A Discontinuous-Galerkin-based Extended Finite Element Method for Fracture Mechanics: Immersed Boundary Perspective and Optimal Convergence

Adrian Lew
Department of Mechanical Engineering, Stanford University

Yongxing Shen
Department of Mechanical Engineering, Stanford University

Abstract: The extended finite element method (XFEM) enables the representation of cracks in arbitrary locations of a mesh. We introduce here a variant of the XFEM rendering an optimally convergent scheme. Its distinguishing features are: a) the introduction of singular asymptotic crack tip fields with support on only a small region around the crack tip, the enrichment region, b) only one and two enrichment functions are added for anti-plane shear and planar problems, respectively, and c) the relaxation of the continuity between the enrichment region and the rest of the domain, and the adoption of a discontinuous Galerkin (DG) method therein. The method is provably stable for any positive value of a stabilization parameter.

1 Introduction: Fracture mechanics plays an essential role in failure analysis of engineering structures. Accurately and efficiently computing the stress field is crucial for predicting the evolution of cracks. In fracture mechanics problems, cracks are usually idealized as geometries of one less dimension than that of the embedding solid, i.e., submanifolds of codimension 1. This idealization gives rise to a stress singularity of the type $1/\sqrt{r}$ for isotropic linear elastic materials, where $r$ is the distance to the crack tip, and leads to suboptimal convergence rates if standard finite element methods (FEMs) with piecewise polynomial spaces are employed.

There are virtually two families of strategies to achieve better convergence behavior: adaptive mesh refinement and extension of the finite element (FE) space with the a priori known singular functions. This introduction will focus on developments along the second family of the aforementioned methods.

With minimal alteration to existing FE programs, Henshell and Shaw [1] and Barsoum [2] independently proposed using isoparametric 8-noded quadrilateral elements while placing some of the mid-side nodes at the quarter positions so as to produce the $\sqrt{r}$ behavior of the displacement field in the physical domain. The quarter-point elements are straightforward to implement. However, the singularity they introduced in the FE space does not capture the angular dependence of the strain field near the crack tip. Consequently, even though convergence is still attained and the accuracy of the solution is enhanced, there is no improvement in the convergence rate with respect to FE discretizations with standard elements; it remains suboptimal.

In 1973, Fix et al. [3] introduced singular functions in combination with standard piecewise polynomial functions for solving Poisson and Helmholtz equations over a cracked domain. The singular functions are continuous and have compact supports centered at the singularity (crack tip). However, they are non-local, which means that their support spans many elements in the mesh. In fact, the support of the singular functions is the same regardless of the mesh size. By doing so, they obtained an optimal convergence rate and accurate approximations to the stress intensity factors (SIFs).

More recently Belytschko and Black [4] and Moës et al. [5] introduced the extended finite element method (XFEM), in which they adopted a partition of unity (PU) technique [6, 7] to incorporate the singularities into the FE basis. Its key advantage is that it requires minimal or no remeshing as the crack propagates. In this context a typical enrichment shape function is the product of a singular function and a standard, continuous, piecewise polynomial shape function. In the early developments of the XFEM, only elements adjacent to (or very close to) the crack tip were enriched with the singular functions.

Since the introduction of the XFEM applied to fracture mechanics problems, related literature grew rapidly,
and numerous variants of the XFEM have been proposed. In 2003, Chessa et al. [8] observed the suboptimal convergence rate of the then standard XFEMs, and attributed this phenomenon to the presence of blending elements, i.e., enriched elements bordering standard elements. Later Laborde et al. [9] and Béchet et al. [10] showed that if the size of the enrichment zone shrinks around the crack as \( h \to 0 \) then the convergence rate will be suboptimal. Hence, they proposed enriching elements over a fixed area independent of the mesh.

A number of strategies has been proposed to overcome the suboptimal convergence rate due to the presence of blending elements. The authors in [9] proposed two methods for removing this artifact: a) adding a bubble quadratic function to the blending elements and b) using the assumed strain method. Another method that still adopted a PU to include the singular enrichment and corrected the suboptimal convergence rate due to the blending elements was proposed by Fries [11]. The author multiplied each singular enrichment basis function by a cutoff function that increases from zero to one in a single ring of elements, the transition layer. While normally this product would have very large derivatives in the transition layer, the particular choices of enrichment functions have small derivatives therein, leading to optimal convergence rate in the numerical examples.

Generalizing the original ideas in [3], Chahine et al. [12] [13] overcame the problem of blending elements by multiplying the singular enrichment functions by a smooth cutoff function independent of the mesh size. The authors also proved the optimal convergence rate of this method.

In [9] the authors framed the problem of the blending elements as one of approximating a function in two overlapping domains, the domains with and without enrichment functions, through a PU approach. It follows from their analysis that the convergence rate should not be affected as the width of the overlapping region, or transition layer, decreases to zero. Consequently, in this limit this type of method would require the approximations on both sides of the interface between the domains with and without enrichment functions to be essentially continuous, i.e., discontinuities are allowed but these should become smaller and smaller as the mesh is refined. The authors then proposed to introduce a basis for the singular enrichment functions without the PU technique, which they labeled “degree of freedom gathering,” and then match the nodal values of the numerical solutions along the interface between the domains with and without enrichment functions.

An idea that is strongly related to those in the previous paragraph is to include the singular enrichment functions on a region that contains the crack tip, and set them to zero outside this region. The resulting enrichment functions are discontinuous at the interface between the two regions, and hence a discontinuous Galerkin (DG) approximation can be adopted therein to weakly enforce continuity. This was proposed in [14], where the authors combined the PU-based XFEM and the interior penalty DG methods in [15] and [16]. One drawback of this approach is that stabilization is required, and the stabilization parameter is problem dependent. A few iterations are needed to find a suitable value.

The idea of introducing discontinuities to weakly enforce continuity of non-polynomial basis functions can also be found in Farhat et al. [17] and Massimi et al. [18] for wave propagation problems, and Yuan and Shu [19] for convection-diffusion problems. In [17] and [18], weak continuity is restored by introducing Lagrange multipliers along the element boundaries, while in [19] this is accomplished through a local DG method.

In this paper we propose a new DG method to connect solutions on the domains with and without enrichment functions. The method is based on the DG methods for elastic problems in [20] and [21], which are constructed by adopting the Bassi-Rebay numerical fluxes. An important feature of this class of methods is that for isotropic, unstressed, linear elastic problems the stabilization term is problem independent. In fact, it is enough for the stabilization parameter to be any positive real number.

Another important feature of the method is that only the two asymptotic displacement fields near the crack tip for modes I and II are adopted as enrichment functions. This enables us to compute the SIF for each mode simply as the coefficient of the enrichment basis function in the numerical solution. We show the convergence of these coefficients to their exact values through numerical examples. While the rate of convergence of the SIFs computed in this way is lower than those computed through the interaction integral [3], the resulting values are surprisingly accurate and require essentially no extra computational cost. Among the methods that introduce enrichments by multiplying by a PU basis, the one by Liu et al. [22] also extracts the SIFs as the coefficients of the enrichment basis functions.

To retain an optimal order of convergence, we set the domain over which the singular enrichment functions are not zero to be the set of all elements that intersect a circle whose radius is independent of \( h \). Consequently, the number of elements over which the enrichment functions are different than zero scales as \( h^{-2} \). For arbitrary enrichment functions this would mean that the number of nonzero en-
tries in the stiffness matrix relating the enrichment functions with the polynomial basis functions scales as $h^{-2}$ as well. However, because the enrichment basis functions are in mechanical equilibrium, we show that most of these entries are identically zero. In fact, the number of such non-zero entries scales as $h^{-1}$, which renders the stiffness matrix surprisingly sparse. Only a polynomial basis function whose support includes either the boundary of the enrichment region or a portion of the face of a curved crack would couple to the enrichment functions through a non-zero pair of entries in the stiffness matrix.

We demonstrate the optimal convergence properties of the method with numerical examples. We compare the results against those obtained by adopting the cutoff function approach [13], since the authors proved that this method attains an optimal convergence rate. For the same mesh, the errors in stresses, displacements and SIFs are substantially lower for the method proposed herein. These differences are not expected to be large if compared with other methods that set the enrichment functions to zero with either a zero-width transition layer as in Laborde et al. [9], or a shrinking transition layer as $h$ tends to zero as in Fries [11], but we did not perform such comparison.

Finally, in Section 2.2 we construct a connection between immersed boundary methods (see, e.g., [23]) and the XFEM by representing the crack path as an immersed boundary rather than an additional enrichment. This perspective is similar but not identical to those introduced by Molino et al. [24], Song et al. [25] and Hansbo and Hansbo [26]. This point of view simplifies the construction of FE spaces over cracked domains, such as spaces $H^1$: $\Omega \cap \emptyset$, $\partial f_\Omega$ where such that the $\Omega$ is a domain with cracks. Each crack is assumed to be a smooth one-dimensional curve. Additionally, we assume that there exists an open neighborhood of each crack such that a) the open neighborhoods of two different cracks do not intersect and, b) once the open neighborhood of each crack or any of its subsets has been removed from $\Omega$, a connected domain remains. These assumptions preclude the important possibilities of intersecting cracks, such as a bifurcation, and of crack kinks. While some of these configurations do not need further modifications of the method to retain an optimal convergence rate, some of them do. We therefore restrict our attention to a less general setting, so as to convey the key new ideas in the simplest terms.

We shall make the standard assumption that crack faces are traction free, i.e., $T = 0$ on $\partial_u \Omega \subseteq \partial \Omega$, where $\partial_u \Omega$ is the set of crack faces. In this way we rule out contact and friction between crack faces.

We assume that $C$ is constant throughout the domain, or at least, in the open neighborhood around each crack tip. Generalizations that retain an optimal rate of convergence to smoothly spatially varying moduli, or to anisotropic materials, are possible provided an asymptotic displacement field near each crack tip is known under those circumstances, and of course, some regularity conditions for the solution are met.

For simplicity, we shall explain the method assuming that there is only one crack tip whose position is denoted by $x_t$, which means we consider an edge crack. The generalization to multiple crack tips is trivial.

Following Stolarska et al. [27], we assume that there exist smooth scalar functions $\phi$ and $g$ over $\Omega$ such that the crack is contained in the level set $\phi = 0$, and along the crack $g$ is strictly monotonous with $g(x_t) = 0$. The crack is then defined as $\partial_g \Omega = \{ x \in \Omega : \phi(x) = 0, g(x) < 0 \}$, see Figure 2(a). By construction, $x_t \notin \partial_g \Omega$.

2.2 Crack Paths as Immersed Boundaries: Adopting the philosophy of the XFEM, we construct a family of quasi-uniform triangulations $\{ T_h \}_{h > 0}$ of $\Omega$ parameterized by the mesh size $h > 0$. We do so without taking into account the locations of the cracks. Therefore we allow for the possibility of the crack cutting through elements. We assume that any node in mesh $T_h$ either belongs to the crack or is a distance of $\epsilon h$ away from any point on the crack for some $\epsilon > 0$ independent of $h$. This condition prevents the crack from cutting elements into arbitrarily small parts, which would render poorly conditioned stiff-
ness matrices. Additionally, we henceforth assume that elements are open subsets of \( \Omega \), and that the intersection of the crack \( \partial_c \Omega \) and any element \( K \) can only be either an empty set or a connected curve, for any \( K \) in \( \{ T_h \} \).

In other words, if any element \( K \) is cut by the crack at all, it can only be cut into at most two parts. A similar but not identical assumption was made in [26], because they followed a convention in which elements are treated as closed sets.

In this paper we adopt the distinct perspective of cracks as immersed boundaries. This perspective is of course equivalent to regarding them as jump discontinuities in the deformation mapping, as in the XFEM. However, it leads to a simple and elegant implementation of the method. Furthermore, this perspective simplifies the construction of enriched FE spaces in cracked domains with a variety of elements, not only of the conforming \( P_1 \) type. Similar albeit not identical ideas have been introduced by Molino et al. [24], Song et al. [25] and Hansbo and Hansbo [26].

In immersed boundary methods the domain \( \Omega \) is overlapped over a mesh that does not fit the boundary, see e.g. [23]. In this way, \( \partial \Omega \) cuts through elements in the mesh, see Figure 1 for an example over a domain with no cracks. A FE space \( V_h \) can be easily constructed over all elements intersected by \( \Omega \). However, only the restriction of functions in \( V_h \) to \( \Omega \) is important for the approximate solution.

A crack is clearly part of the boundary of \( \Omega \). However, in contrast to the example in Figure 1 when immersing \( \Omega \) into \( T_h \) some elements intersect \( \Omega \) on both sides of the crack. Consequently, if the basis functions within the element are continuous, then the kinematics on the two sides of the crack are artificially connected. This artifact can be naturally avoided with a simple modification of \( T_h \) that involves duplicating some nodes and elements to construct a new mesh \( T'_h \).

The basic idea can be explained by splitting the mesh along a curve that contains the crack into two disconnected ones, and then “gluing” them back together along the non-cracked part of this curve, see Figure 2. This separation into two steps is not needed for the final algorithm, but is introduced to facilitate the explanation. In fact, the criteria of duplication are

Element \( K \) is duplicated \iff \( K \setminus \partial_c \Omega \) is not connected,

Node \( a \) is duplicated \iff \( \text{cloud}(a) \setminus \partial_c \Omega \) is not connected,

where \( \text{cloud}(a) \) is defined as the interior of the closure of the domain occupied by all elements in \( T_h \) that have \( a \) as a node (the cloud of a node \( a \) is also the interior of the support of the standard piecewise linear shape function associated with node \( a \)).

We developed a precise algorithm in [28] that is able to construct a new mesh \( T'_h \) with duplicated elements from a mesh \( T_h \) for the uncracked solid. The new mesh \( T'_h \) has elements that occupy the same location in space. However, each one of the two copies can be adopted to describe the kinematics of only one side of the crack, effectively treating each crack face as a regular immersed boundary.

Once \( T'_h \) is defined, a FE space \( V_{h} \) can be defined over it such that

\[
V_h \subseteq \prod_{K \in T'_h} V_h^K \quad = \left\{ v = (v_{K_1}, \ldots, v_{K_{N'}}) : v_K \in V_h^K, \forall K \in T'_h \right\}
\]

where \( V_h^K \) is the FE space over element \( K \) and \( N' \) is the total number of elements in \( T'_h \). For example, a standard choice would be a space that contains functions that are continuous across elements that share a common edge, i.e., elements that have two nodes in common. In this case,

\[
V_h^{\text{conf}} = \left\{ v \in \prod_{K \in T'_h} V_h^K : v|_e = v|_{K'}|_e \right\}
\]

if \( K, K' \) share edge \( e \), for any \( e \).

Notice that \( V_h \) may contain functions that are bi-valued over some regions of the domain. In particular, these will be bi-valued over the domain of all elements \( K \in T_h \) that have been duplicated when constructing \( T'_h \). To specify which of these two values shall be used to approximate the solution of the problem, we define

\[
\ast K = \begin{cases} 
K \setminus \partial_c \Omega & \text{if } K \text{ was not duplicated} \\
K \cap \Omega^\pm & \text{if } K \text{ was duplicated and} \\
K & \text{it shares a node with an element} \\
\end{cases} 
\]

\( K \in T'_h \) such that \( K \subseteq \Omega^\pm \),

for any element \( K \in T'_h \), see Figure 3 for a sketch. Each \( v \in V_h \) then defines a function \( \ast v \) over \( \Omega \) (up to a set of measure zero) such that

\[
\ast v|_{\ast K} = v|_{\ast K}
\]

for all \( K \in T'_h \). The resulting function \( \ast v \) is single-valued almost everywhere in \( \Omega \). Roughly speaking, \( \ast v \) is obtained from \( v \) by restricting its values to the “half” of the element that does not contain duplicated nodes, for all those elements that are completely cut by the crack. Approximation of solutions will then be effectively sought in...
Figure 1: Example of an immersed boundary over a domain with no cracks. The domain $\Omega$ is immersed in a mesh that contains it, in this case the mesh of triangles over the square. Elements intersected by the domain are used to approximate functions inside the domain. These elements are shown in solid lines. In particular, the functions in the finite element space over these elements have a domain larger than $\Omega$, but only their restrictions to $\Omega$ are used.

Figure 2: (a) A crack defined by two implicit functions as $\{x \in \Omega : \phi(x) = 0, g(x) < 0\}$. (b) A mesh of $\Omega$ not taking into account the crack configuration. (c) The domain $\Omega$ is cut into two subdomains by the crack and its extension (dotted curve). Each subdomain is immersed into a submesh. Each submesh serves to represent the deformation of the corresponding subdomain. To obtain a continuous FE approximation of the displacements over the entire domain, and across the dotted curve in particular, some elements appearing in both submeshes but occupying the same physical spaces have to be merged into a single one. These pairs of elements are labeled $K_{i+}$ and $K_{i-}$, $i = 1, 2$. (d) A kinematically admissible deformed configuration as a result of the merging. The kinematics of the discontinuity inside the element that contains the crack tip will then be obtained by enriching the finite element space with the asymptotic crack tip fields.
the space \(*V_h = \{*v : v \in V_h\} = \{v \in V_h\} \) is adopted to approximate the kinematics of the regions \(*K\) and \(*K'\), respectively.

We conclude this description with a few remarks. First, in \(T'_h\) there may be three elements sharing a single edge, since two of these elements may occupy the same spatial location. Second, if the crack lies only along edges of \(T_h\), then \(T'_h\) is simply a mesh that conforms to the boundary, as traditionally done. Third, as mentioned earlier, the construction of \(T'_h\) makes adopting nonstandard FE spaces within each element simple. Finally, a similar algorithm should work in three-dimensional cracked domains.

In the following we specify \(V_h^K\) and adopt a different \(V_h\), in which we introduce some discontinuities in \(V_h^\text{conf}\) as part of the formulation of the numerical method.

### 2.3 Finite Element Spaces with Enrichment

Standard choices for \(V_h^\text{conf}\), such as \(V_h^\text{conf} = \{P_k(K)\}_{k=1}^{2}\), for all \(K \in T_h^\prime\), cannot approximate the singularity in the derivative near the crack tip at an optimal rate in \(h\) (in the \(H^1\)-seminorm). In fact, it is generally not better than \(h^{1/2}\).

One standard way to overcome this is by enriching \(V_h\) with the asymptotic crack-tip solutions \([29]\). For a homogeneous isotropic material these solutions read

\[
\Psi_I = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \left(\frac{\theta}{2}\right) (\kappa - \cos \theta) e_1',
\]

\[
+ \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \left(\frac{\theta}{2}\right) (\kappa - \cos \theta) e_2',
\]

\[
\Psi_{II} = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \left(\frac{\theta}{2}\right) (\kappa + 2 + \cos \theta) e_1',
\]

\[
+ \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \left(\frac{\theta}{2}\right) (2 - \kappa - \cos \theta) e_2',
\]

where \(\mu\) is the shear modulus, and \(\kappa\) is computed from Poisson’s ratio \(\nu\) as \(3 - 4\nu\) for plane strain loading and \((3-\nu)/(1+\nu)\) for plane stress loading. The right-handed orthonormal basis \(\{e_1', e_2'\}\) is such that \(e_1'\) is the tangential direction of the crack at \(\boldsymbol{x}\), pointing away from the crack (see Figure 4). Polar coordinates \(r\) and \(\theta\) are the standard crack-tip coordinates in which \(r\) is the distance from the crack tip and \(\theta = 0\) corresponds to the \(e_1'\) direction. In the case of a straight crack, \(\theta\) ranges from \(-\pi\) to \(\pi\). For curved cracks, the range of \(\theta\) changes with the value of \(r\) in a small enough neighborhood of the crack tip, see \([30]\).

In the scalar case of anti-plane shear crack problems, the enrichment would be constructed out of the mode III asymptotic solution, which reads

\[
\Psi_{III} = \frac{1}{\mu} \sqrt{\frac{2r}{\pi}} \sin \left(\frac{\theta}{2}\right).
\]
The generalization of the method for planar problems to this scalar-field problem is obvious. Hence, in the following we will only discuss the method as applied to planar crack problems.

The enriched FE space is then constructed as

\[ V_h = \ast V_h + \rho_h \text{span} \{ \Psi_I, \Psi_{II} \}. \]

(9)

Here \( \rho_h : \Omega \rightarrow \mathbb{R} \) is a function to be specified next, which may depend on \( h \), and whose role is to localize the enrichment to only a neighborhood around the crack tip.

Finally, in the context of this paper, the term “enrichment” only refers to the set of functions in \( \rho_h \text{span} \{ \Psi_I, \Psi_{II} \} \). This is in contrast to PU-based methods, in which the same term is adopted for the space of functions generated by the multiplication of a generalized Heaviside function as well as the asymptotic behavior at the crack tip by several basis functions of the space, e.g. [5].

2.4 Specifying the enrichment region: We set

\[ \rho_h(x) = \chi_{\Omega_h^E}(x) = \begin{cases} 1 & x \in \Omega_h^E \\ 0 & \text{otherwise}, \end{cases} \]

(10)

where

\[ \Omega_h^E \equiv \{ x \in K : K \in T_h, |K \cap \Omega_h^E| > 0 \}, \]

(11)

is the “enriched” region of the domain, since the enrichment functions are different than zero in all elements therein. Later in this section, we will comment on the choice of \( \Omega_h^E \). For the purpose of definition (11), we recall that we assumed that an element \( K \in \mathcal{T}_h \) is an open set in \( \mathbb{R}^2 \). Clearly this choice introduces discontinuous functions in \( \mathcal{V}_h \), with discontinuities along the set \( \Gamma_h^E = \partial \Omega_h^E \setminus \partial \Omega \), which we term as the enrichment boundary. For later use, we also denote the “unenriched” region by

\[ \Omega_h^U \equiv \Omega \setminus \Omega_h^E. \]

(12)

To obtain an optimally convergent approximation, it should be possible to decrease the size of the discontinuities along \( \Gamma_h^E \) at an appropriate rate with \( h \), since the exact solution is continuous. Consequently, the space \( V_h \) for each \( h \) needs to have a “rich enough” set of discontinuities along \( \Gamma_h^E \). The approximate solution can then be obtained through a DG formulation. One such choice of \( V_h \), and the one we adopt here, is

\[ V_h^{DG} = \{ \mathbf{v} \in \prod_{K \in \mathcal{T}_h'} V_h^K : \mathbf{v}_K|_e = \mathbf{v}_{K'}|_e \}

\]

if \( K, K' \) share edge \( e \), for any \( e \not\subset \partial \Omega_h^E \).}

(13)

The definition of \( \mathcal{V}_h \) follows from (9), i.e.,

\[ \mathcal{V}_h = \ast V_h^{DG} + \rho_h \text{span} \{ \Psi_I, \Psi_{II} \}, \]

(14)

and its description for the forthcoming discussion and numerical examples is completed by specifying \( V_h^K = [P_1(K)]^2 \), where \( K \) is a triangle, for all \( K \in \mathcal{T}_h' \). Under these conditions, any function \( u_h \in V_h^{DG} \) can be uniquely
expressed as
\[ u_h = \chi_{\Omega_h^U} \sum_{a \in S^U} \sum_{i=1}^2 \tilde{\psi}_a u_a e_i + \chi_{\Gamma_h^E} \left( \sum_{a \in S^E} \sum_{i=1}^2 \tilde{\psi}_a u_a^E e_i + K_1 \Psi_I + K_{11} \Psi_{11} \right), \]
where \( K_1, K_{11} \) are scalars, \( \chi_{\Omega} \) denotes the characteristic function of domain \( B \), and \( S^E \) and \( S^U \) denote the sets of nodes in elements included in \( \Omega_h^E \) and \( \Omega_h^U \), respectively. Function \( \tilde{\psi}_a \) is such that \( \tilde{\psi}_a|_K \) is a \( P_1 \) function for each \( K \in T_h \), continuous across elements that share an edge, and equal to 1 at \( a \) and zero on every other node in \( T_h \). By construction, \( u_h \) may be discontinuous across \( \Gamma_h^E \) and possibly bi-valued on elements and edges that intersect the crack \( \partial_{\Omega} \). In contrast, \( \ast u_h \) is single-valued therein, and possibly discontinuous across \( \partial_{\Omega} \).

**Choice of \( \Omega^E \):** As \( \Omega^E \) we choose the intersection of \( \Omega \) with an open disk centered at the crack tip with radius \( r_{\text{cutoff}} > 0 \). We assume that \( r_{\text{cutoff}} \) is such that \( \Omega^E \cap \partial_{\Omega} \) is a connected curve for any mesh \( T_h \). This condition prevents the crack from reentering the enrichment region. Since the hypotheses in Section 2 prevent the crack from intersecting itself, it is always possible to choose \( r_{\text{cutoff}} \) such that for \( h \) small enough a ball of radius \( r_{\text{cutoff}} + h \) centered at the crack tip (which contains \( \Omega^E \)) intersects the crack in a connected curve only. Furthermore, we assume that \( T_h \) is such that \( \partial_{\Omega} \cap \Gamma_h^E \) contains at most one point. This is clearly not satisfied for all meshes, since the crack may overlap with (parts of) edges in \( \Gamma_h^E \). However, it can be accomplished by slightly perturbing the position of only a few nodes provided \( h \) is small enough. This last assumption is not essential, but it greatly simplifies the implementation of the method. An example of \( \Omega^E \) and the resulting \( \Omega_h^E \) are shown in Figure 5.

### 2.5 Discontinuous Galerkin method for the enrichment boundary:

The approximation of continuous solutions with possibly-discontinuous functions needs the craft of special numerical methods, such as one of the many variants of DG methods. The reader is referred to Arnold et al. [31] for a unified analysis of many variants of DG methods. The method proposed herein originates in the one constructed for linear elastic problems in [20], and for the Poisson equation in [32].

We first define some notation. Let \( n \) be the outward unit normal to \( \Omega_h^E \), i.e., pointing towards \( \Omega_h^U \). For a scalar, vector or tensor field \( v \) over \( \Omega \) we denote with \( v^E \) and \( v^U \) its traces on \( \Gamma_h^E \) from \( \Omega_h^U \) and \( \Omega_h^E \), respectively. We also define the jump and average operators on \( \Gamma_h^E \) as
\[ [v] \equiv v^E - v^U, \quad \{v\} \equiv \frac{1}{2} (v^E + v^U), \]

We note that different authors have adopted different conventions for defining these operators, especially for vectorial arguments (see, for example, Arnold et al. [31]). Here we define them in a way that scalars, vectors and tensor fields are treated alike. Additionally, since we shall strongly impose Dirichlet boundary conditions, there is no need to define these operators on \( \partial_{\Omega} \).

**Definition of the DG-derivative:** Following a standard approach in some DG methods (see e.g., [20] [31]), approximations of the derivative of the solution are constructed through a DG-derivative operator on the discontinuous displacement field. This is defined as \( D_{DG} : \mathcal{Y}_h \to (L^2(\Omega))^2 \times 2 \) as
\[ D_{DG} v_h = \nabla v_h + R([v_h]) \quad \text{in } K, \quad \text{for any } K \in T_h, \]
where the (right) lifting operator \( R : (L^2(\Gamma_h^E))^2 \to \mathcal{W}_h \subset (L^2(\Omega))^2 \) is a linear operator that maps a displacement jump across \( \Gamma_h^E \) to a strain defined over \( \Omega \) such that for each element \( K \in T_h \)
\[ \int_{\Omega} R(v_h) : \gamma_h \, dS = - \int_{\Gamma_h^E} v_h \cdot \gamma_h \cdot n \, ds, \]
where
\[ \mathcal{W}_h = \left\{ \gamma_h \in \prod_{K \in T_h} W_h^K : W_h^K = (P_1(*K))^{2 \times 2} \text{ if } *K \subset \Omega_h^U, \right. \]
\[ \left. \text{and } W_h^K = (P_2(*K))^{2 \times 2} \text{ if } *K \subset \Omega_h^E \right\}. \]
Here \( P_k(K) \), for \( k \) a nonnegative integer, denotes the function space consisting of polynomials of degree up to \( k \) over \( K \).

**Variational Principle:** Having defined \( D_{DG} \), we introduce the following bilinear form \( a_h(\cdot, \cdot) : \mathcal{Y}_h \times \mathcal{W}_h \rightarrow \mathbb{R} \):
\[ a_h(u_h, v_h) \equiv \sum_{K \in T_h} \int_{\partial K} D_{DG} u_h : C : D_{DG} v_h \, dS \]
\[ + \beta c_{\min} \int_{\Omega} R([u_h]) : R([v_h]) \, dS, \]
where \( \beta \in \mathbb{R} \) is a stabilization parameter, and \( c_{\min} \) is the minimum eigenvalue of \( C \) among symmetric strains. For
Figure 5: Specification of the enrichment region $\Omega^E$ using a circle. Elements in dark gray are enriched with the near-tip asymptotic displacement fields. Their union is $\Omega^E_h$. Discontinuities in the displacement field are introduced across the boundary of $\Omega^E_h$, in this case also $\Gamma^E_h$, to cutoff the enrichment fields while retaining an optimal convergence rate. A DG formulation is adopted along the elements with an edge along $\Gamma^E_h$, which includes the elements shown in light gray outside $\Omega^E_h$. Some elements may have more than one of its edges across which discontinuities are allowed, as the one shown with a hollow square. Other elements may be both intersected by the crack and have an edge intersected by $\Gamma^E_h$. In this example those are shown with a black circle. These elements come in pairs, one for the kinematics on each side of the crack. In one of these pairs only one of the two elements shares an edge with $\Gamma^E_h$. 

Proceedings of 2011 NSF Engineering Research and Innovation Conference, Atlanta, Georgia

GRANT # 0747089
a stress-free isotropic linear elastic material \( c_{\min} = 2\mu \), where \( \mu \) is the shear modulus. For \( h > 0 \) small enough, the bilinear form is guaranteed to be coercive, i.e., the stiffness matrix is invertible, provided that \( \beta > 0 \), as shown in \([28]\). This is a sufficient but often unnecessary condition. In fact, \( \beta = 0 \) yields a positive-definite matrix in all the numerical examples shown later.

Without loss of generality we assume that \( \partial_d \Omega \cap \partial \Omega^E_h = \emptyset \) (Dirichlet boundary conditions for the case of \( \partial_d \Omega \cap \partial \Omega^E_h \neq \emptyset \) can be imposed as equality constraints for nodal values of \( U \) on \( \partial_d \Omega \)). This is always possible by choosing a small enough value for \( r_{\text{cutoff}} \). Let then \( \mathcal{V}_h \) denote the subset of \( \mathcal{V}_h \) whose members satisfy the Dirichlet boundary condition at all nodes on \( \partial_d \Omega \). The fact that \( \partial \Omega_d \cap \partial \Omega^E_h = \emptyset \) guarantees that the crack enrichments do not affect the value of a function in \( \mathcal{V}_h \) along the Dirichlet boundary. Therefore, \( \mathcal{V}_{h\Omega} \) is simple to construct. Similarly, let \( \mathcal{V}_{h0} \) be the subspace of \( \mathcal{V}_h \) whose members are identically zero at all nodes on \( \partial_d \Omega \).

The numerical method is defined by the following variational problem corresponding to the strong form (1): find \( u_h \in \mathcal{V}_{h\Omega} \), such that

\[
a_h(u_h, v_h) = \int_{\Omega} f \cdot v_h dS + \int_{\partial_h \Omega} T \cdot v_h ds, \quad \forall v_h \in \mathcal{V}_{h0}.
\]

Equation 21 defines the methodology used to define the FE approximation \( u_h \) to \( u \). In [33] we proved that the method defined by (21) is stable for any \( \beta > 0 \).

The numerical integration strategy plays a crucial role in the asymptotic accuracy of the method, which we discussed extensively in [28]. A direct consequence of the integration strategy in [28] is the surprisingly sparse structure of the stiffness matrix, as demonstrated in Figure 6.

2.6 Computation of stress intensity factors: We briefly comment next on the methods used in the numerical examples later to compute SIFs. A neat consequence of the choice of spaces for the method herein is that it is possible to prove that the coefficients of \( \Psi_I \) and \( \Psi_{II} \) converge to the SIFs \( K_I \) and \( K_{II} \) as \( h \) goes to zero. Consequently, the SIFs are obtained directly as part of the primary unknowns.

A similar strategy was adopted by Liu et al. [22], even though they enriched both the leading and some higher-order terms of the asymptotic expansion of the solution near the crack tip. They extracted their SIFs from the leading terms only.

The second and prevailing method for obtaining \( K_I \) and \( K_{II} \) is to compute the interaction integral \([34]\), as described in [10], since \( K_I \) and \( K_{II} \) converge quadratically with \( h \) in this way. We refer the unfamiliar reader to one of the two previous references for insight into their origin.

An attractive feature of the first method over the last one is that when the SIFs are extracted as the coefficients in front of the singularities in the numerical solutions, we are literally using the definition of SIF. In contrast, the second method based in the computation of interaction integrals has a faster expected convergence rate. However, it relies on stricter assumptions on the solution, including the dominance of the near-tip solution over the region of integration, the absence of body forces, and the satisfaction of the zero-traction boundary condition on the crack faces.

3 Numerical examples: We illustrate the performance of the method next with some numerical examples. No stabilization was needed in any of them, i.e., we adopted \( \beta = 0 \) in (20).

3.1 Mode I near-tip field: Our first numerical example tests the convergence properties of the method with both structured and unstructured meshes. For this purpose we considered a square domain \((0, 1) \times (0, 1)\) with an edge crack \([0, 0.55] \times \{0.5\}\) as shown in Figure 7. Dirichlet boundary conditions were imposed on the sides of the square according to the mode I asymptotic field, i.e., \( U = \Psi_I \), with crack length \( a = 0.55 \), Young's modulus \( E = 1000 \), and Poisson’s ratio \( \nu = 0.3 \). We set \( r_{\text{cutoff}} = 0.15 \), and solved the problem under plane strain loading. The coarsest structured and unstructured meshes are depicted in Figures 7(a) and 7(b), respectively. Finer meshes in each case are obtained by subdividing each triangle into four similar ones, recursively.

Before discussing convergence rates, we note that discontinuities do appear in the solutions across \( \Gamma_h \). They are small to be displayed well, and become even smaller as the mesh is refined, but are critical for the optimal convergence of the method.

The convergence curves for displacements and stresses for solutions obtained with the two families of meshes are shown in Figures 8 and 9. To highlight the importance of the quadrature strategy discussed in [28] on the computation of SIFs, we show convergence curves for our method in which all terms in the stiffness matrix and right hand side are instead computed through domain integrals. In this case, for the most singular integrands we adopted the \( 12 \times 12 \) “almost polar” scheme proposed by [9] based on partitioning the element that contains the crack tip \( x_t \) into triangles for each of which \( x_t \) is a vertex. The resulting convergence curves are labeled “Our method (domain integral).” We also show therein the convergence curves for three other methods with the same
Figure 6: The stiffness matrix of the problem is surprisingly sparse. This figure shows a sketch of possible couplings between the degrees of freedom (DOFs) of enrichment functions and those of the polynomial basis. The enrichment region $\Omega^E$ is shown in gray. Of those terms in the stiffness matrix that relate enrichment and polynomial DOFs, the only ones that are different than zero, and hence that need to be computed, are those corresponding to basis functions associated with nodes marked with a circle. As described earlier, some nodes have been duplicated and hence occupy the same spatial location, so a single circle corresponds to both of them. All nodes that do not belong to elements that are either intersected by the crack or that have at least one edge on the enrichment region boundary have their shape functions uncoupled from the enrichment functions. Consequently, the number of entries in the stiffness matrix of this type that need to be computed scales as $O(h^{-1})$. The colors inside the circles indicate how the non-zero terms are computed, see the text for a description.
meshes, namely: a) the (standard) FEM that results from choosing \( \mathcal{V}_h = * \mathcal{V}_{h}^{\text{conf.}} \), in which no singular enrichments are included, b) our method with the quadrature strategy in [28] but with a shrinking enrichment zone, i.e., we chose a different \( \Omega_E \) for each \( h \) by setting \( r_{\text{cutoff}} = h_{\text{max}} \) where \( h_{\text{max}} \) is the maximum length among all triangles in each mesh, see Section 2.4, and c) the cutoff function method by [13] with \( r_0 = 0.003 \) and \( r_1 = 0.15 \), see Section 2.4 as well. In (c) all integrals were performed over the domain by using the strategy in [9], as we did for the curves labeled “Our method (domain integral).”

These results show that not including singular enrichments, or including them but only in an enrichment zone that shrinks with \( h \) gives rise to the same type of suboptimal convergence rate. Naturally, errors in the latter case are smaller, as seen in Figures 8 and 9. Both the cutoff function method and our method render an apparent optimal convergence rate. However, for this example the errors with the method introduced herein are uniformly smaller by roughly one order of magnitude. For this range of mesh sizes there is no distinguishable difference between the quadrature strategies, except for the data point in Figure 9(a) corresponding to the finest mesh. As we discuss next, this is no longer the case when the convergence of the SIFs is considered.

Figure 10 shows the errors in modes I and II SIFs extracted as the coefficients of the enrichment basis functions \( \Psi_I \) and \( \Psi_{II} \), respectively, for each of the methods for which this is possible. When endowed with the quadrature strategy in [28], our method displays a remarkable ability to capture the value of the SIFs as the mesh is refined, in this and subsequent examples. In contrast, when endowed with the domain integration scheme described in [9], the SIFs do not display a converging pattern. The reasons behind this behavior can be traced back to the lack of an appropriate order for the consistency error introduced by the quadrature rule. On the other hand, the cutoff function method which also uses a domain integration scheme does show a converging trend for the SIFs as the mesh is refined, at least for \( K_I \), but the errors of this method are larger for the same mesh and cutoff radius of enrichment.

We next compare the convergence of the SIF in mode I when computed with the interaction integral, see Figure 11. Quadratic convergence is attained for our method as well as the cutoff function method, with the former having an error approximately one order of magnitude smaller for the same mesh. Also plotted is the SIF extracted as the coefficient of \( \Psi_I \) in our method. In this case the two methods of extracting SIFs yield very similar results. However, in general the convergence rate of the method based on the interaction integral will be faster, though more computationally expensive as well.

Finally, we investigated the dependence of the error in the stress field and the mode I SIF as the value of \( r_{\text{cutoff}} \) is changed for our method. These values were computed on an unstructured mesh obtained by performing one subdivision of that in Figure 7(b). A comparison with the cutoff function method is also shown, by varying the value of \( r_1 \) and keeping either the ratio \( r_0/r_1 \) or the value of \( r_0 \) fixed. The results are shown in Figure 12, indicating as expected that the larger \( r_{\text{cutoff}} \) is, the smaller the errors become.

### 3.2 Uniaxial tension:

In this second numerical example we compute the SIF of a planar slab with a crack of length \( a \) in plane strain under uniaxial tension, i.e., the traction vector on the faces parallel to the crack is normal to them and has magnitude \( \sigma \), as shown in Figure 13(a). The Young’s modulus and Poisson’s ratio are 1000 and 0.3, respectively. The deformed configuration obtained with a relatively fine mesh, together with a contour plot of the von Mises stress is shown in Figure 13(b).

The mode I SIF for this example is given by Tada et al. as

\[
K_I = \sigma \sqrt{W} f \left( \frac{a}{W} \right),
\]

where \( W \) denotes the width of the plate and

\[
f \left( \frac{a}{W} \right) = \sqrt{\frac{2 \tan \frac{\pi a}{2W}}{\cos \frac{\pi a}{2W}}} \left[ 0.752 + 2.02 \frac{a}{W} + 0.37 \left( 1 - \sin \frac{\pi a}{2W} \right)^3 \right].
\]

In Table 1 we have tabulated the computed values of \( K_I \) comparing the results reported in Stazi et al. and those of our method with a structured mesh similar to the one in Figure 13(a) and \( r_{\text{cutoff}} = 0.15W \). Both results are then obtained with comparable meshes, approximately the same number of \( P_1 \) DOFs. Regardless of how \( K_I \) is computed, either directly from the DOF conjugate to \( \Psi_I \) or through the interaction integral, the values computed with the method herein are closer to the exact ones.

### 3.3 Mixed-mode loadings:

Our next two examples are two mixed-mode loadings: a planar slab with a slanted edge crack subject to uniaxial tension, see Figure 14 and another one in which one side is clamped and another side is subject to uniform shear, see Figure 15. The Young’s moduli and Poisson’s ratios are 1000 and 0.3 for the first example and 30 and 0.25 for the second example.

We solved these two problems with an unstructured mesh similar to that shown in Figure 7(b). The computed SIFs are tabulated in Tables 2 and 3.
Figure 7: Coarsest structured (left) and unstructured (right) meshes with a crack for the example in Section 3.1. The enrichment zone $\Omega^E_h$ is shown in gray, and the disk of radius $r_{\text{cutoff}}$ that defines $\Omega^E$ is shown with dashed lines.

<table>
<thead>
<tr>
<th>$a/W$</th>
<th>exact</th>
<th>our method (DOF extraction)</th>
<th>our method (interaction integral)</th>
<th>Stazi et al. [30] (interaction integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21</td>
<td>1.1288</td>
<td>1.0909</td>
<td>1.1218</td>
<td>1.0616</td>
</tr>
<tr>
<td>0.22</td>
<td>1.1755</td>
<td>1.1353</td>
<td>1.1691</td>
<td>1.1000</td>
</tr>
<tr>
<td>0.23</td>
<td>1.2235</td>
<td>1.1820</td>
<td>1.2178</td>
<td>1.1321</td>
</tr>
<tr>
<td>0.24</td>
<td>1.2730</td>
<td>1.2320</td>
<td>1.2691</td>
<td>1.1558</td>
</tr>
<tr>
<td>0.28</td>
<td>1.4878</td>
<td>1.4665</td>
<td>1.4833</td>
<td>1.3783</td>
</tr>
<tr>
<td>0.50</td>
<td>3.5426</td>
<td>3.5084</td>
<td>3.4892</td>
<td>3.1299</td>
</tr>
</tbody>
</table>

Table 1: Exact and computed mode I SIFs of the specimen under uniaxial tension shown in Figure 13. The data in columns 3 and 4 are computed using our method with $r_{\text{cutoff}} = 0.15W$. The SIFs in column 3 were obtained by extracting the coefficient of the mode I enrichment $\Psi_I$, while those in column 4 were obtained using the interaction integral method, as described in Section 2.6. The data in column 5 is reported in Stazi et al. [30]. The latter adopted the XFEM in which only elements close to the crack tip are enriched, and the SIF values were computed through the interaction integral based on their numerical solution. The meshes adopted in the two cases have roughly the same number of $P_1$ DOF (approximately 1,280).

Table 2: Modes I and II SIFs normalized by $\sigma\sqrt{\pi a}$ for the solution of the slanted crack in Figure 14. These values were computed with our method in an unstructured mesh with $r_{\text{cutoff}} = 0.15W$. The SIFs were computed both by extracting the enrichment coefficients and by invoking the interaction integral method. A reference solution given by Yau et al. [37] is $K_I = (1.825 \pm 0.030)\sigma\sqrt{\pi a}$, $K_{II} = (0.831 \pm 0.004)\sigma\sqrt{\pi a}$.

<table>
<thead>
<tr>
<th>Number of DOFs</th>
<th>$K_I$ (DOF extraction)</th>
<th>$K_I$ (interaction integral)</th>
<th>$K_{II}$ (DOF extraction)</th>
<th>$K_{II}$ (interaction integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>648</td>
<td>1.872</td>
<td>1.790</td>
<td>0.755</td>
<td>0.813</td>
</tr>
<tr>
<td>2,342</td>
<td>1.839</td>
<td>1.832</td>
<td>0.824</td>
<td>0.826</td>
</tr>
<tr>
<td>8,898</td>
<td>1.849</td>
<td>1.845</td>
<td>0.829</td>
<td>0.830</td>
</tr>
<tr>
<td>34,680</td>
<td>1.851</td>
<td>1.849</td>
<td>0.831</td>
<td>0.831</td>
</tr>
<tr>
<td>136,932</td>
<td>1.852</td>
<td>1.850</td>
<td>0.832</td>
<td>0.831</td>
</tr>
</tbody>
</table>
Figure 8: Convergence curves in terms of $L^2$-errors in (a) displacements and (b) stresses of five methods using a structured mesh family whose coarsest mesh is shown in Figure 7(a) with 144 elements. Finer meshes are obtained by subdividing each triangle into four similar triangles recursively. These methods are: (O) the standard finite element method; (×) our method with a shrinking enrichment zone defined by $r_{\text{cutoff}} = h$; (+) the cutoff function method in [13] with $r_0 = 0.003$ and $r_1 = 0.15$; (□) and (*) our method with $r_{\text{cutoff}} = 0.15$, where (□) and (*) differ in the integration method for the most singular integral: (□) uses the $12 \times 12$ “almost polar” integration scheme proposed by [9] while (*) uses the boundary integration scheme discussed in [28] which originates in [35]. For a fair comparison, (+) and (□) use the same integration schemes.
Figure 9: Convergence curves in terms of $L^2$-errors in (a) displacements and (b) stresses of five methods using an unstructured mesh family whose coarsest mesh is shown in Figure 7(b) with 112 elements. Finer meshes are obtained by subdividing each triangle into four similar triangles recursively. These methods are: (O) the standard finite element method; (×) our method but with the enrichment zone defined by $r_{\text{cutoff}} = h$; (+) the cutoff function method in [13] with $r_0 = 0.003$ and $r_1 = 0.15$; (□) and (*) our method with $r_{\text{cutoff}} = 0.15$, where (□) and (*) differ in the integration method for the most singular integral: (□) uses the 12 × 12 “almost polar” integration scheme proposed by [9] while (*) uses the boundary integration scheme discussed in [28] which originates in [35]. For a fair comparison, (+) and (□) use the same integration schemes.
Figure 10: Normalized errors in SIFs $K_I$ and $K_{II}$ for different methods as extracted as the coefficients of the corresponding enrichment functions in the numerical solutions computed with the family of unstructured meshes stemming from the one in Figure 7(b). Both errors are normalized by $|K_I| = 1$. See the text for the description of each method.
Figure 11: Asymptotic error plots of the mode I SIF obtained by computing the interaction integral, see Section 2.6, for the cutoff function method and our method. The error of the mode I SIF obtained by extracting the enrichment coefficient is also shown for comparison.

Table 3: Modes I and II SIFs of the cracked slab subjected to shear in Figure 15 with $\tau = 1$. The solution was obtained with our method in an unstructured mesh with $r_{\text{cutoff}} = 2.0$. These SIFs are computed both by extracting the enrichment coefficients and by invoking the interaction integral method. A reference solution given by Wilson [38] is $K_I = 34.0, K_{II} = 4.55$.  

<table>
<thead>
<tr>
<th>Number of DOFs</th>
<th>$K_I$ (DOF extraction)</th>
<th>$K_I$ (interaction integral)</th>
<th>$K_{II}$ (DOF extraction)</th>
<th>$K_{II}$ (interaction integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>760</td>
<td>34.2</td>
<td>33.0</td>
<td>3.52</td>
<td>4.50</td>
</tr>
<tr>
<td>2,736</td>
<td>34.9</td>
<td>33.8</td>
<td>4.05</td>
<td>4.53</td>
</tr>
<tr>
<td>10,384</td>
<td>34.8</td>
<td>34.0</td>
<td>4.45</td>
<td>4.53</td>
</tr>
<tr>
<td>40,466</td>
<td>34.3</td>
<td>34.0</td>
<td>4.54</td>
<td>4.54</td>
</tr>
<tr>
<td>159,770</td>
<td>34.2</td>
<td>34.0</td>
<td>4.54</td>
<td>4.54</td>
</tr>
</tbody>
</table>
Figure 12: Relative errors in (a) stress and (b) mode I SIF as the value of $r_{\text{cutoff}}$ is changed over the same mesh for our method, or as the value of $r_1$ was changed for the cutoff function method. The SIF was extracted as the coefficient of $\Psi_I$. (O) cutoff function method with $r_0/r_1 = 0.02$; (×) cutoff function method with $r_0 = 0.003$; (•) our method.
Figure 13: (a) A cracked finite slab under uniaxial tension. (b) Deformed configuration of the same specimen, with a contour plot of the von Mises stress. This result was obtained with a mesh that contains 73,278 elements, resulting in a total of 74,690 DOFs.

Figure 14: (a) A 45° slanted crack in a finite slab subject to tension, with \( a/W = 1/\sqrt{2} \). (b) Deformed configuration of the specimen with a contour plot of the von Mises stress. This solution was obtained with a mesh that contains 135,168 elements and 136,932 DOFs.
that the reference solutions given in the figure captions are also numerical results. One can observe the ability of our method to directly obtain the SIFs for mixed-mode loading configurations without post-processing.

4 Conclusions: We have proposed and demonstrated the performance of an optimally convergent DG-based XFEM. The discontinuity along the enrichment boundary serves the purpose of eliminating the blending elements in earlier versions of the XFEM. Additionally, the enrichment region is chosen so that its size does not go to zero with $h$. This is critical to attain an optimal convergence rate, and since the enrichment functions are lightly coupled with the polynomial basis functions in the enrichment region, this choice has a minor impact in the efficiency of the method. In fact, this choice prevents the condition number of the stiffness matrices in our method from deteriorating too quickly with $h$, which is the case when the enrichments are introduced through a PU on the same enrichment region, see [9]. Another critical advantage of the method is the possibility of obtaining accurate values of the SIFs as primary unknowns of the system of equations.

The numerical results herein indicate that not only does the proposed method render an optimal order of convergence, but that the solutions are remarkably accurate, at least in comparison with the cutoff function method. As mentioned earlier, it is possible that this accuracy is largely the result of the sharp transition of the enrichment functions to zero, so other alternatives in the literature may perform similarly well. This superior accuracy was also encountered in the computation of the SIFs, even when they were extracted as the coefficients of the enrichment functions in the numerical solution.

Finally, we provided an interpretation of cracks as immersed boundaries, which enables the construction of finite element spaces over “arbitrarily” cracked domains. Essentially, it decouples the introduction of the discontinuity in the mesh from the construction of the finite element space over it.

5 Acknowledgements: This work is supported by the NSF Career Award with Grant Number CMMI-0747089.

6 References:


