they introduced in the FE space does not capture the angular dependence of the strain field near the crack tip. Consequently, even though convergence is still attained and the accuracy of the solution is enhanced, there is no improvement in the convergence rate with respect to FE discretizations with standard elements; it remains suboptimal.

In 1973, Fix et al.[3] introduced singular functions in combination with standard piecewise polynomial functions for solving Poisson and Helmholtz equations over a cracked domain. The singular functions are continuous and have compact supports centered at the singularity (crack tip). However, they are non-local, which means that their support spans many elements in the mesh. In fact, the support of the singular functions is the same regardless of the mesh size. By doing so, they obtained an optimal convergence rate and accurate approximations to the stress intensity factors (SIFs).

More recently Belytschko and Black[4] and Moës et al.[5] introduced the extended finite element method (XFEM), in which they adopted a partition of unity (PU) technique [6, 7] to incorporate the singularities into the FE basis. Its key advantage is that it requires minimal or no remeshing as the crack propagates. In this context a typical enrichment shape function is the product of a singular function and a standard, continuous, piecewise polynomial shape function. In the early developments of the XFEM, only elements adjacent to (or very close to) the crack tip were enriched with the singular functions.

Since the introduction of the XFEM applied to fracture mechanics problems, related literature grew rapidly, and numerous variants of the XFEM have been proposed. In 2003, Chessa et al.[8] observed the suboptimal convergence rate of the then standard XFEMs, and attributed this phenomenon to the presence of blending elements, i.e., enriched elements bordering standard elements. Later Laborde et al.[9] and Béchet et al.[10] showed that if the size of the enrichment zone shrinks around the crack as $h \to 0$ then the convergence rate will be suboptimal. Hence, they proposed enriching elements over a fixed area independent of the mesh.

A number of strategies has been proposed to overcome the suboptimal convergence rate due
to the presence of blending elements. The authors in [8] proposed two methods for removing this artifact: a) adding a bubble quadratic function to the blending elements and b) using the assumed strain method. Another method that still adopted a PU to include the singular enrichment and corrected the suboptimal convergence rate due to the blending elements was proposed by Fries [11]. The author multiplied each singular enrichment basis function by a cutoff function that increases from zero to one in a single ring of elements, the transition layer. While normally this product would have very large derivatives in the transition layer, the particular choices of enrichment functions have small derivatives therein, leading to optimal convergence rate in the numerical examples.

Generalizing the original ideas in [3], Chahine et al.[12, 13] overcame the problem of blending elements by multiplying the singular enrichment functions by a smooth cutoff function independent of the mesh size. The authors also proved the optimal convergence rate of this method.

In [9] the authors framed the problem of the blending elements as one of approximating a function in two overlapping domains, the domains with and without enrichment functions, through a PU approach. It follows from their analysis that the convergence rate should not be affected as the width of the overlapping region, or transition layer, decreases to zero. Consequently, in this limit this type of method would require the approximations on both sides of the interface between the domains with and without enrichment functions to be essentially continuous, i.e., discontinuities are allowed but these should become smaller and smaller as the mesh is refined. The authors then proposed to introduce a basis for the singular enrichment functions without the PU technique, which they labeled “degree of freedom gathering,” and then match the nodal values of the numerical solutions along the interface between the domains with and without enrichment functions.

An idea that is strongly related to those in the previous paragraph is to include the singular enrichment functions on a region that contains the crack tip, and set them to zero outside this region. The resulting enrichment functions are discontinuous at the interface between the two regions, and hence a discontinuous Galerkin (DG) approximation can be adopted therein to weakly enforce continuity. This was proposed in [14], where the authors combined the PU-based XFEM and the interior penalty DG methods in [15] and [16]. One drawback of this approach is that stabilization is required, and the stabilization parameter is problem dependent. A few iterations are needed to find a suitable value.

In this paper we propose a new DG method to connect solutions on the domains with and without enrichment functions. The method is based on the DG methods for elastic problems in [17] and [18], which are constructed by adopting the Bassi-Rebay numerical fluxes. An important feature of this class of methods is that for isotropic, unstressed, linear elastic problems the stabilization term is problem independent. In fact, it is enough for the stabilization parameter to be any positive real number.

Another important feature of the method is that only the two asymptotic displacement fields near the crack tip for modes I and II are adopted as enrichment functions. This enables us to compute the SIF for each mode simply as the coefficient of the enrichment basis function in the numerical solution. We show the convergence of these coefficients to their exact values through numerical examples. While the rate of convergence of the SIFs computed in this way is lower than those computed through the interaction integral [5], the resulting values are surprisingly accurate and require essentially no extra computational cost.

To retain an optimal order of convergence, we set the domain over which the singular
enrichment functions are not zero to be the set of all elements that intersect a circle whose radius is independent of $h$. Consequently, the number of elements over which the enrichment functions are different than zero scales as $h^{-2}$. For arbitrary enrichment functions this would mean that the number of nonzero entries in the stiffness matrix relating the enrichment functions with the polynomial basis functions scales as $h^{-2}$ as well. However, because the enrichment basis functions are in mechanical equilibrium, we show that most of these entries are identically zero. In fact, the number of such non-zero entries scales as $h^{-1}$, which renders the stiffness matrix surprisingly sparse. Only a polynomial basis function whose support includes either the boundary of the enrichment region or a portion of the face of a curved crack would couple to the enrichment functions through a non-zero pair of entries in the stiffness matrix.

We have also investigated the minimum requirement of the numerical integration scheme that preserves the optimal convergence property. In fact, we will show that fixed-order domain integration rules for elements near the crack tip have a deleterious effect on the convergence property of an optimally convergent method relying on exact integrations. We then suggest using a boundary integral scheme proposed by [19] for the integrals with unbounded integrands so as to preserve the optimal convergence. In our numerical examples this detrimental effect is only significant in the errors of displacements or stresses for very refined meshes; however, it is much more pronounced when the SIF is obtained as the coefficient of the enrichment functions.

We demonstrate the optimal convergence properties of the method with numerical examples. We compare the results against those obtained by adopting the cutoff function approach [13], since the authors proved that this method attains an optimal convergence rate. For the same mesh, the errors in stresses, displacements and SIFs are substantially lower for the method proposed herein. These differences are not expected to be large if compared with other methods that set the enrichment functions to zero sharply, such as the one by Fries [11] or Laborde et al.[9], but we did not perform such comparison.

A special feature of the DG method introduced here is that, due to the presence of the enrichment functions, the traces of the approximating functions on the two sides of a discontinuity are not necessarily polynomials. To guarantee the stability of the resulting scheme it is then necessary to satisfy an inf-sup condition, which we prove elsewhere [20] to be always satisfied for planar crack problems provided with minor restrictions on the mesh. This observation is backed by the numerical examples shown here. Because of the enrichments, the space of approximations of the derivatives need to be enriched as well in order for the inf-sup condition to be satisfied. While it would be natural to include the derivatives of the singular enrichment functions, this leads to a more computationally expensive method and the inf-sup condition is not easy to prove. Instead, we have found that it is enough to enrich the space of approximation of derivatives with higher order polynomials only on those elements bordering the set where discontinuities may appear. This results in a method that is efficient, accurate and stable.

Finally, in Section 2.2 we construct a connection between immersed boundary methods (see, e.g., [21]) and the XFEM by representing the crack path as an immersed boundary rather than an additional enrichment. This perspective is similar but not identical to those introduced by Molino et al. [22], Song et al. [23] and Hansbo and Hansbo [24]. This point of view simplifies the construction of FE spaces over cracked domains, such as spaces of arbitrary polynomial order or with different degrees of continuity across element boundaries (DG, conforming, Raviart-Thomas elements, etc.)
2. Methodology

2.1. Problem statement

We consider the two-dimensional linear elastostatic problem over a bounded open polygonal domain $\Omega \in \mathbb{R}^2$. It reads find $u : \Omega \rightarrow \mathbb{R}^2$ such that

$$
- \nabla \cdot (C : \nabla u) = f, \quad \text{in } \Omega,
$$

$$
u = U, \quad \text{on } \partial_d \Omega,
$$

$$
(C : \nabla u) \cdot n = T, \quad \text{on } \partial_t \Omega,
$$

where $f : \Omega \rightarrow \mathbb{R}^2$ is the body force, $n$ is the exterior unit normal to $\partial \Omega$, $\partial_d \Omega \cup \partial_t \Omega = \partial \Omega$, $\partial_d \Omega \cap \partial_t \Omega = \emptyset$, $T : \partial \Omega \times \mathbb{R}^2$ are the imposed tractions and $U : \partial_d \Omega \rightarrow \mathbb{R}^2$ are the imposed displacements. The elastic moduli $C$ are those of an initially stress-free isotropic material.

In particular, we are interested in problems in which $\Omega$ is a domain with cracks. Each crack is assumed to be a smooth one-dimensional curve. Additionally, we assume that there exists an open neighborhood of each crack such that a) the open neighborhoods of two different cracks do not intersect and, b) once the open neighborhood of each crack or any of its subsets has been removed from $\Omega$, a connected domain remains. These assumptions preclude the important possibilities of intersecting cracks, such as a bifurcation, and of crack kinks. While some of these configurations do not need further modifications of the method to retain an optimal convergence rate, some of them do. We therefore restrict our attention to a less general setting, so as to convey the key new ideas in the simplest terms.

We shall make the standard assumption that crack faces are traction free, i.e., $T = 0$ on $\partial_c \Omega \subseteq \partial \Omega$, where $\partial_c \Omega$ is the set of crack faces. In this way we rule out contact and friction between crack faces.

We assume that $C$ is constant throughout the domain, or at least, in the open neighborhood around each crack tip. Generalizations that retain an optimal rate of convergence to smoothly spatially varying moduli, or to anisotropic materials, are possible provided an asymptotic displacement field near each crack tip is known under those circumstances, and of course, some regularity conditions for the solution are met.

For simplicity, we shall explain the method assuming that there is only one crack tip whose position is denoted by $x_t$, which means we consider an edge crack. The generalization to multiple crack tips is trivial.

Following [23] but with a different sign convention, we assume that there exist smooth scalar functions $\phi$ and $g$ over $\Omega$ such that the crack is contained in the level set $\phi = 0$, and along the crack $g$ is strictly monotonic with $g(x_t) = 0$. The crack is then defined as $\partial_c \Omega = \{x \in \overline{\Omega} : \phi(x) = 0, g(x) < 0\}$, see Figure 2(a). By construction, $x_t \notin \partial_c \Omega$.

2.2. Crack Paths as Immersed Boundaries

Adopting the philosophy of the XFEM, we construct a family of quasi-uniform triangulations $\{\mathcal{T}_h\}_h$ of $\overline{\Omega}$ parametrized by the mesh size $h > 0$. We do so without taking into account the locations of the cracks. Therefore we allow for the possibility of the crack cutting through elements. We assume that any node in mesh $\mathcal{T}_h$ either belongs to the crack or is a distance of $\epsilon h$ away from any point on the crack for some $\epsilon > 0$ independent of $h$. This condition prevents the crack from cutting elements into arbitrarily small parts, which would render
Figure 1. Example of an immersed boundary over a domain with no cracks. The domain $\Omega$ is immersed in a mesh that contains it, in this case the mesh of triangles over the square. Elements intersected by the domain are used to approximate functions inside the domain. These elements are shown in solid lines. In particular, the functions in the finite element space over these elements have a domain larger than $\Omega$, but only their restrictions to $\Omega$ are used.

poorly conditioned stiffness matrices. Additionally, we henceforth assume that elements are open subsets of $\Omega$, and that the intersection of the crack $\partial_c \Omega$ and any element $K$ can only be either an empty set or a connected curve, for any $K$ in $\{T_h\}_h$. In other words, if any element $K$ is cut by the crack at all, it can only be cut into at most two parts. A similar but not identical assumption was made in [24], because they followed a convention in which elements are treated as closed sets.

In this paper we adopt the distinct perspective of cracks as immersed boundaries. This perspective is of course equivalent to regarding them as jump discontinuities in the deformation mapping, as in the XFEM. However, it leads to a simple and elegant implementation of the method. Furthermore, this perspective simplifies the construction of enriched FE spaces in cracked domains with a variety of elements, not only of the conforming $P_1$ type. Similar albeit not identical ideas have been introduced by Molino et al. [22], Song et al. [23] and Hansbo and Hansbo [24].

In immersed boundary methods the domain $\Omega$ is overlapped over a mesh that does not fit the boundary, see e.g. [21]. In this way, $\partial \Omega$ cuts through elements in the mesh, see Figure 1 for an example over a domain with no cracks. A FE space $V_h$ can be easily constructed over all elements intersected by $\Omega$. However, only the restriction of functions in $V_h$ to $\Omega$ is important for the approximate solution.

A crack is clearly part of the boundary of $\Omega$. However, in contrast to the example in Figure 1, when immersing $\Omega$ into $T_h$ some elements intersect $\Omega$ on both sides of the crack. Consequently, if the basis functions within the element are continuous, then the kinematics on the two sides of the crack are artificially connected. This artifact can be naturally avoided with a simple modification of $T_h$ that involves duplicating some nodes and elements to construct a new mesh $T'_h$.

The basic idea can be explained by splitting the mesh along a curve that contains the crack into two disconnected ones, and then “gluing” them back together along the non-cracked part of this curve, see Figure 2. This separation into two steps is not needed for the final algorithm,
but is introduced to facilitate the explanation.

As aforementioned, the mesh over \( \Omega \) is first split into two disjoint parts along the curve \( \Gamma = \{ x \in \Omega : \phi(x) = 0 \} \), see Figure 2(b). Second, some elements and nodes near the crack are duplicated, effectively replacing \( T_h \) by two disconnected meshes, each one able to have one of the two parts of \( \Omega \) immersed in it, as in Figure 2(c). The duplicated elements are all those intersected by \( \Gamma \), and the duplicated nodes are all those that either belong to these elements or belong to \( \Gamma \). Of course, this duplication results in pairs of elements and pairs of nodes, one in each mesh, that occupy the same spatial location. This enables the kinematics of each one of the two parts of \( \Omega \) to be described with FE spaces built on each one of the two disconnected meshes. To facilitate the construction of FE spaces that render continuous displacements along the non-cracked part of the curve \( \Gamma \), we join the two disconnected meshes into a single one, \( T'_h \), as a third step. The idea is to remove from this union all elements and nodes that may result in bi-valued FE spaces over the non-cracked part of \( \Gamma \). Consequently, we replace each such pair of elements or nodes by the node or element in \( T_h \) from which the two copies were made. Alternatively, such elements and nodes in \( T_h \) do not need to be duplicated in the first place, see Figure 2(d). In fact,

Element \( K \) is duplicated \( \Leftrightarrow \) \( K \setminus \partial_{\text{cr}} \Omega \) is not connected,

Node \( a \) is duplicated \( \Leftrightarrow \) \( \text{cloud}(a) \setminus \partial_{\text{cr}} \Omega \) is not connected,

where \( \text{cloud}(a) \) is defined as the interior of the closure of the domain occupied by all elements in \( T_h \) that have \( a \) as a node \(^1\).

To provide a precise algorithmic description of these ideas, we define \( \Omega_{\pm} = \{ x \in \Omega : \pm \phi(x) > 0 \} \) to be the two open subsets in which \( \Omega \) is split. Additionally, a mesh \( T_h = (\eta, T) \) is described by a map \( \eta : \{1, \ldots, N_{\text{nodes}}\} \to \mathbb{R}^2 \), the set of nodes, and a set \( T \) of \( N \) \( m \)-tuples in \( \{1, \ldots, N_{\text{nodes}}\}^m \), the connectivity table. Here \( N \) and \( N_{\text{nodes}} \) are the total number of elements and nodes in \( T_h \), respectively, and \( m \) is equal to 3 for triangles and to 4 for quadrilaterals. We denote the \( m \)-tuple that contains the connectivity of element \( K \) with \( K \in T \), and use the symbol \( K \) as well for the open set it occupies. Under this definition, a mesh could have nodes and elements that occupy the same spatial location.

Algorithm 1 precisely describes the map \( T_h \mapsto T'_h \) in the case of an edge crack for triangular and quadrilateral elements. The cases in which both internal cracks and edge cracks are present and in which other types of polygonal elements are used are similarly handled.

The new mesh \( T'_h \) also has elements that occupy the same location in space. However, each one of the two copies can be adopted to describe the kinematics of only one side of the crack, effectively treating each crack face as a regular immersed boundary. The results of Algorithm 1 may seem somewhat odd for coarse meshes, since the same node may belong to two or more elements intersecting “distant” parts of the crack. However, such distant parts, belonging to the cloud of the same node, can be shown to be within \( 2h \) away from each other and hence restraining the deformation of these disjoint parts does not affect the convergence property of the method.

\(^1\)The cloud of a node \( a \) is also the interior of the support of the standard piecewise linear shape function associated with node \( a \).
Figure 2. (a) A crack defined by two implicit functions as \( \{ x \in \Omega : \phi(x) = 0, g(x) < 0 \} \). (b) A mesh of \( \Omega \) not taking into account the crack configuration. (c) The domain \( \Omega \) is cut into two subdomains by the crack and its extension (dotted curve). Each subdomain is immersed into a submesh. Each submesh serves to represent the deformation of the corresponding subdomain. To obtain a continuous FE approximation of the displacements over the entire domain, and across the dotted curve in particular, some elements appearing in both submeshes but occupying the same physical spaces have to be merged into a single one. These pairs of elements are labeled \( K_{i+} \) and \( K_{i-} \), \( i = 1, 2 \). (d) A kinematically admissible deformed configuration as a result of the merging. The kinematics of the discontinuity inside the element that contains the crack tip will then be obtained by enriching the finite element space with the asymptotic crack tip fields.

Figure 3. Sketch of the definition of \( *K \) on one of the elements intersected by the crack in the example in Figure 2(a). Element \( K \) is originally in \( T_h \), and its duplicate \( K' \) is generated when constructing \( T'_h \). Elements \( K \) and \( K' \) are adopted to approximate the kinematics of the regions \( *K \) and \( *K' \), respectively.
Algorithm 1 Construction of $T_h'$ from $T_h$

Input: $T_h = (\eta, T)$
Output: $T_h' = (\eta', T')$

Duplicate all nodes
Define $\eta'$: $\{1, \ldots, 2N_{\text{nodes}}\} \to \mathbb{R}^2$ as

\[
\eta'(i) = \begin{cases} 
\eta(i) & \text{if } i \leq N_{\text{nodes}} \\
\eta(i - N_{\text{nodes}}) & \text{otherwise} 
\end{cases} 
\]  

(2.2)

Duplicate elements and modify connectivities
Set $T^{\text{crack}} = \{\hat{K} \in T : \hat{K} \cap \partial_{\Omega} \neq \emptyset\}$ \{Elements that may need modifications\}
Set $A^{\text{dup}} = \{a \in \eta: \text{cloud}(a) \setminus \partial_{\Omega} \text{is not connected}\}$ \{Nodes to be duplicated\}
Set $T' = T$
Set $T^{\pm} = \{\hat{K} \in T : \hat{K} \cap \Omega^{\pm} \neq \emptyset\}$
for all $\hat{K} \in T^{\text{crack}}$ do
  if $\hat{K} \setminus \partial_{\Omega}$ is not connected then \{Duplicate the element\}
    Set $\hat{K}' = \hat{K}$
    for all $a \in \hat{K}$ do \{Change connectivity\}
      if $\phi(\eta(a)) < 0$ and $a \in A^{\text{dup}}$ then
        Replace $a$ by $a + N_{\text{nodes}}$ in $\hat{K}$
      end if
      if $\phi(\eta(a)) \geq 0$ and $a \in A^{\text{dup}}$ then
        Replace $a$ by $a + N_{\text{nodes}}$ in $\hat{K}'$
      end if
    end for
  end if
end for
Set $T' = T' \cup \hat{K}'$ \{Add duplicated element\}
else \{Do not duplicate the element\}
  for all $a \in \hat{K}$ do \{Change connectivity\}
    if $\hat{K} \in T^+$ and $\phi(\eta(a)) < 0$ and $a \in A^{\text{dup}}$ then
      Replace $a$ by $a + N_{\text{nodes}}$ in $\hat{K}$
    end if
    if $\hat{K} \in T^-$ and $\phi(\eta(a)) \geq 0$ and $a \in A^{\text{dup}}$ then
      Replace $a$ by $a + N_{\text{nodes}}$ in $\hat{K}$
    end if
  end for
end if
end for
Set $T_h' = (\eta', T')$
Once $T'_h$ is defined, a FE space $V_h$ can be defined over it such that

$$V_h \subseteq \prod_{K \in T'_h} V^K_h = \{ v = (v_{K_1}, \ldots, v_{K_{N'}}) : v_K \in V^K_h \text{ for all } K \in T'_h \} \tag{2.3}$$

where $V^K_h$ is the FE space over element $K$ and $N'$ is the total number of elements in $T'_h$. For example, a standard choice would be a space that contains functions that are continuous across elements that share a common edge, i.e., elements that have two nodes in common. In this case,

$$V^\text{conf}_h = \left\{ v \in \prod_{K \in T'_h} V^K_h : v_K|_e = v_{K'}|_e \text{ if } K, K' \text{ share edge } e, \text{ for any } e \right\}. \tag{2.4}$$

Notice that $V_h$ may contain functions that are bi-valued over some regions of the domain. In particular, these will be bi-valued over the domain of all elements $K \in T_h$ that have been duplicated when constructing $T'_h$. To specify which of these two values shall be used to approximate the solution of the problem, we define

$$^*K = \begin{cases} K \setminus \partial_e \Omega & \text{if } K \text{ was not duplicated} \\ K \cap \Omega^\pm & \text{if } K \text{ was duplicated and} \\ \text{it shares a node with an element } \tilde{K} \in T'_h \text{ such that } \tilde{K} \subset \Omega^\pm, \end{cases} \tag{2.5}$$

for any element $K \in T'_h$, see Figure 3 for a sketch. Each $v \in V_h$ then defines a function $^*v$ over $\Omega$ (up to a set of measure zero) such that

$$^*v|^{*K} = v|^{*K}$$

for all $K \in T'_h$. The resulting function $^*v$ is single-valued almost everywhere in $\Omega$. Roughly speaking, $^*v$ is obtained from $v$ by restricting its values to the “half” of the element that does not contain duplicated nodes, for all those elements that are completely cut by the crack. Approximation of solutions will then be effectively sought in the space $^*V_h = \{ ^*v : v \in V_h \}$. This is equivalent to, and a generalization of, restricting to $\Omega$ the functions in the FE space in the context of immersed boundary methods.

We conclude this description with a few remarks. First, in $T'_h$ there may be three elements sharing a single edge, since two of these elements may occupy the same spatial location. Second, if the crack lies only along edges of $T_h$ and the crack tip coincides with a vertex, then $T'_h$ is simply a mesh that conforms to the boundary, as traditionally done. Third, as mentioned earlier, the construction of $T'_h$ makes adopting nonstandard FE spaces within each element simple. Finally, a similar algorithm should work in three-dimensional cracked domains.

In the following we specify $V^K_h$ and adopt a different $V_h$, in which we introduce some discontinuities in $V^\text{conf}_h$ as part of the formulation of the numerical method.

### 2.3. Finite Element Spaces with Enrichment

Standard choices for $V_h$, such as $V^\text{conf}_h$ with $V^K_h = [P_k(K)]^2$, $k \in \mathbb{N}$, for all $K \in T'_h$, cannot approximate the singularity in the derivative near the crack tip at an optimal rate in $h$ (in the $H^1$-seminorm). In fact, it is generally not better than $h^{1/2}$.
One standard way to overcome this is by enriching $V_h$ with the asymptotic crack-tip solutions \cite{25}. For a homogeneous isotropic material these solutions read

\begin{align}
\Psi_I &= \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \left(\frac{\theta}{2}\right) (\kappa - \cos \theta) e_1' + \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \left(\frac{\theta}{2}\right) (\kappa - \cos \theta) e_2', \\
\Psi_{II} &= \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \left(\frac{\theta}{2}\right) (\kappa + 2 + \cos \theta) e_1' + \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \left(\frac{\theta}{2}\right) (2 - \kappa - \cos \theta) e_2',
\end{align}

where $\mu$ is the shear modulus, and $\kappa$ is computed from Poisson's ratio $\nu$ as $3 - 4\nu$ for plane strain loading and $(3 - \nu)/(1 + \nu)$ for plane stress loading. The right-handed orthonormal basis $\{e_1', e_2'\}$ is such that $e_1'$ is the tangential direction of the crack at $x_t$ pointing away from the crack (see Figure 4). Polar coordinates $r$ and $\theta$ are the standard crack-tip coordinates in which $r$ is the distance from the crack tip and $\theta = 0$ corresponds to the $e_1'$ direction. In the case of a straight crack, $\theta$ ranges from $-\pi$ to $\pi$. For curved cracks, the range of $\theta$ changes with the value of $r$ in a small enough neighborhood of the crack tip, see \cite{26}.

In the scalar case of anti-plane shear crack problems, the enrichment would be constructed out of the mode III asymptotic solution, which reads

$$\Psi_{III} = \frac{1}{\mu} \sqrt{\frac{2\pi}{r}} \sin \left(\frac{\theta}{2}\right).$$

The generalization of the method for planar problems to this scalar-field problem is obvious. Hence, in the following we will only discuss the method as applied to planar crack problems.

The enriched FE space is then constructed as

$$\mathcal{V}_h = *V_h + \rho_h \text{ span } \{\Psi_I, \Psi_{II}\}.$$
Finally, in the context of this paper, the term “enrichment” only refers to the set of functions in $\rho_h \text{ span } \{\Psi_I, \Psi_{II}\}$. This is in contrast to PU-based methods, in which the same term is adopted for the space of functions generated by the multiplication of a generalized Heaviside function as well as the asymptotic behavior at the crack tip by several basis functions of the space, e.g. [5].

2.4. Specifying the enrichment region

We discuss next some possible choices for $\rho_h$ that have been adopted earlier, and then introduce our choice. Clearly the simplest choice is $\rho_h(x) = 1$ for all $x \in \Omega$. However, this choice makes the basis functions for the enrichment to have support in the entire domain. It therefore reduces the sparsity of the resulting stiffness matrix† and complicates the imposition of boundary conditions. Clearly this is unnecessary, since only the approximation properties near the crack tip are important. A more efficient approach is to choose $\rho_h$ with support on a small neighborhood of the crack tip. As shown by Laborde et al.[9] and Béchet et al.[10], a necessary condition to obtain an optimal convergence rate is that there exists a neighborhood of the crack tip $\Omega_E \subset \Omega$ open in $\mathbb{R}^2$ that is contained in the support of $\rho_h$ for all $h$. In particular, this implies that to retain an optimal convergence rate the support of $\rho_h$ cannot scale with $h$. Consequently, it is not possible to restrict the enrichment to only elements that either contain the crack tip or share an edge with them, for example.

The cutoff function method proposed in [3] and generalized in [12, 13] sets $\rho_h$ as a $C^2$ function of $r$ only, equal to 0 for $r < r_0$, equal to 1 for $r > r_1 > r_0$, and equal to a fifth-order polynomial in $r$ for $r_0 \leq r \leq r_1$. The performance of this approach is very good when $h$ is somewhat smaller than $r_1 - r_0$. This is evidenced by the numerical examples in [13].

Herein we set
\[
\rho_h(x) = \chi_{\Omega_E}^h(x) = \begin{cases} 1 & \text{if } x \in \Omega_E^h \\ 0 & \text{otherwise}, \end{cases}
\] (2.11)
where
\[
\Omega_E^h \equiv \{ x \in K : K \in T_h', |K \cap \Omega_E| > 0 \},
\] (2.12)
is the “enriched” region of the domain, since the enrichment functions are different than zero in all elements therein. For the purpose of definition (2.12), we recall that we assumed that an element $K \in T_h$ is an open set in $\mathbb{R}^2$. Clearly this choice introduces discontinuous functions in $V_h$, with discontinuities along the set $\Gamma_E^h \equiv \partial \Omega_E^h \setminus \partial \Omega$, which we term as the enrichment boundary. For later use, we also denote the “unenriched” region by
\[
\Omega_U^h \equiv \Omega \setminus \Omega_E^h.
\] (2.13)

To obtain an optimally convergent approximation, it should be possible to decrease the size of the discontinuities along $\Gamma_E^h$ at an appropriate rate with $h$, since the exact solution is continuous. Consequently, the space $V_h$ for each $h$ needs to have a “rich enough” set
of discontinuities along $\Gamma_h^F$. The approximate solution can then be obtained through a DG formulation. One choice such of $V_h$, and the one we adopt here, is

$$V_h^{DG} = \{ v \in \prod_{K \in T'_h} V^K_h : v_K|_e = v_{K'}|_e \text{ if } K, K' \text{ share edge } e, \text{ for any } e \not\subset \partial \Omega_h^F \}. \quad (2.14)$$

The definition of $\mathcal{V}_h$ follows from (2.10), i.e.,

$$\mathcal{V}_h = *V_h^{DG} + \rho_h \text{ span}\{\Psi_I, \Psi_{II}\}, \quad (2.15)$$

and its description for the forthcoming discussion and numerical examples is completed by specifying $V_h^h = [P_1(K)]^2$, where $K$ is a triangle, for all $K \in T'_h$. Under these conditions, any function $u_h \in V_h^{DG}$ can be uniquely expressed as

$$u_h = \chi_{\Omega_h^K} \sum_{a \in S^U} \sum_{i=1}^2 \tilde{\psi}_a u_{ai}^E e_i + \chi_{\Omega_h^E} \left( \sum_{a \in S^E} \sum_{i=1}^2 \tilde{\psi}_a u_{ai}^E e_i + K_I \Psi_I + K_{II} \Psi_{II} \right), \quad (2.16)$$

where $K_I, K_{II}, u_{ai}^E$ are scalars, $\chi_B$ denotes the characteristic function of domain $B$, and $S^E$ and $S^U$ denote the sets of nodes in elements included in $\Omega_h^K$ and $\Omega_h^E$, respectively. Function $\tilde{\psi}_a$ is such that $\tilde{\psi}_a|_K$ is a $P_1$ function for each $K \in T'_h$, continuous across elements that share an edge, and equal to 1 at $a$ and zero on every other node in $T'_h$. By construction, $u_h$ may be discontinuous across $\Gamma_h^E$, and possibly bi-valued on elements and edges that intersect the crack $\partial_c \Omega$. In contrast, $*u_h$ is single-valued therein, and possibly discontinuous across $\partial_c \Omega$.

A concise description of the basis functions for $*V_h^{DG}$ is obtained by defining

$$\psi_{\alpha a} = \left( *\tilde{\psi}_a \right) \chi_{\Omega_h^K}, \quad (2.17)$$

for any node $a \in S^\alpha$, with $\alpha = E, U$. It follows from (2.16) that a set of basis functions for $*V_h^{DG}$ is constructed as

$$\{ \psi_{\alpha a} e_i : a \in S^\alpha, \alpha = U, E, i = 1, 2 \}. \quad (2.18)$$

These new basis functions are essentially identical to the original ones. If node $a$ does not lie on the set where discontinuities may appear, i.e. $a \not\subset \Gamma_h^E$, and $a$ does not belong to a duplicated element [for which $*K \neq (K \setminus \partial_c \Omega]$] then $\psi_{\alpha a} = \psi_a$. In contrast, if $a$ belongs to some duplicated element, then $\psi_{\alpha a}$ will simply be the restriction of $\psi_a$ to “one side of the crack.” Additionally, if $a \subset \Gamma_h^E$, then the function $\psi_a$ is split as $\psi_a = \psi_{E a} + \psi_{U a}$, so that each one of them has support only on one side of the discontinuity. Consequently, two additional degrees of freedom (DOFs) are added for each node $a \subset \Gamma_h^E$ because of the discontinuity.

Choice of $\Omega_h^E$. As $\Omega_h^E$ we choose the intersection of $\overline{\Omega}$ with an open disk centered at the crack tip with radius $r_{\text{cutoff}} > 0$. We assume that $r_{\text{cutoff}}$ is such that $\Omega_h^E \cap \partial_c \Omega$ is a connected curve for any mesh $T_h$. This condition prevents the crack from reentering the enrichment region. Since the hypotheses in Section 2 prevent the crack from intersecting itself, it is always possible to choose $r_{\text{cutoff}}$ such that for $h$ small enough a ball of radius $r_{\text{cutoff}} + h$ centered at the crack tip (which contains $\Omega_h^E$) intersects the crack in a connected curve only. Furthermore, we assume that $T_h$ is such that $\partial_c \Omega \cap \Gamma_h^E$ contains at most one point. This is clearly not satisfied for all meshes, since the crack may overlap with (parts of) edges in $\Gamma_h^E$. However, it can be accomplished by slightly perturbing the position of only a few nodes provided $h$ is small enough. This last assumption is not essential, but it greatly simplifies the implementation of the method. An example of $\Omega_h^E$ and the resulting $\Omega_h^F$ are shown in Figure 5.
Figure 5. Specification of the enrichment region $\Omega^E$ using a circle. Elements in dark gray are enriched with the near-tip asymptotic displacement fields. Their union is $\Omega^E_h$. Discontinuities in the displacement field are introduced across the boundary of $\Omega^E_h$, in this case also $\Gamma^E_h$, to cutoff the enrichment fields while retaining an optimal convergence rate. A DG formulation is adopted along the elements with an edge along $\Gamma^E_h$, which includes the elements shown in light gray outside $\Omega^E_h$. Some elements may have more than one of its edges across which discontinuities are allowed, as the one shown with a hollow square. Other elements may be both intersected by the crack and have an edge intersected by $\Gamma^E_h$. In this example these are shown with a black circle. These elements come in pairs, one for the kinematics on each side of the crack. In one of these pairs only one of the two elements shares an edge with $\Gamma^E_h$.

2.5. Discontinuous Galerkin method for the enrichment boundary

The approximation of continuous solutions with possibly-discontinuous functions needs the craft of special numerical methods, such as one of the many variants of DG methods. The reader is referred to Arnold et al.[27] for a unified analysis of many variants of DG methods. The method proposed herein originates in the one constructed for linear elastic problems in [17], and for the Poisson equation in [28].

We first define some notation. Let $n$ be the outward unit normal to $\overline{\Omega^E_h}$, i.e., pointing towards $\Omega^U_h$. For a scalar, vector or tensor field $v$ over $\Omega$ we denote with $v^E$ and $v^U$ its traces on $\Gamma^E_h$ from $\Omega^E_h$ and $\Omega^U_h$, respectively. We also define the jump and average operators on $\Gamma^E_h$ as

$$\llbracket v \rrbracket \equiv v^E - v^U, \quad \{ v \} \equiv \frac{1}{2} (v^E + v^U),$$

(2.19)

We note that different authors have adopted different conventions for defining these
operators, especially for vectorial arguments (see, for example, Arnold et al. [27]). Here we define them in a way that scalars, vectors and tensor fields are treated alike. Additionally, since we shall strongly impose Dirichlet boundary conditions, there is no need to define these operators on \( \partial \Omega \).

**Definition of the DG-derivative.** Following a standard approach in some DG methods (see e.g., [18, 27]), approximations of the derivative of the solution are constructed through a DG-derivative operator on the discontinuous displacement field. This is defined as \( D_{\text{DG}} : \mathcal{V}_h \rightarrow (L^2(\Omega))^{2 \times 2} \) as

\[
D_{\text{DG}} v_h = \nabla v_h + R([v_h]) \quad \text{in } K, \text{ for any } K \in T'_h,
\]

where the (right) lifting operator \( R : (L^2(\Gamma^E_h))^2 \rightarrow \mathcal{V}_h \subset (L^2(\Omega))^{2 \times 2} \) is a linear operator that maps a displacement jump across \( \Gamma^E_h \) to a strain defined over \( \Omega \) such that for each element \( K \in T'_h \)

\[
\int_{\Omega} R(v_h) : \gamma_h \, ds = -\int_{\Gamma^E_h} v_h : \{\gamma_h\} \cdot n \, ds, \quad \forall \gamma_h \in \mathcal{W}_h,
\]

where

\[
\mathcal{W}_h = \left\{ \gamma_h \in \prod_{K \in T'_h} W^K_h : W^K_h = (P_1(*K))^{2 \times 2} \text{ if } *K \subset \Omega^E_h, \text{ and } W^K_h = (P_2(*K))^{2 \times 2} \text{ if } *K \subset \Omega^L_h \right\}.
\]

Here \( P_k(K) \), for \( k \) a nonnegative integer, denotes the function space consisting of polynomials of degree up to \( k \) over \( K \).

We discuss next some features of the DG-derivative operator. First, notice that if \( v_h \in \mathcal{V}_h \) is continuous across \( \Gamma^E_h \), then \( R([v_h]) = 0 \) and \( D_{\text{DG}} v_h = \nabla v_h \), i.e., the standard approximation of a derivative for conforming methods is recovered. Consequently, it is possible to interpret the role of the DG-derivative of a discontinuous displacement field as transforming displacement jumps into “enhanced” strain fields. In fact, the role of the lifting operator shares many traits with that of strain enhancements in the contexts of enhanced-strain methods, see [29].

In practice, the lifting operators can be computed locally within each element, just like the B-bar matrix in enhanced-strain methods, e.g., [30]. To compute the lifting operator over element \( K \in T'_h \), it is enough to consider only all \( \gamma_h \in \mathcal{W}_h \) that may be different than zero in \(*K\). This is possible because functions in \( \mathcal{W}_h \) may be discontinuous across the boundary of \(*K\). Under these conditions (2.21) reduces to

\[
[R(v_h)]^T M_K [\gamma_h] = \int_{*K} R(v_h) : \gamma_h \, ds = -\frac{1}{2} \int_{\Gamma^E_h \cap \partial(*K)} v_h : \gamma_h : n \, ds = [v_h]^T L_K [\gamma_h]
\]

for all such \( \gamma_h \), where \( n \) is the external unit normal to \( \overline{\Omega^E_h} \) and \( \gamma_h|_{\Gamma^E_h \cap \partial(*K)} \) is computed as the trace of \( \gamma_h|_{*K} \) on \( \Gamma^E_h \cap \partial(*K) \). The square brackets around a function indicates its components in a basis of the space they belong to, i.e., \( \mathcal{W}_h \) for \( R(v_h) \) and \( \gamma_h \), and \( \mathcal{V}_h \) for \( v_h \), while \( M_K \) and \( L_K \) are the matrices needed for the equality in (2.23) to hold in these bases. In particular, \( M_K \) is the mass matrix in the basis of \( \mathcal{W}_h \) over \(*K\). The computation of \([R(v_h)]\) given \([v_h] \)
Figure 6. Sketch showing the support (gray area) of the lifting operators of each one of the two scalar \( P_1 \) shape functions associated with node \( a \). One of them, \( \psi_{Ea} \), has support in the enriched region \( \Omega_E^h \) and the other one, \( \psi_{Ua} \), in the unenriched region \( \Omega_U^h \). However, both generate nonzero jumps along the enrichment boundary \( \Gamma_E^h \) shown in thick solid lines, and hence nonzero lifting operators on elements sharing any edge with \( a \) as an endpoint. The crack is not shown in this figure.

then requires the computation of \( M_K^{-1}L_K \). The latter essentially involves the inversion of a small mass matrix over each element and it can be precomputed, just as B-bar matrices, so that it need not be recomputed when solving multiple boundary value problems on the same mesh, or in eventual extensions to the nonlinear setting. We refer the reader to [18] for further details on the implementation of the lifting operator.

The computation of the lifting operators only needs to be performed in a small fraction of the elements. More precisely, only in each element \( K \) for which the length of \( \Gamma_E^h \cap \partial(*K) \) is positive. This reduction follows after noticing that for all other elements the left hand side in (2.23) would be identically zero for every test function \( \gamma_h \) therein, since they are nonzero only within \(*K\). With the same arguments, it is easy to see that \( R[[\psi_{\alpha a}]] \) for any basis function \( \psi_{\alpha a} \) in \(*V_DG^h\) is generally different than zero in at most four elements, two on each side of \( \Gamma_E^h \), see Figure 6. In contrast, \( R[[\Psi_M]] \), \( M = I, II \), may be different than zero over \(*K\) for each element \( K \) having at least one edge such that \( \Gamma_E^h \cap \partial(*K) \) has a positive length.

For the reader familiar with DG formulations, the choice of \( \Psi_h \) such that the condition \( \nabla V_h \subseteq \Psi_h \) (discussed in [27]) is not satisfied is nonstandard. The rationale behind this choice is briefly discussed in Appendix I. It essentially consists of the fact that the mass matrices to invert in the computation of the lifting operators are always those of standard \( P_1 \) or \( P_2 \) elements for elements bordering \( \Gamma_E^h \) and, most importantly, are invertible.

**Variational Principle.** Having defined \( D_{DG} \), we introduce the following bilinear form \( a_h(\cdot, \cdot) : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R} \):

\[
a_h(u_h, v_h) \equiv \sum_{K \in T^h} \int_{*K} D_{DG} u_h : C : D_{DG} v_h \, dS + \beta c_{\text{min}} \int_{\Omega} R([u_h]) : R([v_h]) \, dS, \tag{2.24}
\]

where \( \beta \in \mathbb{R} \) is a stabilization parameter, and \( c_{\text{min}} \) is the minimum eigenvalue of \( C \) among symmetric strains. For a stress-free isotropic linear elastic material \( c_{\text{min}} = 2\mu \), where \( \mu \) is the shear modulus. For \( h > 0 \) small enough, the bilinear form is guaranteed to be coercive, i.e., the stiffness matrix is invertible, provided that \( \beta > 0 \), as shown in [20]. This is a sufficient but often unnecessary condition. In fact, \( \beta = 0 \) yields a positive-definite matrix in all the numerical examples shown later.
Without loss of generality we assume that \( \partial_d \Omega \cap \partial \Omega_h^E = \emptyset \). This is always possible by choosing a small enough value for \( r_{\text{cutoff}} \). Let then \( \mathcal{V}_h \) denote the subset of \( \mathcal{V}_h \) whose members satisfy the Dirichlet boundary condition at all nodes on \( \partial_d \Omega \). The fact that \( \partial_d \Omega \cap \partial \Omega_h^E = \emptyset \) guarantees that the crack enrichments do not affect the value of a function in \( \mathcal{V}_h \) along the Dirichlet boundary. Therefore, \( \mathcal{V}_h \) is simple to construct. Similarly, let \( \mathcal{V}_{h0} \) be the subspace of \( \mathcal{V}_h \) whose members are identically zero at all nodes on \( \partial_d \Omega \).

The numerical method is defined by the following variational problem corresponding to the strong form (2.1): find \( u_h \in \mathcal{V}_h \), such that

\[
a_h(u_h, v_h) = \int_{\Omega} f \cdot v_h d\Omega + \int_{\partial_d \Omega} T \cdot v_h d\Omega, \quad \forall v_h \in \mathcal{V}_{h0}.
\]

Equation (2.25) defines the methodology used to define the FE approximation \( u_h \) to \( u \).

**On the stability of the method.** The global stiffness matrix of the system is non-singular if and only if the bilinear form \( a_h \) is coercive. Clearly, \( a_h(u_h, u_h) \) is non-negative for any \( u_h \in \mathcal{V}_h \). However, there may exist a non-zero \( v_h \in \mathcal{V}_{h0} \) such that \( D_{\text{DG}} v_h = 0 \), i.e., a nonzero deformation that generates no strain. Clearly such \( v_h \) needs to have discontinuities. A sufficient condition for stability is then to guarantee that \( R([u_h]) \) is an injective map from \( \{ [u_h] : u_h \in \mathcal{V}_h \} \) to \( \mathcal{W}_h \). Under these conditions, the stabilization term in \( a_h(u_h, u_h) \) with \( \beta > 0 \) would be strictly positive for any non-zero \( u_h \), since \( R([u_h]) = 0 \) if and only if \( [u_h] = 0 \) on \( \Gamma_h^E \).

Based on its definition (2.21), the lifting operator is injective if and only if for each non-zero function in \( \mathcal{V}_h \) with non-zero jumps it is possible to find a function in \( \{ \mathcal{W}_h \} \cdot n \) such that the right hand side of (2.21) is different than zero. This requirement can be expressed in terms of the following inf-sup condition:

\[
\inf_{v_h \in \mathcal{V}_h} \sup_{\gamma_h \not\in \mathcal{W}_h} \frac{\int_{\Gamma_h^E} [v_h] \cdot \{ \gamma_h \} \cdot n ds}{\| v_h \|_{L^2(\Gamma_h^E)} \| \{ \gamma_h \} \|_{L^2(\Gamma_h^E)}} \geq C > 0,
\]

where \( C \) is independent of \( h \), so that the coercivity constant of the bilinear form does not get arbitrarily close to zero as the mesh is refined.

As we will show in [20], it is possible to prove that the inf-sup condition (2.26) is satisfied for planar crack problems for \( h \) small enough, provided \( r_{\text{cutoff}} \) is chosen to be strictly smaller than the shortest distance from \( x_0 \) to \( \partial \Omega \). A similar conclusion holds for the inf-sup condition arising in anti-plane crack problems. However, in this case the stability analysis occasionally requires the introduction of minor local modifications to \( \Gamma_h^E \), as we shall detail in [20]. In practice, however, these worse-case scenarios considered in the analysis are hardly attained; for example, all the numerical examples in this paper were performed with \( \beta = 0 \).

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\(^4\)Dirichlet boundary conditions for the case of \( \partial_d \Omega \cap \partial \Omega_h^E \neq \emptyset \) can be imposed as equality constraints for nodal values of \( U \) on \( \partial_d \Omega \).
2.6. Numerical integration schemes

The numerical integration strategy plays a crucial role in the asymptotic accuracy of the method. To retain the expected optimal order of convergence of a method that relies on exact integration, the chosen quadrature rule should induce a consistency error of the same order as that of the numerical scheme, see e.g. [31] and the discussion in Appendix II. This requirement is simple to satisfy for problems with smooth solutions, but it needs to be carefully handled in the presence of singularities. Of course, the problem only appears in those elements that either contain the tip $x_t$ or are close to it, since their entries in the stiffness matrix require the integration of functions that scale as $r^{-1}$ or $r^{-1/2}$; see Appendix II for a brief discussion.

As discussed later, a key feature of the method herein is that the stress intensity factors (SIFs) can be directly obtained as the coefficients of the enrichment basis in the numerical solution; see Section 2.7. These values are, however, strongly affected if the integration strategy near the crack tip is not optimally designed.

Addressing these requirements, we propose next an integration scheme based on [19] in which we exploit the fact that the functions in the enrichment basis satisfy mechanical equilibrium to transform domain integrals into boundary integrals. This integration strategy renders an optimally convergent method.

Notably, the derivation of the quadrature formulas unveils a powerful property of the chosen enrichment basis: their coupling with the underlying FE space $V^{DG}_h$ is surprisingly sparse. In fact, the number of nonzero entries in the stiffness matrix coupling the two only scales as $O(h^{-1})$, despite the fact that $\Omega_h^E$ involves $O(h^{-2})$ elements. Had we adopted arbitrary enrichment functions, the number of nonzero entries reflecting their coupling with the underlying FE space would be expected to scale as $O(h^{-2})$ instead. Since the total number of nonzero entries in the stiffness matrix due to coupling of DOFs in $V^{DG}_h$ (the FE space without enrichment) scales as $O(h^{-2})$ as well, the relative computational and memory cost of adding the enrichment functions becomes negligible as $h \to 0$.

We proceed with the derivation of the quadrature strategy next. An entry in the stiffness matrix is obtained by substituting $u_h$ and $v_h$ in the bilinear form (2.24) by two basis functions of $\mathcal{V}_h$. Disregarding the stabilization term, which requires a regular Gauss quadrature rule, any entry in the stiffness matrix is of the form

$$K_{AB} = \sum_{K \in T_h^*} \int_{\star K} (\nabla N_A + R([N_A])) : \mathbb{C} (\nabla N_B + R([N_B])) \, dS, \quad (2.27)$$

where, for each index $A$, $N_A$ is one of the basis functions in $\mathcal{V}_h$, i.e., either $\psi_{\alpha a} e_i$ as a basis function in $V^{DG}_h$ for $a \in S^\alpha$ with $\alpha = U, E$ and $i = 1, 2$, or $\chi_{\Omega_h^E} \Psi_M$, $M = I, II$.

Two observations simplify the computation of many of these terms, and are useful to design
the quadrature rule later. First, notice that

\[
I_{MN} = \sum_{K \in T_h} \int_{*K} \nabla(\chi_{\Omega_h^k} \Psi_M) : C : \nabla(\chi_{\Omega_h^k} \Psi_N) \, dS
\]

\[
= \int_{\Gamma_{Eh} \setminus \partial_\epsilon \Omega} \nabla \Psi_M : C : \nabla \Psi_N \, dS
\]

\[
= -\int_{\Omega_h^k \setminus \partial_\epsilon \Omega} \nabla (\nabla \Psi_M : C) \cdot \Psi_N \, dS + \int_{\partial_{Eh} \cup (\partial_\epsilon \Omega \cup \Gamma_{Eh}^k)} n \cdot (\nabla \Psi_M : C) \cdot \Psi_N \, ds
\]

(2.28)

for \(M, N = I, II\), where the first term is identically zero because the enrichment basis functions are in mechanical equilibrium in the absence of body forces.

We can then write

\[
K_{MN} = I_{MN} + \sum_{K \in T_h} \int_{*K} \left[ \nabla(\chi_{\Omega_h^k} \Psi_M) : C : R \left( \left[ \chi_{\Omega_h^k} \Psi_N \right] \right) \right.
\]

\[
+ \nabla(\chi_{\Omega_h^k} \Psi_N) : C : R \left( \left[ \chi_{\Omega_h^k} \Psi_M \right] \right) \right.
\]

\[
+ R \left( \left[ \chi_{\Omega_h^k} \Psi_M \right] \right) : C : R \left( \left[ \chi_{\Omega_h^k} \Psi_N \right] \right) \] \, dS. \quad (2.29)

The integrand in the boundary integral in \(I_{MN}\) is piecewise smooth and bounded independently of \(h\) over the domain of integration, since: a) points in \(\partial \Omega_{Eh}^k\) lie at a distance greater than \(\min\{r_{\text{cutoff}}, \text{dist}(x, \partial \Omega)\} > 0\) away from the tip, and b) for a smooth crack the value of the integrand is smooth with smooth and bounded derivatives on \(\partial_\epsilon \Omega\), even as \(r \to 0\). Consequently, the contribution over \(\partial \Omega_{Eh}^k\) to the value of \(I_{MN}\) can be computed by a one-dimensional Gauss quadrature over each segment of \(\partial \Omega_{Eh}^k\) that results from its intersection with \(*K\), for each \(K\). We shall specify the quadrature rule later in this section. The contribution to \(I_{MN}\) from the part of the crack inside the enrichment region, \(\partial_\epsilon \Omega \cap \Omega_{Eh}^k\), requires a smooth parametrization of the crack surface, and then any standard quadrature strategy for smooth functions can be applied. We shall not be more specific about it, since a number of options can be adopted to parametrize the surface depending on the application. However, if \(\partial_\epsilon \Omega \cap \Gamma_{Eh}^k\) is a straight segment, then its contribution to \(I_{MN}\) is identically zero, since the enrichments exactly satisfy the traction free boundary conditions on straight crack faces.

To deal with the last three terms in (2.29), we recall that the support of the lifting operator contains only elements \(K\) for which \(*K\) has at least one edge on \(\Gamma_{Eh}^k\) across which discontinuities may appear; see Figure 6. Consequently, integration should be performed only over \(*K\) that shares an edge with \(\Gamma_{Eh}^k\). This layer of elements also remain at a finite distance from the tip for all \(h\), and hence no singularity is encountered in the gradients of the enrichment functions therein. A standard Gauss quadrature rule would then suffice. We specify the rule later in this section.
The second observation is that

\[
I_{M\alpha i} \equiv \sum_{K \in T_h} \int_{\Omega_h^K} \nabla \left( \chi_{\Omega_h} \Psi_M \right) : \nabla (\psi_{\alpha a} e_i) \ dS
\]

\[
= \int_{\Omega_h^E \setminus \partial_{cr} \Omega} \nabla \Psi_M : \nabla (\psi_{\alpha a} e_i) \ dS
\]

\[
= -\int_{\Omega_h^E \setminus \partial_{cr} \Omega} \nabla \cdot \left( \nabla \Psi_M : \mathbb{C} \right) (\psi_{\alpha a} e_i) \ dS
\]

\[
+ \int_{\partial (\Omega_h^E \setminus \partial_{cr} \Omega)} \mathbf{n} \cdot \left( \nabla \Psi_M : \mathbb{C} \right) (\psi_{\alpha a} e_i) \ ds
\]

\[
= \begin{cases} 0 & \text{if } \alpha = U, \\ \int_{\Omega_h^E \cup (\partial_{cr} \Omega \cap \Omega_h^E)} \mathbf{n} \cdot \left( \nabla \Psi_M : \mathbb{C} \right) (\psi_{\alpha a} e_i) \ ds & \text{if } \alpha = E, \end{cases}
\]

for \( M = I, \Pi \) and any node \( a \in S^\alpha \) with \( \alpha = E, U \) and spatial direction \( i = 1, 2 \). Remarkably, in addition to \( I_{MU\alpha i} = 0, I_{ME\alpha i} = 0 \) whenever the node \( a \in S^E \) is such that \( \text{cloud}(a) \cap \left[ \Omega_{hE} \cup (\partial_{cr} \Omega \cap \Omega_h^E) \right] = \emptyset \). This is because for such nodes \( \psi_{Ea} = 0 \) on \( \partial \Omega_h^E \cup (\partial_{cr} \Omega \cap \Omega_h^E) \), see Figure 7.

The coupling between enrichment functions and basis functions in \( *V_h^{DG} \) is measured by

\[
K_{Maai} = I_{M\alpha i} + \sum_{K \in T_h} \int_{\Omega_h^K} \left[ \nabla \left( \chi_{\Omega_h} \Psi_M \right) : \mathbb{C} : R \left( [\psi_{\alpha a} e_i] \right) \right]
\]

\[
+ \nabla \psi_{\alpha a} e_i : \mathbb{C} : R \left( \left[ \chi_{\Omega_h} \Psi_M \right] \right)
\]

\[
+ R \left( \left[ \chi_{\Omega_h} \Psi_M \right] \right) : \mathbb{C} : R \left( [\psi_{\alpha a} e_i] \right) \ dS.
\]

As before, the last three terms in (2.31) can only be different than zero on those elements \( K \) such that \( *K \) has at least one edge on \( \Gamma_h^E \), since the lifting operators can only be different than zero therein. Consequently, it follows that \( K_{MU\alpha i} = 0 \) unless \( a \) belongs to one such element. This also shows that \( K_{ME\alpha i} = 0 \) unless \( a \) belongs to \( K \) with \( *K \subset \Omega_h^E \) such that either \( \overline{K} \cap \partial_{cr} \Omega \neq \emptyset \) or \( *K \) shares an edge with \( \Omega_h^E \). This is surprising, because it would not happen with arbitrary enrichment functions. Since \( K_{Maai} \) may be different than zero only for nodes \( a \) within a distance of \( h \) away from either \( \partial \Omega_h^E \) or \( \partial_{cr} \Omega \cap \Omega_h^E \), and the lengths of these curves remain finite as \( h \to 0 \), the number of nonzero entries in \( K_{Maai} \) scales as \( O(h^{-1}) \), as mentioned earlier.

The computation of the nonzero values of \( K_{Maai} \) then involves the design of quadrature rules to compute the non-zero values of \( I_{M\alpha i} \) by integrating over a one-dimensional domain. As earlier, standard Gauss quadrature rules can be adopted over both \( \partial \Omega_h^E \) and \( \partial_{cr} \Omega \cap \Omega_h^E \). The latter requires the choice of a parametrization for the crack surface, so we shall not be specific about the quadrature therein. It is important to note, however, that for a smooth crack the integrand near the crack tip behaves as \( r^{1/2} \) as \( r \to 0 \), so a low-order quadrature rule could still be adopted over the parametric domain. Furthermore, the integral in (2.30) over \( \partial_{cr} \Omega \cap \Omega_h^E \) is identically zero if \( \partial_{cr} \Omega \cap \Omega_h^E \) is a straight segment.
Figure 7. The stiffness matrix of the problem is surprisingly sparse. This figure shows a sketch of possible couplings between the degrees of freedom (DOFs) of enrichment functions and those of the polynomial basis. The enrichment region $\Omega^E_h$ is shown in gray. Of those terms in the stiffness matrix that relate enrichment and polynomial DOFs, the only ones that are different than zero, and hence need to be computed, are those corresponding to basis functions associated with nodes marked with a circle. As described in Algorithm 1, some nodes have been duplicated and hence occupy the same spatial location, so a single circle corresponds to both of them. All nodes that do not belong to elements that are either intersected by the crack or that have at least one edge on the enrichment region boundary have their shape functions uncoupled from the enrichment functions. Consequently, the number of entries in the stiffness matrix of this type that need to be computed scales as $O(h^{-1})$. The colors inside the circles indicate how the non-zero terms are computed, see the text for a description.

Figure 7 shows with a graphical example of how each term $K_{Maai}$ is computed through a combination of surface and domain integrals. Therein, $K_{Maai}$ may be different than zero only for those nodes marked with a circle. If a node $a$ is marked with either a filled circle or a half-filled circle, then the last three terms in (2.31) may be different than zero, and are computed by integration over the domain. For straight cracks, the term $I_{MEai}$ is different than zero only for nodes with filled circles, in which case it can be computed through a boundary integral over $\partial \Omega^E_h \cap \text{supp}(\psi_{Ea})$, where $\text{supp}(\psi_{Ea})$ denotes the support of $\psi_{Ea}$. In contrast, for curved cracks, $I_{MEai}$ may also be different than zero for nodes marked with either an open circle or a letter $b$, and an integral over part of the crack faces should be computed.

**Quadrature rules.** A single quadrature rule is chosen over each element or edge of an element over which quadrature is necessary. The considerations behind the choice of each quadrature rule are: a) the consistency error introduced by the quadrature rule, which should not deteriorate the convergence rate, b) the need to satisfy the stability condition (2.26), which translates in integrating jumps and lifting operators to a high-enough order, and c) computational efficiency, which generally implies that quadrature rules should be of the lowest
Figure 8. A sketch illustrating the minimum quadrature orders for domain integration (open circles) and boundary integration (filled circles) for computing the entries in the stiffness matrix, considering both the order of the consistency error and the stability requirements. The enrichment boundary $\Gamma_h^E$ and the crack $\partial_C \Omega$ are both drawn with a thick stroke. For any element $K \in \mathcal{T}_h$, integration to compute entries in the stiffness matrix are performed only over $*K$, which may be a triangle or a quadrilateral. In the latter case the quadrilateral is partitioned into two triangles only for the purpose of integration, see the dashed lines. The same idea applies to a segment on $\Gamma_h^E$ cut by the crack. Elements in dark gray are enriched elements with one edge in $\Gamma_h^E$. Therein, the product of two DG-derivatives may involve quadratic polynomials for each lifting operator, and to guarantee stability, it is convenient to adopt a six-point quadrature rule to exactly integrate fourth-order polynomials. Elements in light gray use a three-point quadrature rule to exactly integrate quadratic polynomials. They are elements in $\Omega_h^K$ that share an edge with $\Gamma_h^E$, for which the lifting operators are linear polynomials. The rest of the elements are colored in white. They only need a one-point quadrature rule, since only constant strains are involved in computing the elemental contribution to the stiffness matrix. Boundary integrals are performed by integrating over edges. Edge segments along $\Gamma_h^E$ need a three-point Gauss quadrature rule for stability, so that products of jumps and averages of quadratic polynomials, i.e., quartic polynomials, are integrated exactly. In the case of a curved crack, integration along the crack is needed. The quadrature order depends on the parametrization of the crack path, which is not shown on this figure.
2.7. Computation of stress intensity factors

We briefly comment next on the methods used in the numerical examples later to compute SIFs. A neat consequence of the choice of spaces for the method herein is that it is possible to prove that the coefficients of $\Psi_I$ and $\Psi_{II}$ converge to the SIFs $K_I$ and $K_{II}$ as $h$ goes to zero. Consequently, the SIFs are obtained directly as part of the primary unknowns.

A similar strategy was adopted by Liu et al. [32], even though they enriched with both the leading and some higher-order terms of the asymptotic expansion of the solution near the crack tip. They extracted their SIFs from the leading terms only.

The second and prevailing method for obtaining $K_I$ and $K_{II}$ is to compute the interaction integral[33], as described in [10], since $K_I$ and $K_{II}$ converge quadratically with $h$ in this way. For completeness, we summarize next the formulae we use in numerical examples later to compute SIFs. We refer the unfamiliar reader to one of the two previous references for insight into their origin.

Let $A_{\text{inner}}$ and $A_{\text{outer}}$ be two simply connected sets in $\overline{\Omega}$ with piecewise smooth boundaries such that $x_t \in A_{\text{inner}} \subset A_{\text{outer}} \subset \overline{\Omega}$. Set $A = A_{\text{outer}} \setminus A_{\text{inner}}$ and let $q : A \to \mathbb{R}$ be a smooth function such that $q(\partial A_{\text{outer}}) = 0$ and $q(\partial A_{\text{inner}}) = 1$.

Given two displacement fields $u^{(a)} : \Omega \to \mathbb{R}^2$, $a = 1, 2$, the interaction integral between the two, taking into account our DG formulation, is defined as

$$ I \left[ u^{(1)}, u^{(2)} \right] = \int_A \left[ \sigma^{(1)} \cdot \left( D_{\text{DG}} u^{(2)} \cdot e \right) - W^{(1, 2)} e \right] \cdot \nabla q \, dS , \tag{2.32} $$

where $W^{(1, 2)} = \sigma^{(1)} : D_{\text{DG}} u^{(2)}$, $e$ is the unit vector tangential to the crack at the crack tip and pointing towards the uncracked material, and $\sigma^{(a)} = \mathbb{C} : D_{\text{DG}} u^{(a)}$, $a = 1, 2$.

When each one of the two displacements fields is a solution of the linear elasticity problem with no body forces and free traction boundary conditions on $\partial_{x} \Omega$, it holds that

$$ I \left[ u^{(1)}, u^{(2)} \right] = \frac{2}{E'} \left( K^{(1)}_I K^{(2)}_I + K^{(1)}_{II} K^{(2)}_{II} \right) , \tag{2.33} $$

where $K^{(a)}_M$, $M = I, II$ and $a = 1, 2$ is the SIF for mode $M$ under displacement field $u^{(a)}$, and $E' = E$ for plane stress and $E' = E/(1 - \nu^2)$ under plane strain.

In the numerical examples we set $u^{(2)} = \Psi_M$ to compute $K_M$, with $M = I, II$, since for each one of these two choices one of the SIFs is zero and the other one is one. If the function $q$ is chosen independently of $h$, then the values of $I[u_h, u^{(2)}]$ converge to $I[u, u^{(2)}]$ with second order in $h$, where $u$ is the exact solution of the problem. The approximate values of $K_M$, $K_{Mh}$, follow from (2.33).

In the convergence studies shown later in this paper we consider an initial mesh $T_{h0}$ and finer meshes obtained by recursively subdividing each element in the mesh into four similar ones. We define $q$ in the entire domain using $T_{h0}$. We set $q$ to be the piecewise linear function over $T_{h0}$ such that its nodal values are 0 at all nodes that either lie on $\partial \Omega$ or outside the open ball of radius $r_q > 0$ centered at $x_t$, and 1 otherwise. In all later numerical examples we have set $r_q = 2r_{\text{cutoff}}$. In terms of our previous definitions, we have set

$$ A_{\text{inner}} = \{ x \in \overline{\Omega} : q(x) = 1 \} $$

$$ A_{\text{outer}} = \{ x \in \overline{\Omega} : q(x) > 0 \} \tag{2.34} $$
Figure 9. Contour plot of the scalar field $q$ involved in computing the interaction integral over the coarsest mesh, for one of the numerical examples in Section 3. The function $q$ does not change as the mesh is refined. This is a necessary condition for the computed SIFs to converge to their exact values quadratically with $h$, see the text and Béchet et al.[10].

We have effectively defined $A$ to be the region in which $0 < q < 1$, and it is comprised of all those elements whose nodes do not all have the same value for $q$. While for the first mesh $A$ is only one element thick, see Figure 9, it becomes several element thick upon subdivision. As investigated by Béchet et al.[10], fixing the $q$-field as $h \to 0$ is a necessary condition for obtaining a second-order convergence rate for the interaction integral.

An attractive feature of the first method over the last one is that when the SIFs are extracted as the coefficients in front of the singularities in the numerical solutions, we are literally using the definition of SIF. In contrast, the second method based in the computation of interaction integrals has a faster expected convergence rate. However, it relies on stricter assumptions on the solution, including the dominance of the near-tip solution over the region of integration, the absence of body forces, and the satisfaction of the zero-traction boundary condition on the crack faces.

2.8. Relation to Other Methods

The method introduced here shares many features with other variants of the XFEM. The key distinguishing aspects are: a) we introduce only two asymptotic crack tip fields for plane strain problems, and only one for anti-plane shear ones, b) we do not adopt a PU approach to introduce them, as traditionally done in XFEMs, c) we restrict the support of the singular enrichments to a small region around the crack tip by introducing a discontinuity, and d) the region with non-zero singular enrichment around each crack tip is chosen so that it contains a ball of a fixed radius for all $h$. These features endow the method with optimal order of convergence. Let us discuss the differences with other strategies next.

First, we would like to contrast our choice of a two-dimensional enrichment space with
the more common choice of an eight-dimensional one resulting from enriching each Cartesian
component of the displacement field with the space
\[
\text{span}\left\{ \sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta \right\}.
\]

The adoption of the smaller enrichment space is possible because all singular asymptotic crack
tip fields for the class of problems considered here can be obtained as linear combinations of
\(\Psi_I\) and \(\Psi_{II}\), see, e.g., Westergaard[25].

The second important feature of our approach is that we do not introduce a PU to incorporate the singular enrichments. Instead, we include the non-local vector functions \(\Psi_I\) and \(\Psi_{II}\) in our extended FE basis. The advantages of using non-local functions instead of their
PU are twofold besides the obvious consideration of rendering a much lower total number
of unknowns. First, as investigated by Laborde et al.[9], the PU method leads to a worse-conditioning matrix than its non-local enrichment counterpart. Second, non-local enrichments
have a higher chance of satisfying the inf-sup condition discussed in Subsection 2.5.

The inclusion of singular enrichments with a small support in the domain is important to
reduce the number of DOFs to which they couple, and to simplify the imposition of Dirichlet
boundary conditions. Given that only the behavior at the crack tip is important, the enrichment
functions can be modified away from it in essentially any way by choosing the function \(\rho_h\).
The choice of a smooth cutoff function as in [12, 13] results in enrichment functions with
compact support but gradients that scale with \((r_1 - r_0)^{-1}\), following the notation in Section
2.4. Approximations are accurate then only when \(h\) is somewhat smaller than \(r_1 - r_0\).

The introduction of a discontinuity instead alleviates this requirement, as shown in the
numerical examples later, and can be regarded as the limiting case \(r_1 \rightarrow r_0\). Optimally
convergent approximations in this case can be obtained in a number of ways (e.g.,[9], [11]), and
a DG formulation is one of them. This was first noticed by Gracie et al [14]. The latter differs
from our proposed method in two aspects: a) the former is known to be stable for \(\beta\) large
enough, usually of the order of the elastic moduli, while ours is stable for any \(\beta > 0\), and b)
the former adopts a PU to include the enrichments. In terms of stability, the point-matching
method in [9] is also non-conforming, so its stability properties need to be investigated. In
contrast, the way to construct the enrichment functions in [11] renders approximation spaces
that are in \(H^1(\Omega)\), and hence inherit the stability of the original problem.

Finally, some early versions of the XFEM, and some more recent contributions, only enrich
on a few layers of elements around the crack tip, typically one or two. This leads to suboptimal
convergence rates. More precisely, this strategy results in the same convergence rate as if the
singular enrichments were not included. It does provide more accurate solutions though.

3. Numerical examples

We illustrate the performance of the method next with some numerical examples. No
stabilization was needed in any of them, i.e., we adopted \(\beta = 0\) in (2.24).

\*termed “degree-of-freedom-gathering” approach by the authors
3.1. Mode I near-tip field

Our first numerical example tests the convergence properties of the method with both structured and unstructured meshes. For this purpose we considered a square domain \((0, 1) \times (0, 1)\) with an edge crack \([0, 0.55] \times \{0.5\}\) as shown in Figure 10. Dirichlet boundary conditions were imposed on the sides of the square according to the mode I asymptotic field, i.e., \(U = \Psi_I\), with crack length \(a = 0.55\), Young’s modulus \(E = 1000\), and Poisson’s ratio \(\nu = 0.3\). We set \(r_{\text{cutoff}} = 0.15\), and solved the problem under plane strain loading. The coarsest structured and unstructured meshes are depicted in Figures 10(a) and 10(b), respectively. Finer meshes in each case are obtained by subdividing each triangle into four similar ones, recursively.

Before discussing convergence rates, we note that discontinuities do appear in the solutions across \(\Gamma_E\). They are small to be displayed well, and become even smaller as the mesh is refined, but are critical for the optimal convergence of the method.

The convergence curves for displacements and stresses for solutions obtained with the two families of meshes are shown in Figures 11 and 12. To highlight the importance of the quadrature strategy discussed in Section 2.6 on the computation of SIFs, we show convergence curves for our method in which all terms in the stiffness matrix and right hand side are instead computed through domain integrals. In this case, for the most singular integrands we adopted the \(12 \times 12\) “almost polar” scheme proposed by [9] based on partitioning the element that contains the crack tip \(x_t\) into triangles for each of which \(x_t\) is a vertex. The resulting convergence curves are labeled “Our method (domain integral).” We also show therein the convergence curves for three other methods with the same meshes, namely: a) the (standard) FEM that results from choosing \(V_h = V_{\text{conf}}^h\), in which no singular enrichments are included, b) our method with the quadrature strategy in Section 2.6 but with a shrinking enrichment zone, i.e., we chose a different \(\Omega_E^h\) for each \(h\) by setting \(r_{\text{cutoff}} = h_{\text{max}}\) where \(h_{\text{max}}\) is the maximum length among all triangles in each mesh, see Section 2.4, and c) the cutoff function method by [13] with \(r_0 = 0.003\) and \(r_1 = 0.15\), see Section 2.4 as well. In (c) all integrals were performed over the domain by using the strategy in [9], as we did for the curves labeled “Our method (domain integral).”

These results show that not including singular enrichments, or including them but only in an enrichment zone that shrinks with \(h\) gives rise to the same type of suboptimal convergence rate. Naturally, errors in the latter case are smaller, as seen in Figures 11 and 12. Both the cutoff function method and our method render an apparent optimal convergence rate. However, for this example the errors with the method introduced herein are uniformly smaller by roughly one order of magnitude. For this range of mesh sizes there is no distinguishable difference between the quadrature strategies, except for the data point in Figure 12(a) corresponding to the finest mesh. As we discuss next, this is no longer the case when the convergence of the SIFs is considered.

Figure 13 shows the errors in modes I and II SIFs extracted as the coefficients of the enrichment basis functions \(\Psi_I\) and \(\Psi_{II}\), respectively, for each one of the methods for which this is possible. When endowed with the quadrature strategy in Section 2.6, our method displays a remarkable ability to capture the value of the SIFs as the mesh is refined, in this and subsequent examples. In contrast, when endowed with the domain integration scheme described in [9], the SIFs do not display a converging pattern. The reasons behind this behavior can be traced back to the lack of an appropriate order for the consistency error introduced by the quadrature rule.
On the other hand, the cutoff function method which also uses a domain integration scheme does show a converging trend for the SIFs as the mesh is refined, at least for $K_I$, but the errors of this method are larger for the same mesh and cutoff radius of enrichment.

We next compare the convergence of the SIF in mode I when computed with the interaction integral, see Figure 14. Quadratic convergence is attained for our method as well as the cutoff function method, with the former having an error approximately one order of magnitude smaller for the same mesh. Also plotted is the SIF extracted as the coefficient of $\Psi_I$ in our method. In this case the two methods of extracting SIFs yield very similar results. However, in general the convergence rate of the method based on the interaction integral will be faster, though more computationally expensive as well.

Finally, we investigated the dependence of the error in the stress field and the mode I SIF as the value of $r_{\text{cutoff}}$ is changed for our method. These values were computed on an unstructured mesh obtained by performing one subdivision of that in Figure 10(b). A comparison with the cutoff function method is also shown, by varying the value of $r_1$ and keeping either the ratio $r_0/r_1$ or the value of $r_0$ fixed. The results are shown in Figure 15, indicating as expected that the larger $r_{\text{cutoff}}$ is, the smaller the errors become.

### 3.2. Uniaxial tension

In this second numerical example we compute the SIF of a planar slab with a crack of length $a$ in plane strain under uniaxial tension, i.e., the traction vector on the faces parallel to the crack is normal to them and has magnitude $\sigma$, as shown in Figure 16(a). The Young’s modulus and Poisson’s ratio are 1000 and 0.3, respectively. The deformed configuration obtained with a relatively fine mesh, together with a contour plot of the von Mises stress is shown in Figure 16(b).

The mode I SIF for this example is given by Tada et al.[34] as

$$K_I = \sigma \sqrt{W F} \left( \frac{a}{W} \right), \quad (3.1)$$
Figure 11. Convergence curves in terms of $L^2$-errors in (a) displacements and (b) stresses of five methods using a structured mesh family whose coarsest mesh is shown in Figure 10(a) with 144 elements. Finer meshes are obtained by subdividing each triangle into four similar triangles recursively. These methods are: (O) the standard finite element method; ($\times$) our method with a shrinking enrichment zone defined by $r_{\text{cutoff}} = h$; (+) the cutoff function method in [13] with $r_0 = 0.003$ and $r_1 = 0.15$; (□) and (∗) our method with $r_{\text{cutoff}} = 0.15$, where (□) and (∗) differ in the integration method for the most singular integral: (□) uses the $12 \times 12$ “almost polar” integration scheme proposed by [9] while (∗) uses the boundary integration scheme discussed in Section 2.6 which originates in [19].

For a fair comparison, (+) and (□) use the same integration schemes.

Figure 12. Convergence curves in terms of $L^2$-errors in (a) displacements and (b) stresses of five methods using an unstructured mesh family whose coarsest mesh is shown in Figure 10(b) with 112 elements. Finer meshes are obtained by subdividing each triangle into four similar triangles recursively. These methods are: (O) the standard finite element method; (×) our method but with the enrichment zone defined by $r_{cutoff} = h$; (+) the cutoff function method in [13] with $r_0 = 0.003$ and $r_1 = 0.15$; (□) and (*) our method with $r_{cutoff} = 0.15$, where (□) and (*) differ in the integration method for the most singular integral: (□) uses the $12 \times 12$ “almost polar” integration scheme proposed by [9] while (*) uses the boundary integration scheme discussed in Section 2.6 which originates in [19]. For a fair comparison, (+) and (□) use the same integration schemes.


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Figure 13. Normalized errors in SIFs $K_I$ and $K_{II}$ for different methods as extracted as the coefficients of the corresponding enrichment functions in the numerical solutions computed with the family of unstructured meshes stemming from the one in Figure 10(b). Both errors are normalized by $|K_I| = 1$. See the text for the description of each method.
where $W$ denotes the width of the plate and

$$f \left( \frac{a}{W} \right) = \frac{\sqrt{2} \tan \frac{\pi a}{2W}}{\cos \frac{\pi a}{2W}} \left[ 0.752 + 2.02 \frac{a}{W} + 0.37 \left( 1 - \sin \frac{\pi a}{2W} \right)^3 \right]. \quad (3.2)$$

In Table I, we have tabulated the computed values of $K_I$ comparing the results reported in Stazi et al.\cite{26} and those of our method with a structured mesh similar to the one in Figure 10(a) and $r_{\text{cutoff}} = 0.15W$. Both results are then obtained with comparable meshes, approximately the same number of $P_1$ DOFs. Regardless of how $K_I$ is computed, either directly from the DOF conjugate to $\Psi_I$ or through the interaction integral, the values computed with the method herein are closer to the exact ones.

### 3.3. Mixed-mode loadings

Our next two examples are two mixed-mode loadings: a planar slab with a slanted edge crack subject to uniaxial tension, see Figure 17, and another one in which one side is clamped and another side is subject to uniform shear, see Figure 18. The Young’s moduli and Poisson’s ratios are 1000 and 0.3 for the first example and 30 and 0.25 for the second example.

We solved these two problems with an unstructured mesh similar to that shown in Figure 10(b). The computed SIFs are tabulated in Tables II and III, respectively. Note that the reference solutions given in the figure captions are also numerical results. One can observe the ability of our method to directly obtain the SIFs for mixed-mode loading configurations without post-processing.
Figure 15. Relative errors in (a) stress and (b) mode I SIF as the value of $r_{cutoff}$ is changed over the same mesh for our method, or as the value of $r_1$ was changed for the cutoff function method. The SIF was extracted as the coefficient of $\Psi_I$. (O) cutoff function method with $r_0/r_1 = 0.02$; (×) cutoff function method with $r_0 = 0.003$; (∗) our method.
Figure 16. (a) A cracked finite slab under uniaxial tension. (b) Deformed configuration of the same specimen, with a contour plot of the von Mises stress. This result was obtained with a mesh that contains 73,278 elements, resulting in a total of 74,690 DOFs.

<table>
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<th>$a/W$</th>
<th>exact</th>
<th>our method (DOF extraction)</th>
<th>our method (interaction integral)</th>
<th>Stazi et al. [26] (interaction integral)</th>
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</table>

Table I. Exact and computed mode I SIFs of the specimen under uniaxial tension shown in Figure 16. The data in columns 3 and 4 are computed using our method with $r_{cutoff} = 0.15W$. The SIFs in column 3 were obtained by extracting the coefficient of the mode I enrichment $\Psi_I$, while those in column 4 were obtained using the interaction integral method, as described in Section 2.7. The data in column 5 is reported in Stazi et al. [26]. The latter adopted the XFEM in which only elements close to the crack tip are enriched, and the SIF values were computed through the interaction integral based on their numerical solution. The meshes adopted in the two cases have roughly the same number of $P_1$ DOF (approximately 1,280).
(a) A 45° slanted crack in a finite slab subject to tension, with $a/W = 1/\sqrt{2}$. (b) Deformed configuration of the specimen with a contour plot of the von Mises stress. This solution was obtained with a mesh that contains 135,168 elements and 136,932 DOFs.

Table II. Modes I and II SIFs normalized by $\sigma \sqrt{\pi a}$ for the solution of the slanted crack in Figure 17. These values were computed with our method in an unstructured mesh with $r_{\text{cut off}} = 0.15W$. The SIFs were computed both by extracting the enrichment coefficients and by invoking the interaction integral method. A reference solution given by Yau et al.\cite{35} is $K_I = (1.825 \pm 0.003)\sigma \sqrt{\pi a}$, $K_{II} = (0.831 \pm 0.004)\sigma \sqrt{\pi a}$.

<table>
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<tr>
<th>Number of DOFs</th>
<th>$K_I$ (DOF extraction)</th>
<th>$K_I$ (interaction integral)</th>
<th>$K_{II}$ (DOF extraction)</th>
<th>$K_{II}$ (interaction integral)</th>
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</table>
Figure 18. (a) Cracked slab in which one face is clamped and another one is subject to uniform shear, with \( L = 16.0 \), \( W = 7.0 \), and \( a = 3.5 \). (b) Deformed configuration of the specimen, with a contour plot of the von Mises stress. This result was obtained with a mesh that contains 157,696 elements and 159,770 degrees of freedom.

Table III. Modes I and II SIFs of the cracked slab subjected to shear in Figure 18, with \( \tau = 1 \). The solution was obtained with our method in an unstructured mesh with \( r_{\text{cutoff}} = 2.0 \). These SIFs are computed both by extracting the enrichment coefficients and by invoking the interaction integral method. A reference solution given by Wilson[36] is \( K_I = 34.0 \), \( K_{II} = 4.55 \).

<table>
<thead>
<tr>
<th>Number of DOFs</th>
<th>( K_I ) (DOF extraction)</th>
<th>( K_I ) (interaction integral)</th>
<th>( K_{II} ) (DOF extraction)</th>
<th>( K_{II} ) (interaction integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>760</td>
<td>34.2</td>
<td>33.0</td>
<td>3.52</td>
<td>4.50</td>
</tr>
<tr>
<td>2,736</td>
<td>34.9</td>
<td>33.8</td>
<td>4.05</td>
<td>4.53</td>
</tr>
<tr>
<td>10,384</td>
<td>34.8</td>
<td>34.0</td>
<td>4.45</td>
<td>4.53</td>
</tr>
<tr>
<td>40,466</td>
<td>34.3</td>
<td>34.0</td>
<td>4.54</td>
<td>4.54</td>
</tr>
<tr>
<td>159,770</td>
<td>34.2</td>
<td>34.0</td>
<td>4.54</td>
<td>4.54</td>
</tr>
</tbody>
</table>
4. Conclusions

We have proposed and demonstrated the performance of an optimally convergent DG-based XFEM. The discontinuity along the enrichment boundary serves the purpose of eliminating the blending elements in earlier versions of the XFEM. Additionally, the enrichment region is chosen so that its size does not go to zero with \( h \). This is critical to attain an optimal convergence rate, and since the enrichment functions are lightly coupled with the polynomial basis functions in the enrichment region, this choice has a minor impact in the efficiency of the method. In fact, the method possesses a superior condition number for the resulting stiffness matrices than most other versions of the XFEM, due to the adoption of two non-local enrichment functions instead of the many that result from adopting a PU approach. Another critical advantage of the method is the possibility of obtaining accurate values of the SIFs as primary unknowns of the system of equations.

The numerical results herein indicate that not only does the proposed method render an optimal order of convergence, but that the solutions are remarkably accurate, at least in comparison with the cutoff function method. As mentioned earlier, it is possible that this accuracy is largely the result of the sharp transition of the enrichment functions to zero, so other alternatives in the literature may perform similarly well. This superior accuracy was also encountered in the computation of the SIFs, even when they were extracted as the coefficients of the enrichment functions in the numerical solution.

We also provided an interpretation of cracks as immersed boundaries, which enables the construction of finite element spaces over “arbitrarily” cracked domains. Essentially, it decouples the introduction of the discontinuity in the mesh from the construction of the finite element space over it.

The seemingly standard replacement of exact integrals by quadrature rules, which is a trivial step in the context of smooth problems, needs careful crafting in this context for the optimal convergence rate to be retained. The deleterious effect of a suboptimal quadrature rule on the convergence curves of stresses and displacements was detected only for very refined meshes. However, it severely affected the value of the SIFs obtained as the coefficients of the enrichment functions already for coarse meshes.

Finally, our choice of numerical flux in the DG setup leads to a problem-independent stabilization parameter for the method to be stable. The numerical examples indicate that more often than not the absence of such stabilization term does not result in a singular matrix in this context. Unlike DG methods using only polynomial basis functions, our method relies on the satisfaction of an inf-sup condition by the mesh and the enrichment zone in order to have a coercive stiffness matrix. As discussed, this inf-sup condition is satisfied by planar and anti-plane crack problems with minor restrictions on the enrichment boundary.

REFERENCES


APPENDIX

I. Remark on the choice of the space containing the lifting operators

In this section we comment on an interesting difference between the DG method for linear elasticity herein and others introduced earlier, e.g. [17] and [18]. In these references, the definition of the DG-derivative (2.20) is obtained following a classical DG way, see e.g. [18] or [27]. For a scalar field \( u_h \in V_h \), possibly discontinuous across element boundaries, the DG-derivative is a function \( D_{DG}u_h \in \nabla V_h + W_h \) that satisfies

\[
\int_{\Omega} D_{DG} u_h : \gamma_h \, dS = \int_{\Omega} \nabla u_h \cdot \gamma_h \, dS - \int_{\Gamma_h} \{u_h\} \cdot n \, ds,
\]

for all \( \gamma_h \in W_h \), where \( \Gamma_h \) is the set of all element boundaries interior to the domain. When condition \( \nabla V_h \subseteq W_h \) is satisfied, then (2.20) follows from (I.1) after replacing by the definition of the lifting operator (2.21) with \( \Gamma_h \) playing the role of \( \Gamma_E \).

Stability considerations similar to those involved in the formulation of the inf-sup condition (2.26) then require \( V_h \subseteq W_h \). Consequently, if \( V_h \) restricted to each element is equal to \( P_k \), it is possible to choose \( W_h \) equal to \( P_k \) when restricted to each element as well. This is a convenient choice because the mass matrices to be inverted to compute the lifting operators are small; they involve only the DOFs local to the element.

However, in our case with non-local basis functions for the displacements, a constraint of the form \( \nabla \chi_{\Omega_h} \subseteq \Psi_h \) leads to severe technical difficulties. Under this constraint, it would be necessary for \( \Psi_h \) to contain the gradients of the enrichments functions \( \chi_{\Omega_E} \nabla \Psi_M \), \( M = I, II \). Because these functions have support over possibly many elements, the resulting mass matrices in \( \Psi_h \) can become very large and hence costly to invert when computing the lifting operators. In fact, since the number of elements in \( \Omega_E \) scales as \( h^{-2} \), the size of these matrices scale as \( h^{-2} \times h^{-2} \) as \( h \to 0 \).

A second possibility in defining \( \Psi_h \) would have been to include in it all functions of the form \( \chi_K \nabla \Psi_M \), for all \( K \in \mathcal{T}_h^E \) and \( M = I, II \), just as in PU methods. In this case the mass matrices involved in the computation of the lifting operators are small. However, their condition number grows as the mesh is refined, leading to severely ill-conditioned matrices and large numerical errors in the computation of the lifting operators. We illustrate this behavior with an example next.
Figure 19. Sequence of elements whose mass matrices become progressively ill conditioned as $h \to 0$, see text. The edge AB lies in $\Gamma_E$ for each $h$. Consequently, the mass matrix of each one of these elements is the one to be inverted to compute the value of the lifting operator.

Table IV. Condition number for the $14 \times 14$ mass matrix (for computing the lifting operator) of the element shown in Figure 19 for different values of $h$ when using $(P_1(K))^{2 \times 2}$, $\nabla \Psi_I$, and $\nabla \Psi_{II}$, as basis functions. Here we have set $\kappa = 1.8$ (corresponding to $\nu = 0.3$ for plane strain) and $\mu = 1/\sqrt{8\pi}$. A much better and mesh-independent conditioning of the mass matrix can be obtained by using basis functions in $(P_2(K))^{2 \times 2}$, as we have adopted in (2.22).

<table>
<thead>
<tr>
<th>$h$</th>
<th>Condition number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$9.0 \times 10^2$</td>
</tr>
<tr>
<td>$2^{-1}$</td>
<td>$2.0 \times 10^2$</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>$5.4 \times 10^0$</td>
</tr>
<tr>
<td>$2^{-3}$</td>
<td>$1.1 \times 10^3$</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>$1.9 \times 10^9$</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>$3.0 \times 10^{10}$</td>
</tr>
</tbody>
</table>

Consider a crack that coincides with the negative $x$-axis of a Cartesian coordinate system, with the crack tip located at the origin, see Figure 19. For each $h$, assume that there is a triangular element $K = \ast K \subset \Omega_h$ with nodes A $(-1, 0)$, B $(-1 + h/2, -1 - h/2)$, and C $(-1 + h, 0)$ such that the segment AB is always contained in $\Gamma_E$ as the mesh is refined, see Figure 19. In computing the lifting operators for this element with the space $V_h$ indicated in the last paragraph, it is necessary to invert the mass matrix of the basis made of the $P_1$ basis functions for each component of the displacement gradient field, or a total of twelve functions, together with the functions $\nabla \Psi_I$ and $\nabla \Psi_{II}$ restricted to this element. The condition numbers of this $14 \times 14$ elemental mass matrix for selected values of $h$ are listed in Table I, computed with material properties $\kappa = 1.8$ (corresponding to $\nu = 0.3$ for plane strain) and $\mu = 1/\sqrt{8\pi}$. The condition number of this sequence of mass matrices progressively deteriorates as $h \to 0$, a consequence of the fact that as the mesh is refined, the enrichment gradients over this element become progressively better approximated by linear combinations of the $P_1$ shape functions. This ill-conditioned mass matrix leads to a large error in the computation of the lifting operators.

Based on these considerations we adopted a solution that does not satisfy $\nabla V_h \subseteq \mathbb{V}_h$; instead we included quadratic shape functions in $\mathbb{V}_h$ in some elements of the mesh. Notice that in our case (I.1) is still satisfied, but $D_{DG} u_h$ as defined in (2.20) may not be its only solution.
II. On the asymptotic consistency error introduced by numerical integration

To further justify the quadrature strategy proposed in Section 2.6, we will examine the asymptotic consistency error introduced by certain numerical integration schemes, in particular the standard Gauss quadrature rules, for integrating stiffness matrix entries in which the integrand scales as $r^{-1}$ or $r^{-1/2}$, where $r$ is the distance to the crack tip.

The effect of numerical integration on the numerical scheme is to replace the bilinear form $a_h(\cdot, \cdot)$ defined in (2.24) by an approximate one $\tilde{a}_h(\cdot, \cdot)$, and likewise for the right hand side in (2.25). This induces a consistency error, so the exact solution does not satisfy (2.25) exactly when stated in terms of the approximate bilinear form $\tilde{a}_h(\cdot, \cdot)$ and the approximate right hand side. A sufficient condition for the resulting method to possess optimal convergence is that the consistency error induced by the numerical quadrature

$$e_h = a_h(u, v_h) - \tilde{a}_h(u, v_h)$$

must decay to zero at a certain rate, where $u$ is the exact solution and $v_h \in \mathcal{V}_h$ is a test function in the FE space. In the case in which $v_h$ is continuous, this requirement can be expressed as [37]

$$|e_h| \leq Ch|v_h|_{H^2(\Omega)}$$

where $C > 0$ is a mesh-independent constant that does not depend on $v_h$. The analogous expression for our space $\mathcal{V}_h$ admitting test functions discontinuous across $\Gamma_h^E$ is given by

$$|e_h| \leq Ch \left( \|\nabla v_h\|_{L^2(\Omega_h^E)} + \|\nabla v_h\|_{L^2(\Omega_h^E)} + \|R([v_h])\|_{L^2(\Omega)} \right).$$

In either case, the adoption of fixed-order quadrature rules cannot be expected to satisfy this requirement, as we demonstrate with an example next.

Consider a problem in which $\partial_\tau \Omega$ aligns with the mesh such that $K = *K$ for all $K \in \mathcal{I}_h$, and the crack tip $x_t$ coincides with some node $a$. Let $u = \Psi_I$ and $v_h = (\Psi_I - \hat{I}_h \Psi_I) \chi_{\Omega_h^E}$, where $\hat{I}_h$ is the classical interpolation operator such that $\Psi_I = \hat{I}_h \Psi_I$ at all nodes of $\mathcal{I}_h$.

The right hand side of (II.2) can be simplified based on a few observations. First, it is obvious that $\|\nabla v_h\|_{L^2(\Omega_h^E)} = 0$; and since $\Psi_I$ and its derivatives are smooth and bounded on $\Gamma_h^E$, $\|v_h\| = \|\Psi_I - \hat{I}_h \Psi_I\|_{\Gamma_h^E}$ is a small quantity whose $L^2$-norm scales as $h^2$. Therefore $\|R([v_h])\|_{\Omega_h^E}$ decays to zero as $h^{3/2}$, faster than $\|\nabla v_h\|_{L^2(\Omega_h^E)}$ does. In fact, since $\Psi_I \chi_{\Omega_h^E} \in (H^{3/2-\delta}(\Omega_h^E))^2$ for any $\delta > 0$, it can be shown that

$$\|\nabla v_h\|_{L^2(\Omega_h^E)} \leq Ch^{3/2-\delta},$$

for some $C > 0$ independent of $h$. As a result, (II.2) reduces to

$$|e_h| \leq Ch^{3/2-\delta}.$$
Since $\delta$ can be arbitrarily small, the consistency error induced by the quadrature should decay as
\begin{equation}
|e_h| \leq C h^{3/2}
\end{equation}
for this particular choice of functions $v_h$.

Now consider a sequence of meshes obtained by recursively subdividing each triangle into four similar ones, and let $K_h \in T_h$ be an element with the crack tip as one of its nodes (see Figure 20 as an example of a coarse mesh). The coordinates of the other two nodes can then be written as $(c_1 h, c_2 h)$ and $(c_3 h, c_4 h)$, $|c_i| \leq 1$, $i = 1, 2, 3, 4$ in the system of Cartesian coordinates local to the crack tip (see Figure 4). The contribution of $K_h$ to $a_h$ is
\begin{equation}
a_{K_h} = \int_{K_h} \nabla \Psi_I : \mathbb{C} : \nabla(\Psi_I - I_h \Psi_I) \, dS,
\end{equation}
where we have used that for this element the lifting operator is identically zero.

We then substitute (2.7) in (II.6), invoke the definition of $I_h \Psi_I$, and factor out $r$ to obtain
\begin{equation}
a_{K_h} = \int_{K_h} \left( \frac{\varphi_1(\theta)}{r} - \frac{\varphi_2(\theta)}{\sqrt{hr}} \right) \, dS,
\end{equation}
where $\varphi_1(\theta)$ and $\varphi_2(\theta)$ are trigonometric functions of $\theta$ depending on $\mu$, $\kappa$, and $c_i$, $i = 1, 2, 3, 4$, but independent of $h$.

We use the classical reference element $\hat{K}$ defined by the vertices with Cartesian coordinates $(\xi, \eta) = (1, 0), (0, 1), (0, 0)$ and let $F_{K_h} \hat{K} = K_h$. Then for any point $(\xi, \eta) \in \hat{K}$, the Jacobian $j = \det F_{K_h} = c_5 h^2$, $r = h \sqrt{c_6 \xi^2 + c_7 \eta^2}$, $\theta = \theta_K(\xi, \eta)$, where
\begin{align*}
c_5 &\equiv c_1 c_4 - c_2 c_3, \quad c_6 \equiv (c_1 - c_2)^2, \quad c_7 \equiv (c_3 - c_4)^2, \\
\theta_K(\xi, \eta) &\equiv \arctan \frac{c_2 \xi + c_4 \eta}{c_1 \xi + c_3 \eta}.
\end{align*}

We can rewrite the integral in (II.2) as one over the reference element
\begin{equation}
a_{K_h} = h \int_{\hat{K}} s(\xi, \eta) \, d\xi d\eta,
\end{equation}
where
\begin{equation}
s(\xi, \eta) \equiv c_5 \left[ \frac{\varphi_1(\theta_K(\xi, \eta))}{(c_6 \xi^2 + c_7 \eta^2)^{1/2}} - \frac{\varphi_2(\theta_K(\xi, \eta))}{(c_6 \xi^2 + c_7 \eta^2)^{1/2}} \right].
\end{equation}
Notice that the only dependence of $a_{K_h}$ on $h$ is through the prefactor.

The effect of a certain quadrature rule with $n_q$ sampling points is replacing $a_{K_h}$ by $\tilde{a}_{K_h}$ through a quadrature rule for $s(\cdot, \cdot)$ over $\hat{K}$:
\begin{equation}
\tilde{a}_{K_h} = h \sum_{l=1}^{n_q} s(\xi_l, \eta_l) w_l,
\end{equation}
where $(\xi_l, \eta_l) \in \hat{K}$, $\sum_{l=1}^{n_q} w_l = |\hat{K}|$.

Therefore, the contribution from element $K_h$ to the consistency error $e_{K_h} = a_{K_h} - \tilde{a}_{K_h}$ is given by
\begin{equation}
e_{K_h} = he_{\hat{K}},
\end{equation}
where

\[ e^K \equiv \int_{K} s(\xi, \eta) \, d\xi \, d\eta - \sum_{l=1}^{n_q} s(\xi_l, \eta_l) w_l. \]  

(II.12)

Notice that the value of \( e^K \) does not depend on \( h \). Additionally, if the chosen quadrature rule was designed to exactly integrate polynomials of a certain degree, the value of \( e^K \) will be different than zero for most values of \( c_i \), since the integrand \( s(\xi, \eta) \) is made of trigonometric and rational functions.

If we next assume that the integrals in the rest of the domain are performed exactly, then \( |e_h| = |e_{K_h}| = h e_K \), which is in clear contradiction with the consistency condition (II.5). The replacement of the exact integration in the rest of the domain by a quadrature rule does not change this conclusion, as the numerical example in Figure 21 shows.

We conclude by highlighting at least two strategies to overcome this problem:

1. Adopting an adaptive integration scheme such as the DECUHR[39] investigated by Xiao and Karihaloo [40].
2. Transforming the domain integration into a boundary integration as in Ventura et al.[19]. In this case, the involved integrands for computing \( a_h \) are always bounded and smooth as \( h \) tends to zero, and \( e_h \) can be controlled to decay to the required order. We adapted this method for our numerical scheme in Section 2.6.
Figure 21. Consistency errors defined in (II.1) resulting from adopting fixed-order quadrature rules as functions of $h$ for $u = \Psi_I$ as the exact solution and $v_h = (\Psi_I - \Pi_h \Psi_I) \chi_{\Omega_E^h}$ as a particular sequence of test functions (see the text) for some mesh family $\{T'_h\}$ whose coarsest mesh is shown in Figure 20. Finer meshes are obtained by recursively subdividing each triangular element into four similar ones. The consistency error for this choice of $v_h$ must converge to zero at a rate of $h^{3/2}$ for the method to converge optimally. For isolating the effects of the singularity, the enrichment region $\Omega_E^h$ is set such that $\Omega_E^h = \Omega$, i.e., every element in every mesh in $\{T'_h\}_h$ is enriched. The elastic constants are set to be $\lambda = \mu = 1$. The side length $b$ in Figure 20 is set to be $\sqrt{2}$. Numerical integrations over each triangular element are performed using the 6-point ($\times$), 12-point (O), and 33-point (□) Gauss integration rules for the reference triangle[38], respectively. While a higher order quadrature yields smaller errors, the order of the consistency error is the same for all these quadrature rules, and none of them meets the requirement for optimal convergence.