Adaptive stabilization of discontinuous Galerkin methods for nonlinear elasticity: Analytical estimates

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Abstract

This is the second of two papers in which we motivate, introduce and analyze a new type of strategy for the stabilization of discontinuous Galerkin (DG) methods in nonlinear elastic problems. The foremost goal behind it is to enhance the robustness of the method without deteriorating the accuracy of the resulting solutions. Its distinctive property is that for nonlinear elastic problems the stabilization term is solution dependent, and hence it is termed an adaptive stabilization mechanism. The key contribution of this paper is the construction of a stabilization strategy for which the method is perfectly stable, since the stabilization parameters can be explicitly computed as part of the numerical solution. This is accomplished through the main result of this paper, which consists of a theorem that provides lower bounds for the size of the stabilization parameters. Numerical examples confirm the guaranteed stability of the resulting method. However, they also show that the computed lower bounds overestimate the minimum amount of stabilization needed, negatively affecting the approximation properties of the method. This results underscores the fact that a better understanding of the stabilization mechanisms is needed in order to construct a method that is both robust and efficient.

Key words: discontinuous Galerkin methods, nonlinear elasticity, stabilization, adaptivity.

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1. Introduction

In this paper we motivate, introduce and analyze a new type of strategy for the stabilization of discontinuous Galerkin (DG) methods in nonlinear elastic problems, extending the ideas introduced in [19]. The foremost goal behind it is to enhance the robustness of the method without deteriorating the accuracy of the resulting solutions. Its distinctive property is that the stabilization term depends on the solution, and hence it is termed an adaptive stabilization mechanism.

In [9] a class of DG methods for nonlinear elasticity was introduced and a standard stabilization term, consisting in a quadratic penalization of the discontinuities on the element boundaries, was adopted. A single stabilization parameter scales the magnitude of this term and controls the size of the discontinuities in the numerical solution. The effects of this and other types of stabilization terms on the performance of several DG methods for linear elliptic problems was analyzed in [8]. The numerical examples therein demonstrate that the discontinuities are in fact used to enhance the accuracy of solutions. The same results were observed for nonlinear elasticity problems in [9] and [19]. In fact, when this is the case DG approximations may render solutions that are comparable in accuracy with a conforming method of the same order. However, the larger the stabilization parameter, the smaller the discontinuities in the numerical solution. If a very large value for the stabilization term is adopted the resulting solutions are nearly conforming, and naturally the performance of the method is no longer attractive.

While it can be argued that this trait is common to many DG approximations for a variety of different problems, it is particularly exacerbated in the context of nonlinear elasticity, since the coercivity constant responsible for the linearized stability of the problem depends on the solution. When computing a sequence of nonlinear elastic deformations through a prescribed loading path, the value of the stabilization parameter needed is essentially determined by the least linearly stable configuration. If not carefully chosen, this value may lead to overpenalized discontinuities for most other configurations. Since it is challenging and costly to determine what a suitable amount of stabilization is, it is convenient to design a stabilization strategy for which the approximation properties of the method are substantially less sensitive to the choice of the stabilization parameter, and that performs well for a wide range of nonlinear elastic deformations.

Discontinuous Galerkin formulations for linear elasticity have been presented and analyzed in [17,18,13,10,12,21], while [15,7,16] have done so for nonlinear elastic problems. By taking advantage of the minimum potential energy principle that defines the algorithm in [9], Ortner and Buffa [7] elegantly proved the convergence of a slight generalization of this method for convex nonlinear strain energy densities. Key to their result was the modification of the stabilization term so that it has similar growth properties as the strain energy density. In this way, they could prove that sets of discontinuous Galerkin functions with bounded potential energy are compact. This is in fact a statement of stability for the algorithm, which implies that the resulting discontinuities may be very large, but will be bounded. By experience, however, we found that large discontinuities are often related to large errors, and finer meshes may be needed to reach an acceptable solution.

In [19] we formulated an adaptive stabilization strategy for the method in [9], and demonstrated its performance through numerical examples. Therein, however, we heuris-
tically determined appropriate values for the stabilization parameters which led to good approximation properties for a wide range of nonlinear elastic deformations. This proved to be fairly simple to do, often accomplished in very few trials. This is in sharp contrast with uniform stabilization strategies, for which the quality of the approximation changes more markedly with the stabilization parameter. We provide evidence of this behavior in Section 4 of this manuscript, through a numerical example in linearized elasticity. Ideally, however, it would be desirable to construct a strategy in which the values of the stabilization parameters are automatically determined.

This is what we do in this manuscript. We construct a stabilization term for which estimates for the lower bounds of the stabilization parameters are obtained, and such that these can be computed for any given problem. This last trait is essential, since the mere existence of such lower bounds is easily obtained in a number of different ways. In other words, it is not difficult to show that the method will be stable provided the stabilization term is large enough. However, it is difficult to determine exactly how large it should be. The method provided in this paper is guaranteed to be robust no matter how the structure is loaded. However, when we tested the performance of this new strategy, the numerical results show that the lower bounds generally overestimate the amount of stabilization needed, to the extent that discontinuities are not well exploited to enhance the accuracy of the approximation. These numerical examples clearly manifest an important weakness that may be shared by other DG methods for elliptic problems in the literature, namely, the fact that the values of the stabilization parameter for which stability can be proved may be too large for the method to be efficient. This is to our knowledge the first attempt to utilize rigorous estimates of the size of the stabilization term for this type of problems, and it is the key contribution of the current manuscript. Additionally, the type of stabilization term considered here shares many traits with that in [19]. In fact, with a minor modification of the proof of the main theorem we obtain the stability of this latter strategy.

The formulation of the adaptive stabilization strategy for nonlinear elasticity relies on a stability analysis we perform for the linearized elasticity problem. What distinguishes this analysis from that for classical linear problems, such as the diffusion equation [6] or the stress-free linear elastic case of [13], is the fact that the elastic moduli may have negative eigenvalues in a part of or in the entire domain, whereas the bilinear form that defines the problem may still be coercive in, for example, $H^1_0$. As the numerical examples in [19] suggest, the regions of the domain where these eigenvalues are negative are most often the ones in which signs of incipient instability in the numerical solution are found. These entail large discontinuities and relative rotations between neighboring elements. If the location of incipient instability happens to be restricted to a small region of the domain, it is natural to more strongly penalize the appearance of discontinuities there than elsewhere. This was the motivation for the adaptive stabilization strategy we proposed in [19], and this idea naturally emerges in the proof of the main theorem in this manuscript. It is also what distinguishes this strategy from that in [5] for Darcy flow, in which the spatially varying stabilization term is scaled with the local value of the permeability coefficient.

In studying the stability of a DG scheme for linearized nonlinear elasticity it is important to notice that the bilinear form that defines the exact problem need not be coercive. In fact, those configurations at which the linearization renders an indefinite bilinear form are often of interest, since they are elastically linearly unstable. Consequently, we should
not expect the bilinear form of the DG scheme to be coercive in these cases. Our main result in Sec. 5.3 provides lower bounds for the stabilization parameters of the DG scheme to be stable, provided the exact linearized problem is elastically linearly stable. Moreover, the numerical examples in [19] demonstrate that if the coercivity constant for the DG method is positive but small, clear signs of incipient instability are found in the numerical solutions. Therefore, the theorem in Sec. 5.3 provides a mean to guarantee a lower bound for the coercivity constant of the method through the stabilization parameters.

The idea behind the proof is similar to that used in [7,16]. We first define a projection operator that to any given piecewise discontinuous function in the DG space assigns a continuous one. We can then split the bilinear form of the problem into a part that depends only on this continuous projection, and another one that is identically zero over continuous functions. The coercivity of the former is then guaranteed if the exact linearized problem is elastically linearly stable, while the coercivity of the latter can be easily asserted if enough penalization on the size of the discontinuities is added.

The proposed stabilization strategy is then extended to the nonlinear elastic case in Sec. 5.5 by recourse to an incremental variational principle at each step of a loading path, as done in [19].

A numerical example is shown in Sec. 6, which compares the error as a function of the total number of degrees of freedom between the DG scheme and a competing conforming one. The resulting method is, as expected, perfectly robust, i.e., no numerical instabilities appear. As noted earlier, however, the numerical example shows that the lower bounds for the stabilization parameters we obtained are too conservative. This overestimation tends to produce nearly conforming approximations, and hence while the method is robust and automatic, it shows a relatively poor computational performance.

We begin the paper with a brief review of the nonlinear elasticity problem in Sec. 2, and of the linearized nonlinear elasticity problem in Sec. 2.1. We introduce the basic framework for a DG discretization, such as the definition of the DG-derivative and the numerical fluxes used, in Sec. 3. A numerical example motivating the need for alternative stabilization strategies, and underlining the need for an automatic scheme, is shown in Sec. 4. The stabilization strategy and the main result are presented in Sec. 5, followed by the numerical example in Sec. 6.

Henceforth boldface symbols, such as $\varphi$ and $F$, represent tensorial quantities. Their components with respect to a Cartesian basis in $\mathbb{R}^d$ are denoted as $\varphi_i$, $F_{ij}$, respectively. The number of subscripts reflects the order of the tensor, and lower case subscripts indicate components in the spatial configuration while upper case subscripts indicate components in the reference configuration. With $\| \cdot \|$ we indicate the Euclidean norm of a vector or a second-order tensor, $\| \cdot \|_{k,\Omega}$ indicates the $H^k(\Omega)$-norm, and $| \cdot |_{k,\Omega}$ indicates the $H^k(\Omega)$-seminorm.

2. Nonlinear elasticity problem

Let $B_0 \subset \mathbb{R}^d$ be an open, bounded and connected polyhedral domain. The nonlinear elasticity problem consists in finding $\varphi \in V$, $\varphi : B_0 \rightarrow \mathbb{R}^d$, which is a stationary point of the potential energy functional

$$I[\varphi] := \int_{B_0} W(\nabla \varphi) \, dV - \int_{\partial B_0} T : \varphi \, dS$$

(2.1)
in $V$. Here $V \subset W^{1,1}(B_0; \mathbb{R}^d)$ is the set of all admissible deformation mappings $\varphi$. Functions in $V$ should satisfy Dirichlet boundary conditions $\varphi = \mathcal{B}$ on $\partial dB_0 \subset \partial B_0$, where $\partial dB_0$ has positive (Hausdorff) measure. Neumann-like boundary conditions in the form of prescribed surface tractions $T$ are applied on $\partial T B_0 = \partial B_0 \setminus \partial \tau B_0$. The function $W : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ returns a scalar non-negative value, it is known as the strain energy density of the material, and defines its mechanical response. To prevent a local interpenetration of matter, it is customary to consider functions for which $W(F) = +\infty$ if $\det F \leq 0$.

For simplicity we shall henceforth assume that $T = 0$, since the conclusions of the paper carry over to the general case. For a more comprehensive discussion about this problem we refer the reader to [14].

The existence of solutions to the nonlinear elasticity problem depends crucially on the properties of $W$, the choice of boundary conditions, and the selection of the space $V$ (see e.g., [2]). We shall not worry about these difficult issues herein, and simply assume that we are concerned with particular problems for which solutions exist and are smooth.

### 2.1. Linearized nonlinear elasticity problem

The linearized nonlinear elasticity problem at a deformation $\varphi$ consists in finding $u \in [H^1_0(B_0)]^d$ such that

$$
\mathbf{B}(u, v) := \int_{B_0} \nabla u : \mathcal{A}(\nabla \varphi) : \nabla v \, dV = \int_{B_0} f \cdot v \, dV, \tag{2.2}
$$

for all $v \in [H^1_0(B_0)]^d$. Here

$$
H^1_0(B_0) = \{ u \in H^1(B_0) : u|_{\partial dB_0} = 0 \}, \tag{2.3}
$$

the body force per unit volume is given by $f \in [L^2(B_0)]^d$ and

$$
\mathcal{A}(F) = \frac{\partial^2 W}{\partial F^2}(F) \tag{2.4}
$$

denotes the fourth-order tensor of elastic moduli, a function of the spatial position through the value of $\nabla \varphi$. Henceforth we shall assume that $W$ is smooth, and hence its second derivatives in Eq. (2.4) commute and $\mathcal{A}$ possesses major symmetries, namely, $A_{iJkL} = A_{kLij}$. In the special case in which $\varphi$ is stress free, then $\mathcal{A}$ also has minor symmetries, i.e., $A_{iJkL} = A_{JikL}$. This type of problems are encountered in approximating small deformations of the body from a configuration $\varphi$ under small perturbations of the loads or boundary conditions. In fact, under appropriate smoothness assumptions, the bilinear form for this problem is precisely the second variation of the potential energy $I(\varphi)$.

If $\|\mathcal{A}(\nabla \varphi(X))\| \in L^\infty(B_0)$, the domain $B_0$ has a smooth boundary ($C^1$), and $\partial dB_0 \equiv \partial B_0$, then the linearized nonlinear elasticity problem has a unique solution in $[H^1_0(B_0)]^d$ provided the bilinear form $\mathbf{B}$ is coercive there (see, e.g., Thm. 4.2 in [20]), i.e., there exists $\kappa > 0$ such that

$$
\int_{B_0} \nabla u : \mathcal{A}(\nabla \varphi) : \nabla u \, dV \geq \kappa \| u \|_{1, B_0} \tag{2.5}
$$

for all $u \in [H^1_0(B_0)]^d$. Furthermore, if the elastic moduli $\mathcal{A}$ and the domain are smooth enough then $u \in [H^2(B_0) \cap H^1_0(B_0)]^d$, see [20].
As discussed in [9], the elastic moduli $\mathbf{A}$, when regarded as linear operators in $\mathbb{R}^{d \times d}$, cannot be expected to be positive definite so that the coercivity of $\mathbf{B}$ can be guaranteed. In fact, the value of

$$\lambda_{\text{min}}(X) = \min_{0 \neq g \in \mathbb{R}^{d \times d}} \frac{\mathbf{A}(\nabla \varphi(X)) : g}{g : g}$$

(2.6)

may be negative everywhere in the domain and the problem still be coercive.

3. DG Discretization

Let $T_h$ be a conforming triangulation of the domain $B_0$. For simplicity, each element $E \in T_h$ is assumed to be a $d$-simplex, i.e., a triangle when $d = 2$ and a tetrahedron when $d = 3$. Let $e$ denote an arbitrary face of an element $E$, and set $\Gamma := \cup_{E \in T_h} \partial E$, the union of all element faces in the triangulation. We assume that $\Gamma$ can be decomposed into three mutually exclusive subsets, namely,

$$\Gamma^\circ = \{ e \subset \partial E \setminus \partial B_0 : E \in T_h \},$$

$$\Gamma^d = \{ e \subset \partial E \cap \partial B_0 : E \in T_h \},$$

$$\Gamma^\tau = \{ e \subset \partial E \cap \partial \tau B_0 : E \in T_h \},$$

such that $\Gamma = \Gamma^\circ \cup \Gamma^d \cup \Gamma^\tau$. Since the mesh was assumed to be conforming, each face $e \in \Gamma^\circ$ is shared by precisely two elements, which we name as $E^+$ and $E^-$. In other words $e = \overline{E^+} \cap \overline{E^-}$. The unit normal vector $\mathbf{N}$ to the face $e \in \Gamma^\circ$ is chosen to be the unit outward normal to $E^-$. For faces $e \in \Gamma \setminus \Gamma^\circ$, $\mathbf{N}$ simply denotes the unit outward normal to $B_0$.

In the following we will consider a quasi-uniform family of triangulations $\{T_h\}_h$ with $h \downarrow 0$, where

$$h = \max \{ \text{diam } B_E : E \in T_h \}. \quad (3.1)$$

Such a family of triangulations satisfies that there exist positive constants $\rho^-$ and $\rho^+$ such that

$$\rho^- h^d \leq |E| \leq \rho^+ h^d \quad (3.2)$$

for every element $E \in \{T_h\}_h$, where $|E|$ is the measure of $E$.

Next we introduce the finite element spaces

$$V_h = \{ v \in L^2(B_0) : v|_E \in P^k(E) \ \forall E \in T_h \} \quad (3.3)$$

$$W_h = \{ w \in L^2(B_0) : w|_E \in P^\ell(E) \ \forall E \in T_h \}, \quad (3.4)$$

for $1 \leq k \leq \ell$, where $P^k(E)$ is the space of polynomials of degree at most $k$ on $E$. Functions in $V_h$ and $W_h$ are allowed to have discontinuities across element interfaces belonging to $\Gamma^\circ$. The fact that $k \leq \ell$ guarantees that $V_h \subset W_h$ and the compatibility condition $\nabla V_h \subset W_h^d$, which together are sufficient conditions for the resulting method to have only displacements as unknowns [13]. The DG space $V_h^d$ will be used to approximate the deformation mapping $\varphi$, and the space $W_h^{d \times d}$ will be used to approximate its gradient. Since every element in $\{T_h\}_h$ is a $d$-simplex, each element is affine-equivalent to a single reference element $\hat{E}$.

The definition of the DG methods later in the manuscript are greatly facilitated by the introduction of the DG-derivative operator. To this end, we define the lifting operator $\mathbf{R} : L^2(\Gamma) \rightarrow W_h^d$ so as to satisfy that
\[
\int_{\Omega} \mathbf{R}(v) \cdot \mathbf{z} \, dV = -\int_{\Gamma} v \otimes \mathbf{N} \cdot \{ \{ \mathbf{z} \} \} \, dS,
\]

for all \( \mathbf{z} \in W_h^d \). The average and jump operators used here for functions \( v \in V_h \) and \( \mathbf{z} \in W_h^d \) are defined as follows. For a face \( e \in \Gamma^\circ \), let

\[
[v] = v^- - v^+ \quad \text{and} \quad \{ \{ \mathbf{z} \} \} = \frac{1}{2}(\mathbf{z}^- + \mathbf{z}^+),
\]

The superscripts + and − indicate that the functions are evaluated from within \( E^+ \) and \( E^- \), respectively, where \( e = \overline{E^+} \cap \overline{E^-} \). For a face \( e \in \Gamma \setminus \Gamma^\circ \), the traces of the functions \( v \) and \( \mathbf{z} \) are single-valued. For faces \( e \in \Gamma^d \) we define

\[
[v] = 0 \quad \text{and} \quad \{ \{ \mathbf{z} \} \} = \mathbf{z}
\]

whereas for faces \( e \in \Gamma^\tau \) we define

\[
[v] = v \quad \text{and} \quad \{ \{ \mathbf{z} \} \} = \mathbf{z}
\]

and leave \( \{ v \} \) and \( [ \mathbf{z} ] \) undefined, as they are not needed later on. The DG-derivative operator \( D_{DG} : V_h \rightarrow W_h^d \) we shall be concerned with in this paper is then defined as

\[
D_{DG} v = \nabla v + \mathbf{R}( [ [ v ] ] ) + \mathbf{R}( [ v ] ),
\]

with the choice of

\[
\tilde{\varphi}_{\partial}(\varphi) = \begin{cases} 
0 & \text{on } \Gamma^\circ \\
\varphi & \text{on } \Gamma^d \\
\varphi & \text{on } \Gamma^\tau
\end{cases}
\]

as the numerical traces. Finally, we note that the operator \( D_{DG} \) is defined only for scalar functions in \( V_h \), however, it is easily generalized to vector fields in \( V_h^d \) by applying it to each one of the Cartesian components, see [9].

A key property of the lifting operator \( \mathbf{R} \) is that for a quasiuniform family of triangulations \( \{ T_h \} \) there exist constants \( C^- \) and \( C^+ \) independent of \( h \) such that

\[
C^- \| \mathbf{R}( [ [ v_h ] ] ) \|_{0, \mathcal{B}_0} \leq \frac{1}{h} \| v_h \|_{0, \Gamma} \, dS \leq C^+ \| \mathbf{R}( [ [ v_h ] ] ) \|_{0, \mathcal{B}_0}
\]

for any \( v_h \in V_h \), a fact essentially stated and proved in [5]. Equation (3.8) follows after a trivial modification of the proof therein to include the jumps on \( \partial\Omega \) and the definition of the lifting operator in our setting.

Remark: The definition of \( D_{DG} \) provided here is not the most general one; it is only the one we shall use in here. We refer the interested reader to [9,1] for the most general case. Also, for notational convenience we have adopted a slightly different definition for the jump on \( \Gamma^\tau \) than the one used in [9]. Both formulations are exactly equivalent.

4. Motivation for adaptive stabilization

In this section, we provide a numerical example that gives a glimpse into the challenges that arise when using DG for nonlinear elasticity problems, and some of the benefits that can be obtained when using a nontraditional form of stabilization. In this example we consider a linearized elasticity problem with homogeneous boundary conditions whose bilinear form is coercive in \( [ H^1_0(\mathcal{B}_0) ]^2 \), and hence a standard conforming method will be as well, but such that a DG-discretization is only stable if properly stabilized.
Two DG-discretizations of the linearized elasticity problem are considered next. The two schemes are based on that in [9], but differ in the stabilization term. First, a formulation with a standard stabilization term. Its bilinear form is

$$B_{\text{standard}}(u_h, v_h) = \sum_{E \in T_h} \left[ \int_E D_{\text{DG}} u_h : \mathbb{A} : D_{\text{DG}} v_h \; dV + \beta \int_E \mathbb{R}([u_h]) : \mathbb{R}([v_h]) \; dV \right]. \quad (4.1)$$

for some $\beta > 0$. The second alternative contains the adaptive stabilization term from [19], given by the bilinear form

$$B_{\text{adaptive}}(u_h, v_h) = \sum_{E \in T_h} \left[ \int_E D_{\text{DG}} u_h : \mathbb{A} : D_{\text{DG}} v_h \; dV + \beta \int_E \left( \lambda_X + \epsilon \right) \mathbb{R}([u_h]) : \mathbb{R}([v_h]) \; dV \right], \quad (4.2)$$

for some $\beta > 0$ and $\epsilon > 0$. For a point $X \in B_0$, the quantity $\lambda_X$ is defined as

$$\lambda_X := \max \left\{ 0, - \min_{0 \neq g \in \mathbb{R}^d} \frac{g : \mathbb{A}(X) : g}{g : g} \right\}. \quad (4.3)$$

We shall return to $\lambda_X$ in section 5.

In both cases, the discrete problem consists in finding $u_h \in V_h$ such that

$$B_x(u_h, v_h) = \int_{B_0} f \cdot v_h \; dV$$

for all $v_h \in V_h$, where $B_x$ stands for either one of the bilinear forms in equations 4.1 or 4.2.

The idea behind the stabilization term in equation (4.2) is to penalize discontinuities in the regions of the mesh where eigenvalues of $\mathbb{A}(X)$ are negative. The introduction of such a term was motivated with numerical examples in [19], since for many problems the locations where the discontinuities were most highly visible coincided with regions where $\lambda_X$ was positive. If $\mathbb{A}(X)$ contains only non-negative eigenvalues, the adaptive stabilization term reduces to the standard stabilization term.

The problem we study next qualitatively represents a typical linearized elasticity problem that may be obtained by considering small perturbations to a configuration of a nonlinear elastic body. The domain of the problem is the square $B_0 = [0, 1]^2$, and it is fixed all along $\partial B_0$. The body is loaded by a smooth body force defined by

$$f = -0.05 \sin(\pi X_1) \sin(\pi X_2) e, \quad (4.4)$$

where $e$ is a unit vector in the $X_1$ direction.

The elastic moduli are given by

$$A_{ijkl}(X) = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{jk} \delta_{li} + \delta_{ik} \delta_{jl} \right) - p(X) \delta_{ik} \delta_{jl}, \quad (4.5)$$

which corresponds to an isotropic linear elastic materials. In the following we have adopted $\lambda = 1.4$ and $\mu = 0.36$. The compressive isotropic stress (pressure) $p(X)$ is included to represent the fact that the configuration at which the linearized problem is obtained may be stressed. The elasticity tensor becomes nondefinite when $p(X) > 0$. To illustrate a case in which the negative eigenvalues of $\mathbb{A}$ are restricted to a small part of the domain, we adopted

$$p(X_1, X_2) = 0.30 \sin(2\pi(X_1 - 0.25)) \sin(2\pi(X_2 - 0.25)), \quad (4.6)$$
for $X \in [0.25, 0.75] \times [0.25, 0.75]$.

Under these circumstances we have that $\lambda_X = -p(X)$; it is enough to notice that $A(X) : \omega = -p(X) : \omega$ for any skew-symmetric tensor $\omega$. A contour plot of $\lambda_X$ is shown in Fig. 1(a). For the chosen material properties and boundary values, the linearized problem is coercive for all $u \in [H^1_0(B_0)]^2$. This follows from

$$
\int_{B_0} \nabla u : A(X) : \nabla u \, dV \geq 2\mu|\mathbf{e}|^2_{0,B_0} - \|p(X)\|_{\infty,B_0}|u|^2_{1,B_0} \geq (\mu - \|p(X)\|_{\infty,B_0})|u|^2_{1,B_0},
$$

(4.7)

where we $\mathbf{e}$ is the symmetric part of $\nabla u$. The second line in equation (4.7) follows from Korn’s inequality for vector fields that are zero on the boundary of the domain, which in this case reads $2|\mathbf{e}|^2_{0,B_0} \geq |u|^2_{1,B_0}$ for any $u \in [H^1_0(B_0)]^2$, see [11]. In contrast, when $\beta = 0$ the bilinear forms (4.1) and (4.2) are generally not coercive in $u_h \in V_h^2$, a fact that can be checked numerically for each choice of $V_h^2$. If a large enough value for $\beta$ is adopted, either $B_{\text{standard}}$ or $B_{\text{adaptive}}$ are coercive and good approximations are obtained. One such solution with $B_{\text{standard}}$ is shown in Fig. 1(b), which depicts the isocontours of the $F_{22}$-component of the deformation gradient, $F_{ij} = \delta_{ij} + (u_h)_{i,j}$. Finally, notice that for this problem the bilinear form $B_{\text{adaptive}}$ will more severely penalize discontinuities in the central region of the domain, where $\lambda_X > 0$.

![Contour plot of $\lambda_X$](image)

(a) Contours of $\lambda_X$, which is greater than zero only in $[0.25, 0.75]^2$.

![Component of $F_h$](image)

(b) The $F_{22}$ component of $F_h$ obtained from a DG approximation with standard stabilization.

Fig. 1. Model elasticity problem with homogeneous Dirichlet boundary conditions and a smoothly varying body force. The elasticity tensor possesses negative eigenvalues in $[0.25, 0.75]^2$, but the bilinear form of the exact problem is still coercive. Nevertheless, this problem is challenging to solve when its solution is approximated with DG, unless properly stabilized.

The idea behind the following numerical experiments is to show the dependence of the accuracy of solutions on the choice of the stabilization parameters. Out of two different methods, we say that one is more robust than the other if it displays a lesser dependence of the solution error on the choice of the stabilization parameter.

We examined three different cases: (a) standard stabilization, in which we adopted $B_{\text{standard}}$ for the computations and, (b) and (c), in which solutions were obtained with $B_{\text{adaptive}}$ and the choices of $\epsilon = 0.0$ and $\epsilon = 0.1$, respectively. In all cases we compared
the $L^2(B_0)$ error of the solution for $u_h$ and $\nabla u_h$ obtained with a mesh with element size $h = 1/15$, relative to a very fine DG approximation with standard stabilization for which $h = 1/100$ and $\beta = 1$ (shown in Fig.1(b)).

The results of the experiment are depicted in Fig. 2, which shows the accuracy of solutions as a function of the stabilization parameter $\beta$. As evidenced therein, the method with standard stabilization is rather sensitive to changes in $\beta$. In the case of the displacements we observe that by decreasing $\beta$ the $L^2(B_0)$-error decreases by 75%. Any further decrease in $\beta$ resulted in a loss of coercivity. In the case of the displacement gradients this difference amounts to over 30%. Figure 2 also shows that for $\beta > 10^3$ there is very little change in the error of the solutions, signaling the beginning of a regime in which the resulting approximations have virtually no discontinuities and strongly resemble a conforming method. At this point, all potential benefits of a DG approximation over a conforming one have been lost.

![Graph showing solution accuracy as a function of the stabilization parameter $\beta$.](image)

(a) $L^2(B_0)$-error in displacements, normalized by its maximum. (b) $L^2(B_0)$-error in the displacement gradients, normalized by its maximum.

Fig. 2. Solution accuracy as a function of the stabilization parameter $\beta$. Three cases are considered, one with standard stabilization, Eq. (4.1), and two with an adaptive stabilization scheme, Eq. (4.2), which differ on the amount of standard stabilization they include ($\epsilon = 0$ and $\epsilon = 0.1$, respectively). The results show that standard stabilization schemes are rather sensitive to the choice of $\beta$, at least in contrast with a purely adaptive scheme ($\epsilon = 0.0$), for which the sensitivity is greatly reduced.

A very different behavior is observed when the adaptive stabilization term is adopted with $\epsilon = 0$. For small values of $\beta$, the $L^2(B_0)$-error in displacements is only slightly greater than that obtained with the method with standard stabilization. However, for large values of $\beta$ the adaptive scheme provides solutions that are more than twice as accurate. A similar behavior is observed for the displacement gradients, in which the difference is less marked, but for which the adaptive strategy provides nearly insensitive approximations to the value of $\beta$. As expected when $\epsilon = 0.1$, the performance carries traits of both previous cases, displaying a larger range of values of $\beta$ for which solutions are still fairly accurate than the method with standard stabilization. It also displays the best approximations of the deformation gradient for some values of $\beta$, albeit only by a thin margin.

The reason behind the robustness displayed by the adaptively stabilized method is that discontinuities are only penalized on a small subset of $B_0$, only locations where $\lambda_X > 0$. For this example, this is enough to recover stable approximations.
These results are qualitatively representative of extensive numerical studies performed with adaptive stabilization. The importance of a robust stabilization strategy is magnified in the context of nonlinear elastic problems [19]. In this case it is often necessary to solve a sequence of quasistatic nonlinear elasticity problems throughout a prescribed loading path. At each step of the loading program the elastic moduli $A$ change, and so does the set in $B_0$ for which $\lambda_X < 0$. In these circumstances our experience indicate that it is generally very challenging to find a value of $\beta$ for a standard stabilization term that would render the method linearly stable while simultaneously retaining good accuracy. In contrast, this task was considerably simpler when the adaptive stabilization strategy was adopted. The key reason behind this observation is embedded in Fig. 2: the accuracy of solutions is much less dependent on the choice of stabilization parameter in this case.

Even though this adaptive stabilization strategy has greatly enhanced our ability to reliably and automatically simulate nonlinear elastic problems, we would like to construct a completely automatic stabilization scheme, which we do next. The foremost objective is to be able to prescribe values of $\beta$ for which the method is guaranteed to be stable; it is not enough to just conclude that the method will be stable for large enough $\beta$. We shall see that in constructing such scheme traits of the adaptive stabilization term showed here naturally appear. Of course, a key feature of any automatic stabilization method should be to avoid overpenalizing the discontinuities so that these can be used to improve the accuracy of the approximation. While we succeed in constructing an automatic scheme, we do not succeed in this later goal. This remains an open research question.

5. Adaptive stabilization with analytical lower bounds

We next turn to building a novel stabilization strategy characterized by the facts that: a) it is adaptive and, b) it prescribes values of the stabilization parameters that can be computed and for which the resulting method is guaranteed to be linearly stable.

This new stabilization mechanism departs from the one in [9] in three essential aspects. First, we employ a projection operator which maps a piecewise discontinuous function in $V_h$ to a continuous function over $B_0$. The objective in using such a projection is to inherit the coercivity properties of the continuous problem, namely, Eq. (2.5). Second, the stabilization parameters change from element to element. Finally, the stabilization is adaptive, which means that throughout the loading process the values of the stabilization parameters and the stabilization term within each element are automatically chosen so as to preserve the overall coercivity of the problem. This adaptive strategy follows naturally from the forthcoming analysis, which shows that the values of the stabilization parameters are strongly dependent on the minimum eigenvalue of the elasticity tensor $A$. As the loading of the body evolves, the value of $A$ throughout the mesh changes, and the method automatically picks larger values for the stabilization parameters in the elements in which numerical instabilities are more likely to occur. In this sense, this method shares many features with the one in [19] and discussed in Sec. 4, designed precisely to avoid the occurrence of such numerical pathologies.

The rest of this section is organized as follows. In Sec. 5.1 we introduce the projection operator mentioned above. Then, we describe a DG method for the linearized nonlinear elasticity problem with adaptive stabilization. In Sec. 5.3 we state our main theorem which provides sufficient conditions for the stability of this method, and later we give
a rigorous proof of the theorem. We later extend the above method to the nonlinear elasticity problem by recourse to an incremental variational principle.

5.1. The projection operator

We introduce next a projection operator that maps a piecewise discontinuous function in $V_h$ to a continuous one over the entire domain $B_0$. Let $C_h \subset V_h$ be the space of all continuous functions in $V_h$ that are zero on $\Gamma_d$, and let $P : V_h \rightarrow C_h$ be a linear projection operator, $P \circ P = P$, such that $P v = v$ for all $v \in C_h$. The definition of $P$ on vector fields in $V^d$ is trivial by just applying $P$ componentwise.

For example, if $V_h$ is the space of elementwise linear functions in $T_h$, then $C_h := \{ v \in H^1(B_0) : v_E = v|_E \in P^1(E) \; \forall E \in T_h, v|_{\Gamma_d} = 0 \}$.

Following Brenner [3,4], we define the map $P : V_h \rightarrow C_h$ as follows. Let $V(E)$ be the set of all vertices of an element $E \in T_h$ that do not belong to $\Gamma_d$. Then $P v \in C_h$ is defined by
\[
(Pv)(p) = \frac{1}{|\chi_p|} \sum_{E \in \chi_p} v_E(p) \quad \text{for all } p \in V(E),
\]
(5.1)

where $\chi_p = \{ E \in T_h : p \in \partial E \}$ is the set of elements sharing $p$ as a common vertex, and $|\chi_p|$ is the cardinality of $\chi_p$.

The operator $P$ sends a discontinuous function to a continuous one by averaging at each node over all neighbors. When restricted to $C_h$, $P$ coincides with the identity operator and hence $P^2 = P \circ P = P$, from where it follows that $P$ is a projection onto $C_h$. Of course, the averaging technique we employ in (5.1) is not the only possibility.

Definition (5.1) is also valid for Lagrange finite elements with $P^k$ polynomials, provided $C_h$ is defined as the set of continuous functions over $T_h$ that consists of $P^k$ polynomials in each element that are zero on $\Gamma_d$. Similar projection operators were introduced in and analyzed by Ortner and Suli [16] and Buffa and Ortner [7] for the purposes of analyzing certain class of DG methods for nonlinear second-order elliptic problems, and in [21]. In fact, it follows from [16] and [4] that for a quasiuniform family of triangulations there exists a constant $C$ independent of $h$ such that
\[
\sum_E |u - P u|^2_{1,E} \leq C \int_{\Gamma} h^{1-2k} |[u]|^2 \; dS
\]
(5.2)
for all $u \in V_h$.

5.2. The method for the linearized nonlinear elasticity problem

The starting point behind the construction of a discretization of the linearized nonlinear elasticity problem with analytical lower bounds consists in using the projection operator to decompose any function in $v \in V_h$ as a sum of two functions, $v = P v + (v - P v)$, where $P v$ is continuous over the entire body $B_0$. The essential idea is to inherit the coercivity of the continuous problem for $P v$ and then observe that, since the remaining terms in the bilinear form depend linearly and quadratically on the discontinuous part $(v - P v)$, these can be stabilized by penalizing the discontinuities.
Let $\varphi_h \in V^d$ be a deformed configuration, with corresponding DG-derivative $F = D_{DG}\varphi_h \in W^d_{DG,d}$, such that $A(F(X))$ is essentially bounded in $\cup_{E \in T_h} E$. The proposed discretization of the linearized nonlinear elasticity problem at $\varphi_h$ consists in finding $u_h \in V^d_h$ such that

$$B_h(u_h, v_h) = \int_{B_0} f \cdot v_h \, dV_0$$

for all $v_h \in V^d_h$, where the bilinear form $B_h$ is given by

$$B_h(u, v) = B_{h,1}(u, v) + B_{h,2}(u, v) + B_{h,3}(u, v)$$

for $u, v$ in $V^d_h$, with

$$B_{h,1}(u, v) = \sum_{E \in T_h} \int_{E} (\nabla u + R([u])) : A(F) : (\nabla v + R([v])) \, dV$$

$$B_{h,2}(u, v) = \sum_{E \in T_h} \int_{E} (\nabla(u - P u) + R([u])) : (\beta^E F A(F) + \lambda_X \beta^E \mathbb{I}) : (\nabla(v - P v) + R([v])) \, dV$$

$$\beta^E \mathbb{I}$$

is defined as follows.

Here $\beta^E_1, \beta^E_2$ and $\beta$ are positive parameters to be specified later. A proper choice of these parameters will be given in Thm. 5.1. The symbol $\mathbb{I}$ represents the fourth-order identity tensor, that is, $\mathbb{I} = g$ for all $g \in \mathbb{R}^{d \times d}$. The quantity $\lambda_X$ is a measure of the smallest eigenvalue of the tensor $A(F(X))$ as a linear operator in $\mathbb{R}^{d \times d}$, and is defined as follows. For a point $X \in B_0$ we set

$$\lambda_X := \lambda(A(F(X)))$$

with

$$\lambda(A) := \max\left\{0, -\min_{\mathbb{R}^{d \times d} \setminus 0} \frac{g \cdot A \cdot g}{g \cdot g}\right\}.$$  

If $A(F(X))$ is positive semi-definite then $\lambda_X$ is zero, and if $A(F)$ has one or more negative eigenvalues then $\lambda_X$ is equal to the absolute value of the smallest eigenvalue of $A(F(X))$.

A few remarks are appropriate at this point:

(i) The term $B_{h,1}$ is precisely the bilinear form used for the linearized nonlinear elasticity problem in [9] with no stabilization. Alternatively, it results from Eq. (2.2) by replacing the gradient operator by its approximation, namely, the DG-derivative.

(ii) For the second term, $B_{h,2}$, note that the tensor $\beta^E_1 A(F) + \lambda_X \beta^E_2 \mathbb{I}$ is a shifting of $A(F)$ by a multiple of its smallest eigenvalue. It is positive semi-definite whenever $\beta^E_2 \geq \beta^E_1 > 0$, and strictly positive definite if $\lambda_X > 0$ and $\beta^E_2 > \beta^E_1 > 0$. Thus, in the latter case, we have that $B_{h,2}(u, u) > 0$, for all $u \in V^d_h$. For this reason, it can be considered as a stabilization term. The amount contributed to the positivity of $B_h(u, u)$ is determined by the parameters $\beta^E_1$ and $\beta^E_2$, which we shall see are in turn determined by $\lambda_X$. For this reason, we consider $B_{h,2}$ as an adaptive stabilization term.

(iii) For similar reasons as above, the term $B_{h,3}$ is a stabilization term which controls the jumps by penalizing them. It should be mentioned, though, that most often than not this term is not needed in practice when linear triangles and tetrahedra are used. In the numerical examples we considered so far we took $\beta = 0$ and we have observed that the resulting DG method is still stable.
Next, for \( u \in V_h^d \), let
\[
\| u \|^2 := \| \mathbf{P} u \|^2_{1,B_0} + \frac{1}{h} \| [ u ] \|^2_{0,\Gamma}
\] (5.8)
be the norm with respect to which we will carry out our stability analysis. Let us see that \( \| \cdot \| \) is a norm on \( V_h^d + [ H^1_0(B_0)]^d \). Suppose that for some \( u \in V_h^d + [ H^1_0(B_0)]^d \) we have that \( \| u \| = 0 \), then by the second term on the right-hand side of Eq. (5.8) we see that \( [ u ] = 0 \) on all faces in \( \Gamma \setminus \Gamma^r \), hence \( u \) is continuous over all \( B_0 \). In particular, \( u = 0 \) on \( \Gamma_d \), and hence \( u \in C^0 \). This implies that \( \mathbf{P} u = u \), then by the first term on the right-hand side of Eq. (5.8) we see that \( u \) is identically constant on \( B_0 \), since \( B_0 \) is connected. Finally, the Dirichlet boundary condition ensures that this constant is zero.

5.3. The coercivity result

We state and discuss the main result next.

**Theorem 5.1** Let \( \varphi_h \in V_h^d \) be a deformation mapping and \( \mathbf{F} = D_{DG} \varphi_h \) its DG-derivative defined by Eq. (3.7), such that \( \mathcal{A}(\mathbf{F}(X)) \) is essentially bounded in \( \cup_{E \in T_h} E \).

Let the bilinear form \( B_h(\cdot, \cdot) \) be defined by Eq. (5.4). Suppose that the linearized nonlinear elasticity problem is linearly stable, i.e. there exists a constant \( \kappa > 0 \) such that
\[
\int_{B_0} \nabla u : \mathcal{A}(\mathbf{F}) : \nabla u \, dV \geq \kappa \| u \|^2_{1,B_0} \quad \text{for all } u \in [ H^1_0(B_0)]^d
\] (5.9)

Let \( \kappa_h \in (0, \kappa) \) be an arbitrary constant. For an element \( E \in T_h \) we set
\[
\Lambda_E = \text{ess sup}_{X \in E} \lambda_X, \quad \text{and} \quad \Lambda = \max_{E \in T_h} \Lambda_E
\]
where \( \lambda_X \) is defined by Eq. (5.7). Let \( \eta > 0 \) be such that
\[
\eta < \frac{\kappa - \kappa_h}{\kappa + \Lambda}
\] (5.10)
and let \( \beta_1^E, \beta_2^E, \) and \( \beta \) be such that
\[
\beta_1^E > \frac{1}{\eta} - 1, \quad \beta_2^E > \beta_1^E + 1 + \frac{\Lambda_E}{\kappa - \eta(\kappa + \Lambda_E) - \kappa_h}, \quad \beta \geq \kappa_h.
\] (5.11)

Then,
\[
B_h(u_h, u_h) \geq \kappa_h \| u_h \|^2 \quad \text{for all } u_h \in V_h^d.
\] (5.12)

**Remarks:**

(i) The range of stability of the DG method is determined by the constant \( \kappa_h > 0 \), that is, the larger the coercivity constant \( \kappa_h \) the more stable the DG method is guaranteed to be. The hypotheses of the theorem limits this parameter to be at most as large as \( \kappa \), the coercivity constant for the corresponding continuous linearized elasticity problem. Observe that as \( \kappa_h \) increases \( \beta_1^E, \beta_2^E, \) and \( \beta \) increases. In other words, to gain more stability of the method we have to penalize the jumps more.

In practice, we can only compute an approximation for \( \kappa \), for that matter we can determine \( \kappa_h \) only approximately. Let us denote this approximation by \( \kappa^c \). Moreover, rather than requiring the condition (5.9) for every \( u \in [ H^1_0(B_0)]^d \), we only request that the condition is satisfied for all \( u \in V_h^d \). Hence, computation of \( \kappa^c \) reduces to solving a global eigenvalue problem. Note that we necessarily have

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\( \kappa^c \geq \kappa \), and the difference between these two values depend on how well we can approximate functions in \( H^1_0(B_0) \) with functions in \( V_h \). Our choice of \( \kappa_h \in (0, \kappa^c) \) still guarantees the stability for the linearized elasticity problem. It may, however, miss the onset of buckling, as long as the corresponding conforming method does.

(ii) The computation of \( \Lambda_E = \text{ess sup}_{X \in E} \lambda_X \), and hence of \( \Lambda = \max_{E \in T_h} \Lambda_E \), can be performed locally on each element. But we need to compute \( \lambda_X \) at every point in \( E \) for every \( E \in T_h \). However, from a computational perspective, since \( \lambda_X \) enters the formulation only through an integral, and since the integrals are computed by a quadrature rule, it suffices to compute them only at the quadrature points.

(iii) The choice of \( \eta \) also plays an important role in the amount of stabilization added. Notice that if \( \eta \) is chosen very close to its upper bound in Eq. (5.10), then the value of \( \beta^E \) necessarily becomes very large in at least one of the elements, the one in which \( \Lambda = \Lambda_E \). On the other hand, a value of \( \eta \) very close to zero renders a very large \( \beta^E \), which adds stabilization in the entire domain. While there may be a way to find an optimal value, in the numerical examples shown later we have simply set \( 2\eta = (\kappa - \kappa_h)/(\kappa + \Lambda) \).

(iv) While the bilinear form defined in Eq. (5.4) is guaranteed to be coercive provided the conditions in Thm. 5.1 are satisfied, it has one major drawback that severely hinders the computational performance of the method, namely, the sparsity of the resulting stiffness matrix. One key advantage of standard DG methods is that the resulting stiffness matrices have a well-defined narrow bandwidth that is largely independent of the mesh. The fixed bandwidth of the matrix is the result of each element interacting only with its face-sharing neighbors, and not with all of its node-sharing neighbors, as is the case for conforming approximations. The introduction of the projections into the stabilization term has the undesirable result of inheriting the less sparse structure of conforming approximations. Basically, each degree of freedom in a given element is connected with all degrees of freedom of any node-sharing neighbor through a nonzero entry in the appropriate location in the matrix. For a given mesh, the resulting method then possesses both the larger number of degrees of freedom characteristic of DG approximations and the sparse structure of conforming methods.

The use of this projection in the stabilization term is not essential for stability. Simply increasing the size of \( B_{h,3} \) enough preserves the stability and does not deteriorate the sparsity of the matrix, as follows from equation (5.2). Even though the resulting lower bounds for the stabilization parameters in this later context may still be computed in practice, they will introduce more coercivity than the scheme contemplated here. The loss of computational efficiency because of a poorer sparsity pattern is inessential to the conclusions of this paper, so we preferred to keep the sharper estimates in this study.

(v) Finally, notice that if \( \Lambda = 0 \) and we choose \( \kappa_h > 0 \), then \( \eta \) can be very close to 1 and \( \beta^E \) needs only be greater than zero, while \( \beta^E \lambda_X = 0 \). This is the linear elastic case studied in [13].
5.4. The proof

In this subsection, we give a detailed proof Thm. 5.1. Before embarking on the stability analysis we state and prove a preliminary lemma, namely, a generalized Young’s inequality.

Lemma 5.1 (Generalized Young’s inequality) Let \( g^1 \) and \( g^2 \) be two arbitrary tensors in \( \mathbb{R}^{d \times d} \). Then for any \( \eta, \xi > 0 \) we have that

\[
2g^1 : A : g^2 \geq -\eta g^1 : A : g^1 - \frac{1}{\eta} g^2 : A : g^2
- \lambda \left[ (\eta + \xi) g^1 : g^1 + \left( \frac{1}{\eta} + \frac{1}{\xi} \right) g^2 : g^2 \right],
\]

where \( \lambda = \lambda(A) \) is given by Eq. (5.7).

Proof. By the definition of \( \lambda \) we have that

\[
\left( \sqrt{\eta} g^1 + \frac{g^2}{\sqrt{\eta}} \right) : A : \left( \sqrt{\eta} g^1 + \frac{g^2}{\sqrt{\eta}} \right) \geq -\lambda \left( \sqrt{\eta} g^1 + \frac{g^2}{\sqrt{\eta}} \right) : \left( \sqrt{\eta} g^1 + \frac{g^2}{\sqrt{\eta}} \right).
\]

The left-hand side of the above inequality is equal to

\[
\eta g^1 : A : g^1 + \frac{1}{\eta} g^2 : A : g^2
\]

where we made use of the major symmetries of \( A \). Inserting this into Eq. (5.14) we obtain

\[
2g^1 : A : g^2 \geq -\eta g^1 : A : g^1 - \frac{1}{\eta} g^2 : A : g^2
- \lambda \left( \eta g^1 : g^1 + \frac{1}{\eta} g^2 : g^2 + 2 g^1 : g^2 \right).
\]

Applying the classical Young’s inequality to the term \( 2g^1 : g^2 \) we get

\[
2g^1 : A : g^2 \geq -\eta g^1 : A : g^1 - \frac{1}{\eta} g^2 : A : g^2
- \lambda \left( \eta g^1 : g^1 + \frac{1}{\eta} g^2 : g^2 + \xi g^1 : g^1 + \frac{1}{\xi} g^2 : g^2 \right)
\]

for any \( \xi > 0 \). Now, collecting similar terms together we obtain (5.13). \( \square \)

Remark: Note that, if \( A = I \), the identity tensor, then \( \lambda = 0 \) and we recover the classical Young’s inequality.

Next we give an outline of the proof of the estimate (5.12). We begin by writing \( B_h(u_h, u_h) \) as the sum \( B_{h,1}(u_h, u_h) + B_{h,2}(u_h, u_h) + B_{h,3}(u_h, u_h) \). Next, we obtain an estimate on \( B_{h,1}(u_h, u_h) \) by decomposing \( \nabla u_h \) as a sum of a continuous and a discontinuous function, through the use of the projection operator \( P \). After expanding the resulting expression, an application of the generalized Young’s inequality will bring this term in a form which can be combined with \( B_{h,2}(u_h, u_h) \). After combining these two terms, and invoking the assumptions on the choice of the parameters \( \beta^c \) and \( \beta^F \), we will obtain a suitable estimate on \( B_{h,1}(u_h, u_h) + B_{h,2}(u_h, u_h) \). Finally, by adding \( B_{h,3}(u_h, u_h) \) and invoking the assumption on \( \beta \), we will obtain (5.12).
Proof: (Theorem 5.1) We begin with estimating $B_{h,1}(u_h, u_h)$. To this end we decompose $\nabla u_h$ as $\nabla u_h = \nabla \mathcal{P} u_h + \nabla (u_h - \mathcal{P} u_h)$ and proceed as follows:

$$B_{h,1}(u_h, u_h) = \int_{B_0} \nabla \mathcal{P} u_h : \mathbb{A} : \nabla \mathcal{P} u_h \, dV$$

$$+ \sum_{E \in \mathcal{T}_h} \int_E 2 \nabla \mathcal{P} u_h : \mathbb{A} : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV$$

$$+ \sum_{E \in \mathcal{T}_h} \int_E (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV. \quad (5.17)$$

Applying the generalized Young’s inequality to the integrand in the second term on the right-hand side above we get

$$2 \nabla \mathcal{P} u_h : \mathbb{A} : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \geq -\eta \nabla \mathcal{P} u_h : \mathbb{A} : \nabla \mathcal{P} u_h$$

$$- \frac{1}{\eta} (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : \mathbb{A} : (\nabla (u_h - \mathcal{P} u_h) + R([u_h]))$$

$$- \lambda_{\mathcal{X}} \left[ (\eta + \xi_{\mathcal{X}}) \nabla \mathcal{P} u_h : \nabla \mathcal{P} u_h \right]$$

$$- \lambda_{\mathcal{X}} \left[ \frac{1}{\eta} + \frac{1}{\xi_{\mathcal{X}}} \right] (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : (\nabla (u_h - \mathcal{P} u_h) + R([u_h]))$$

where at each point $\mathbf{X} \in \bigcup_{E \in \mathcal{T}_h} E$, we take the same $\eta > 0$ but the parameter $\xi_{\mathcal{X}} > 0$ is allowed to change from point to point. Inserting this into (5.17) yields

$$B_{h,1}(u_h, u_h) \geq T_1 + T_2 + T_3 + T_4, \quad (5.18)$$

where

$$T_1 = \int_{B_0} (1 - \eta) \nabla \mathcal{P} u_h : \mathbb{A} : \nabla \mathcal{P} u_h \, dV,$$

$$T_2 = -\sum_{E \in \mathcal{T}_h} \int_E \lambda_{\mathcal{X}} (\eta + \xi_{\mathcal{X}}) \nabla \mathcal{P} u_h : \nabla \mathcal{P} u_h \, dV,$$

$$T_3 = \sum_{E \in \mathcal{T}_h} \int_E \left( 1 - \frac{1}{\eta} \right) (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : \mathbb{A} : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV,$$

$$T_4 = -\sum_{E \in \mathcal{T}_h} \int_E \lambda_{\mathcal{X}} \left( \frac{1}{\eta} + \frac{1}{\xi_{\mathcal{X}}} \right) (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV.$$

Using the linearized stability hypothesis (5.9) we get

$$T_1 \geq \sum_{E \in \mathcal{T}_h} \int_E \kappa (1 - \eta) \nabla \mathcal{P} u_h : \nabla \mathcal{P} u_h \, dV.$$

Thus,

$$T_1 + T_2 \geq \sum_{E \in \mathcal{T}_h} \int_E \left[ \kappa (1 - \eta) - \lambda_{\mathcal{X}} (\eta + \xi_{\mathcal{X}}) \right] \nabla \mathcal{P} u_h : \nabla \mathcal{P} u_h \, dV. \quad (5.19)$$

Combining this estimate with the terms $T_3$ and $T_4$ we obtain

$$B_{h,1}(u_h, u_h) \geq B_{h,1}^m(u_h, u_h) + B_{h,1}^M(u_h, u_h), \quad (5.20)$$

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where

\[
B_{h,1}^m(u_h, u_h) := \sum_{E \in T_h} \int_E m(\eta, \xi_X) \nabla \mathcal{P} u_h : \nabla \mathcal{P} u_h \, dV,
\]

and

\[
B_{h,1}^M(u_h, u_h) := - \sum_{E \in T_h} \int_E (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : \bar{M}(\eta, \xi_X) : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV.
\]

Here,

\[
m(\eta, \xi_X) = \kappa (1 - \eta) - \lambda_X (\eta + \xi_X)
\]

and

\[
\bar{M}(\eta, \xi_X) = \left( \frac{1}{\eta} - 1 \right) \kappa (F) + \lambda_X \left( \frac{1}{\eta} + \frac{1}{\xi_X} \right) \mathbb{I}.
\]

Combining (5.22), and (5.5b) (with \(u\) and \(v\) replaced by \(u_h\)) we get that

\[
B_{h,1}^M(u_h, u_h) + B_{h,2}(u_h, u_h) = \sum_{E \in T_h} \int_E (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : \bar{M}(\eta, \xi_X) : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV
\]

where

\[
\bar{M}(\eta, \xi_X) := \left( \frac{\beta_F^E}{\eta} - 1 \right) \kappa (F) + \lambda_X \left( \frac{\beta^F}{\eta} - 1 \right) \mathbb{I}.
\]

Inserting this into (5.20) yields

\[
B_{h,1}(u_h, u_h) + B_{h,2}(u_h, u_h) \geq \sum_{E \in T_h} \int_E m(\eta, \xi_X) \nabla \mathcal{P} u_h : \nabla \mathcal{P} u_h \, dV
\]

\[
+ \sum_{E \in T_h} \int_E (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) : \bar{M}(\eta, \xi_X) : (\nabla (u_h - \mathcal{P} u_h) + R([u_h])) \, dV.
\]

This estimate is valid for arbitrary \(\xi_X > 0\). Since \(\lambda_X \leq \Lambda\), we have that

\[
\frac{\kappa - \kappa_h}{\kappa + \lambda} (\kappa + \lambda_X) \leq \kappa - \kappa_h
\]

and hence

\[
\kappa - \frac{\kappa - \kappa_h}{\kappa + \lambda} (\kappa + \lambda_X) - \kappa_h \geq 0.
\]

Since \(0 < \eta < \frac{\kappa - \kappa_h}{\kappa + \lambda}\) by hypothesis, we see that

\[
\kappa - \eta (\kappa + \lambda_X) - \kappa_h > 0.
\]

Therefore, for \(\lambda_X > 0\)

\[
\frac{\kappa - \eta (\kappa + \lambda_X) - \kappa_h}{\lambda_X} > 0.
\]

For \(\lambda_X = 0\) we note that the quantity \(\kappa - \eta (\kappa + \lambda_X) - \kappa_h\) depends continuously on \(\lambda_X\), and increases as \(\lambda_X\) decreases since \(\eta\) is strictly positive. Taking the limit in (5.28) we deduce that

\[
\lim_{\lambda_X \to 0} \frac{\kappa - \eta (\kappa + \lambda_X) - \kappa_h}{\lambda_X} = +\infty.
\]
Hence, we can always pick $\xi_X$ such that
\[ 0 < \xi_X < \frac{\kappa - \eta (\kappa + \lambda X) - \kappa h}{\lambda X} \] (5.29)
for almost all $X \in B_0$. Assuming that $\xi_X$ obeys these bounds we see that
\[ m(\eta, \xi_X) = \kappa (1 - \eta) - \lambda X (\eta + \xi_X) > \kappa h \] (5.30)
for almost every $X$ in $B_0$. Thus
\[ \text{ess inf}_{X \in B_0} m(\eta, \xi_X) \geq \kappa h \] (5.31)
which together with Eq. (5.27) implies
\[ B_{h,1}(u_h, u_h) + B_{h,2}(u_h, u_h) \geq \kappa h \sum_{E \in T_h} \int_E \nabla P u_h : \nabla \mathcal{P} u_h \, dV \]
\[ + \sum_{E \in T_h} \int_E (\nabla (u_h - \mathcal{P} u_h) + \mathcal{R}([u_h])) : \mathcal{M}(\eta, \xi_X) : (\nabla (u_h - \mathcal{P} u_h) + \mathcal{R}([u_h])) \, dV. \]
\[ \geq \kappa h |\mathcal{P} u_h|_{1, B_0}^2 + \Lambda_{\text{min}}(\mathcal{M}) \| \nabla (u_h - \mathcal{P} u_h) + \mathcal{R}([u_h]) \|_{0, B_0}^2 \] (5.32)
where, as we shall see next,
\[ \Lambda_{\text{min}}(\mathcal{M}) := \text{Minimum eigenvalue of } \mathcal{M}(\eta, \xi_X) \text{ in } \cup_{E \in T_h} E \geq 0. \] (5.33)
Now, we know, by the very definition of $\lambda_X$, that $A(F) + \lambda_X \mathbb{I}$ is a positive semi-definite tensor. As a consequence, $\alpha_1 A(F) + \alpha_2 \lambda_X \mathbb{I}$ is positive semi-definite for any $\alpha_2 \geq \alpha_1 > 0$.
The hypotheses (5.11) imply that
\[ \beta_1^E - \frac{1}{\eta} + 1 > 0 \]
and that there exists $\xi_X$ that satisfies Eq. (5.29) and such that
\[ \beta_2^E - \frac{1}{\eta} - \frac{1}{\xi_X} > \beta_1^E - \frac{1}{\eta} + 1. \]
Hence,
\[ \mathcal{M}(\eta, \xi_X) = \left( \beta_1^E - \frac{1}{\eta} + 1 \right) A + \lambda_X \left( \beta_2^E - \frac{1}{\eta} - \frac{1}{\xi_X} \right) \mathbb{I} \]
is a positive semi-definite tensor, and so $\Lambda_{\text{min}}(\mathcal{M}) \geq 0$. Thus we can drop the last term on the right-hand side of Eq. (5.32) and deduce that
\[ B_{h,1}(u_h, u_h) + B_{h,2}(u_h, u_h) \geq \kappa h |\mathcal{P} u_h|_{1, B_0}^2. \] (5.34)
Since, by hypothesis (5.11)
\[ B_{h,3}(u_h, u_h) \geq \beta_h \| u_h \|_{0, \Gamma}^2 \geq \frac{\kappa h}{h} \| u_h \|_{0, \Gamma}^2 \]
we get that
\[ B_h(u_h, u_h) = B_{h,1}(u_h, u_h) + B_{h,2}(u_h, u_h) + B_{h,3}(u_h, u_h) \]
\[ \geq \kappa h \left( |\mathcal{P} u_h|_{1, B_0}^2 + \frac{1}{h} \| u_h \|_{0, \Gamma}^2 \right) \]
\[ = \kappa h \| u_h \|^2 \]
Remarks:
(i) As it is clear from the proof we have blatantly dropped the second term in inequality (5.32) by proving it is non-negative. While we have not succeeded in proving that such a term controls the size of the discontinuities, we have not been able to find an example where it does not either. The numerical examples have all been performed with $\beta = 0$ and have all resulted in linearly stable discretizations. The role of $\beta_1^E$ and $\beta_2^E$ is then simply to modify the action of the bilinear form on the discontinuous part of $u_h, u_h - \mathcal{P}u_h$, so that it does not subtract any of the coercivity guaranteed by $\mathcal{P}u_h$.

(ii) Notice that the use of condition (5.9) is the key to overcome the possibility of indefinite elastic moduli $A$ in the domain. It is precisely because $A$ may not be positive definite that we cannot guarantee that the term $T_3$ in (5.18) is positive, and may need to have $\beta_1^E \neq 0$ to balance it.

(iii) The bilinear form $B_{h,3}$ can be replaced with

$$C^+ \beta \int_{B_0} R([u]): R([v]) \, dV,$$

as follows from equation (3.8).

(iv) We chose $\mathcal{P}$ as defined in equation (5.1) for the projection operator because it is simple to compute and implement. It is clear from the proof that the stability of the algorithm and the lower bounds for the stabilization parameters follow for any choice of the projection operator; the methods will of course differ. In particular, a very convenient projection is one defined as the operator $\hat{\mathcal{P}} : V_h \mapsto C_h$ such that

$$\sum_{E \in T_h} \int_E (\nabla(u_h - \hat{\mathcal{P}}u_h) + R([u])) : A : \nabla v_h \, dV = 0$$

(5.35)

for all $v_h \in C_h$. In this case we can sidestep the use of generalized Young’s inequality, since the second term on the right hand side of equation (5.17) is identically zero. A straightforward stability coercivity estimate for $B_{h,1}$ is then

$$B_{h,1}(u_h, v_h) = \int_{B_0} \nabla u_h : A : \nabla v_h \, dV$$

$$+ \sum_{E \in T_h} \int_E (\nabla(u_h - \hat{\mathcal{P}}u_h) + R([u_h])) : A : (\nabla(u_h - \hat{\mathcal{P}}u_h) + R([u_h])) \, dV$$

$$\geq - \sum_{E \in T_h} \int_E (\nabla(u_h - \hat{\mathcal{P}}u_h) + R([u_h])) : \lambda_X I : (\nabla(u_h - \hat{\mathcal{P}}u_h) + R([u_h])) \, dV,$$

where the last inequality follows by taking advantage of the fact that $A + \lambda_X I$ is positive semi-definite. The lower bounds for the stabilization parameters then follow as $\beta_1^E = 0, \beta_2^E \geq 1, \beta \geq \kappa_h$. However, the computation of $\mathcal{P}u_h$ requires the inversion of a large system of equations, and hence $\mathcal{P}u_h$ depends on essentially all degrees of freedom in the mesh. This makes such an approach unfeasible.

(v) The stability of the adaptive stabilization method introduced in [19] and exemplified in Sec. 4 for large enough $\beta$ follows immediately from the fact that there exists $C_h > 0$ such that
\[ C_h \sum_{E \in T_h} \int_E R([u_h]) : (\lambda_X + \epsilon) I : R([u_h]) \, dV \geq \\
\sum_{E \in T_h} \int_E (\nabla (u_h - \hat{P} u_h) + R([u_h])) : \beta^E_2 \lambda_X I : (\nabla (u_h - \hat{P} u_h) + R([u_h])) \, dV, \]

(5.36)

for all \( u_h \in V^d_h \), for any \( \epsilon > 0 \). This inequality is obtained by noticing that the left hand side term is the square of a norm in the finite dimensional quotient space \( V^d_h / C_h \) induced by the projection \( \hat{P} \), while the right hand side is a quadratic form over it.

5.5. Application to the nonlinear elasticity problem.

We apply the idea of adaptive stabilization to the nonlinear elasticity problem next. To this end, we define a quasistatic loading path from the reference, unstressed configuration to the solution sought in equilibrium prescribed deformation mapping \( \varphi \) on \( \partial_B \). This loading path can either serve just as a tool to reach the desired deformation, or it could actually correspond to a real loading process to be studied. For example, in the former case it is enough to consider a sequence of \( n \geq 1 \) loading steps, such that in loading step \( k \) the prescribed deformation mapping is equal to \( \hat{\varphi}^k(X) = (X(n - k) + k\varphi)/n \).

The operator \( D_{DG} \) contains the values of the prescribed displacements through the numerical traces, and hence the operator itself changes at each loading step\(^3\). We shall denote with \( D_{DG}^k \) the operator that uses \( \hat{\varphi}^k \) for the computation of the numerical traces. Additionally, we let \( A^k = A(D_{DG}^k \hat{\varphi}^k) \), and \( \lambda^k_X = \lambda(A^k(X)) \).

The proposed adaptively stabilized method arises from an incremental variational principle. If configuration \( \varphi^k \) is known, then configuration \( \varphi^{k+1} \in V^d_h \) follows as the stationary point in \( V^d_h \) of the functional

\[
I^{k+1}_h[\varphi_h] = \sum_{E \in T_h} \int_E W(D_{DG}^{k+1} \varphi_h) \, dV + \frac{1}{2} \sum_{E \in T_h} \int_E (\nabla (\varphi_h - \hat{P} \varphi_h) + R([\varphi_h - \hat{v}(\varphi_h)])) : (\beta_1^{E,k} A^k + \lambda_X \beta_2^{E,k}) I : (\nabla (\varphi_h - \hat{P} \varphi_h) + R([\varphi_h - \hat{v}(\varphi_h)])) \, dV \\
+ \beta^k \int_{\Gamma} [\varphi_h - \hat{v}(\varphi_h)] : [\varphi_h - \hat{v}(\varphi_h)] \, dS. \quad (5.37)
\]

The first and second variations of this functional are trivially computed, as similarly done in [19], so we shall skip them here. It suffices to say that at \( \varphi_h = \hat{\varphi}^k \) the second variation is coercive in \( V^d_h \), by the result in Thm. 5.1, provided that \( \beta_1^{E,k}, \beta_2^{E,k} \) and \( \beta^k \) are chosen to satisfy conditions (5.11). The method is then linearly stable at \( \varphi^k \), and it generally remains linearly stable in a neighborhood of \( \varphi^k \). If the size of the loading steps are chosen small enough, it is often the case that the method remains linearly stable at each step of the Newton-Raphson iteration, and at the converged solution \( \varphi^{k+1} \).

In our computations we calculate \( \kappa^c \) at each quasistatic loading step. Moreover, rather than requiring condition (5.9) for every \( u \in [H^1_0(B_0)]^d \), we only request it to be satisfied for all \( u \in V^d_h \). This operation entails the computation of the lowest eigenvalue of the

\(^3\) Of course, it is possible to redefine \( D_{DG} \) to avoid this.
stiffness matrix obtained by taking the stabilization parameters large enough to mimic a conforming approximation. We also note that, by experience, when far from buckling $\kappa_c$ does not change drastically at every quasistatic loading step, so it is possible to use the same value of $\kappa_c$ for several steps before recomputing it.

Once the solution $\varphi^k$ is known, it is necessary to compute $\lambda_C, \Lambda, \kappa, \kappa_c$, and pick a value for $\kappa_h \in (0, \kappa_c)$. Then we can choose any parameter $\eta$ by Eq. (5.10), and consequently values of $\beta^{E,k}_1, \beta^{E,k}_2, \beta^k$, needed to compute $\varphi^{k+1}$. Because these values are computed with the latest quasistatic solution, they adaptively change in such a way that the stability of the linearized nonlinear elasticity problem at each configuration $\varphi^k$ is guaranteed.

Finally, we note that the choice of the incremental variational principle depends on the chosen loading path, and hence the numerical approximation to the deformation mapping may strongly depend on it as well. In [19] we have shown with numerical experiments that this dependence can be safely neglected in all cases we looked at, for the approach studied therein. Although we shall not repeat these studies here, we expect a similar behavior to hold for the new method in this manuscript.

6. Numerical example

We next showcase the performance of the adaptive method with analytical lower bounds with an example, in which we study its convergence behavior for a two-dimensional problem. The reference configuration is a square with unit length made of a compressible neo-Hookean material, whose strain energy density is

$$W(F) = \frac{E}{2(1+\nu)} \left( \frac{\nu}{1-2\nu} \log(\text{det}(F))^2 - \log(\text{det}(F)) + \frac{\text{tr}(F^T F - I)}{2} \right),$$

if $\text{det}(F) > 0$, and $W(F) = +\infty$ otherwise. Here $E$ and $\nu$ are material constants, which for this example are assumed to be $E = 1$ and $\nu = 0.25$. The bottom side is held fixed while the top surface is mapped into a parabola, mimicking an indentation problem. This same problem was used in [19] to test the performance of the adaptive stabilization strategy therein.

As noted earlier, the results in this example show that the lower bounds in Thm. 5.1 turn out to be too conservative. We therefore consider here the limiting case, in which we adopted $\kappa_h = 0, 2\eta = \kappa_c/(\kappa_c + \Lambda)$ and $\beta^{E,k}_1, \beta^{E,k}_2, \beta^k$ equal to their lower bounds. We compared the performance of this method with the results already shown in [19] for the adaptive stabilization strategy therein and a competing conforming method of the same order. In all cases we used piecewise linear triangles to interpolate displacements, as well as deformation gradient in the case of the DG schemes.

In this example we examine the number of degrees of freedom required to achieve a given accuracy level in the approximation, for both displacements and the deformation gradient. Since the solution to this nonlinear elasticity problem has no analytical solution, a highly refined conforming mesh was used to approximate the exact solution, $\varphi_{\text{exact}}$ and $F_{\text{exact}}$. The highly refined mesh consists of 79,202 linear conforming triangles. Figure 3(a) shows the value of $\|\varphi_h - \varphi_{\text{exact}}\|_{0,B_0}$ as a function of the total number of degrees of freedom for the three different methods. All three methods roughly converge at an asymptotic rate of $h^2$. Similarly, Fig. 3(b) shows the corresponding plot for the
Fig. 3. Convergence plots for the displacements (left) and deformation gradient (right) as a function of the number of degrees of freedom for an indentation problem. A comparison between a conforming piecewise linear approximation and two adaptively stabilized DG methods is shown. The adaptively stabilized (Sec. 4) method refers to the method in [19] and described in Sec. 4, while the adaptively stabilized method (Sec. 5) refers to the method with guaranteed lower bounds in Sec. 5. The poor performance of the latter, however, is a direct indication that discontinuities are being heavily penalized, and that the lower bounds provided by Thm. 5.1 are too conservative.

decomposition gradient, with the vertical axis showing the value of $\|F_h - F_{exact}\|_{0,B_0}$. For linear conforming elements, the deformation gradient is simply $F_h = \nabla \varphi_h$, and for DG, the deformation gradient is given by $F_h = D_{DG} \varphi_h$. Again, all three methods roughly converge at a rate of $h$.

As already shown in [19], for this example the stabilization strategy therein produces results that are comparable to those of the conforming method, being less efficient in terms of displacements, but more efficient in terms of the deformation gradient. For the method introduced here, however, the performance in both cases is quite poor, displaying a displacement error approximately six times larger than that of a conforming method, and four times larger for the deformation gradient. When plotted as a function of $1/h$ in Fig. 4, the DG approximations perform better than the conforming one, which shows that accuracy is gained due to the presence of discontinuities. This gain, however, is rather small when the lower bounds in theorem 5.1 are adopted.

The method herein shows a slightly slower convergence rate for coarse meshes. This is a reflection of the fact that as the mesh is refined the elastic moduli $\lambda$ are better resolved, and the values of $\beta_1^E$ and $\beta_2^E$ steadily grow. As seen in Fig. 4, the method recovers the $h^2$ convergence rate of the conforming method once the stabilization parameters essentially render the DG approximation very close to a conforming one.

Further numerical examples carried out with the adaptive stabilization method with analytical lower bounds effectively confirmed that, as expected, the method is perfectly robust. Nevertheless, they have also shown that the bounds in theorem 5.1 are consistently overestimating the minimum amount of stabilization needed for the method to be stable, occasionally by a few orders of magnitude. Under near incompressibility conditions, in which $\Lambda$ has at least one very large positive eigenvalue, its presence in the stabilization term makes the appearance of locking very likely, since discontinuities are easily overpenalized.
7. Summary

The foremost goal of this paper was to construct a DG method for nonlinear elasticity that is fully automatic, and examine its performance. To this end we determined and proved analytical lower bounds for the stabilization parameters of a new type of stabilization term. We did so for the linearized nonlinear elasticity problem at any configuration of the body, and extended it to the nonlinear elastic range.

Crucial to the creation of this method was the use of a projection operator to decompose the discontinuous Galerkin approximation space into a direct sum of continuous and discontinuous functions. In this way, the coercivity of the original linearized nonlinear elasticity problem is inherited by the continuous part. By crafting a suitable penalization for the discontinuous part, the bilinear form of the proposed method can always be guaranteed to be coercive whenever the one of the exact problem is. The adaptive stabilization term for this method involves the elastic moduli at each point in the body with its eigenvalues uniformly shifted so that it becomes positive semi-definite, and a local choice of the stabilization parameters within each element. We also obtained that the adaptively stabilized method in [19] is stable for a large enough value of the stabilization parameter, by a simple modification of the proof of the main theorem.

To preserve the linearized stability of this method in the context of nonlinear elastic problems a sequence of incremental variational principles along a quasistatic loading path was proposed. At each loading step then the method is linearly stable in a neighborhood of the last-step solution. By crafting the size of each loading step it is then possible to transverse the entire loading path without encountering numerical instabilities. This was confirmed in numerical examples.

The numerical examples also highlighted a key drawback of the resulting method, namely, that the stability estimates in Thm. 5.1 are too conservative, resulting in nearly conforming solutions and hence a substantial lost of computational efficiency. Of course,
the claim that the computed lower bounds are too coarse is only applicable to the method presented here. It is remarkable, however, that there is little or none documented verification of the numerical performance of other lower bounds for the stabilization parameters in the literature. In constructing the method presented here we purposely tried to minimize the number of sequential estimates used to get the bounds on the stabilization parameters, even at the cost of including the projection operator in the stabilization term. The numerical results underscore the need to better understand how the stabilization mechanism works, if an efficient and robust method is to be constructed.

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References


