DISCONTINUOUS GALERKIN FOR NONLINEAR ELASTICITY

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Abstract. This paper is concerned with recent results on the formulation of Discontinuous Galerkin (DG) methods for nonlinear elastic problems. Nonlinear elasticity is key to the study of deformable bodies made of polymeric, biological, martensitic and elasto-plastic materials. Widely used in the study of fluid mechanics and other hyperbolic phenomena, DG has more recently proved itself as a robust tool in the study of linear elliptic problems. Here, we provide additional examples of a class of DG methods for nonlinear elasticity the authors have recently proposed and analyzed. Additionally, the main results of the analysis are reviewed. The paper contains several numerical examples that highlight the performance benefits of DG in comparison with traditional conforming finite element methods.
1 PROBLEM DESCRIPTION

The nonlinear elasticity problem begins with an elastic body in its initial configuration, \( B_0 \in \mathbb{R}^d \). In the presence of applied tractions, \( \mathbf{T} \) on \( \partial_r B_0 \), and imposed displacements, we wish to find the deformation mapping \( \varphi \), that minimizes the total potential energy, \( I[\varphi] \), where

\[
I[\varphi] = \int_{B_0} W(\mathbf{X}, \mathbf{F}) \, dV - \int_{\partial_r B_0} \mathbf{T} \cdot \varphi \, dS. \tag{1}
\]

The essential parameter here being the body’s strain energy density, \( W(\mathbf{X}, \mathbf{F}) : B_0 \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R} \), where \( \mathbf{X} \) is the reference position, and \( \mathbf{F} \) is the deformation gradient. In general, materials with complex constitutive behavior, such as muscles, blood vessels, elastoplastic materials, and martensitic materials, often have very complex strain energy densities that can be very difficult to model. For our purposes, we assume that \( W(\mathbf{F}) \) exists, is indifferent to rigid body rotations, and is well-behaved in general. Materials with these properties are often considered to be simple, a classic example is that of a compressible Neo-hookean material, which is used to model soft materials such as rubber, and can be written

\[
W(\mathbf{F}) = \frac{\lambda}{2} (\log(\det \mathbf{F}))^2 - \mu \log(\det \mathbf{F}) + \frac{\mu}{2} \mathbf{F} : \mathbf{F}, \tag{2}
\]

where \( \lambda, \mu > 0 \) are material constants.

2 DISCRETIZATION

The formulation of discontinuous Galerkin (DG) methods for nonlinear elasticity begins with the discretization of \( B_0 \) into a finite element mesh, \( T_h \). For each element \( E \in T_h \), the deformation mapping \( \varphi \) is approximated with the discrete space, \( V_E^h \). The finite element space is constructed as \( V^h = \Pi_{E \in T_h} V_E^h \), which allows functions in \( V^h \) to be discontinuous across element faces. This is a sharp contrast to traditional finite element methods for nonlinear elasticity where admissible configurations in \( V^h \) are strictly continuous. If we consider a uniaxial tension test in \( \mathbb{R}^2 \), we see in figure 1(a) a case where \( V^h \) is continuous across element faces, and in figure 1(b) a case where \( V^h \) is discontinuous across element faces. For this particular example, an extremely coarse mesh has been used, making the resulting discontinuities in the discontinuous Galerkin solution very prominent. Here we also see that by adding discontinuities as additional degrees of freedom the solution is in fact much nicer than that of the continuous solutions. The real challenge when developing a finite element method with discontinuous Galerkin is how to incorporate the discontinuities into \( \mathbf{F} \), the deformation gradient.

3 THE DG DERIVATIVE AND A ONE FIELD VARIATIONAL PRINCIPLE

DG methods are often presented in a mixed formulation that involves an additional field of unknowns in \( \Gamma \). Where \( \Gamma \) is the set of element faces, consisting of Dirichlet boundaries, Neumann boundaries, as well as all interior faces. From figure 1(b), we can see that such a method would no doubt inevitably require loops over exactly three faces for each element. For extremely fine meshes, this could become impractical. Wishing to avoid this we have chosen to present the formulation as a one field variational principle.

Since of course, the discontinuities reside in \( \Gamma \), and at some point they will need to be considered when evaluating \( \mathbf{F} \), we do not expect to completely avoid a loop over the element faces, we simply wish to minimize this as much as possible. With a one field variational principle, we show in [3] that this computation can, in fact, be performed only once.
This elegant and simple formulation begins with defining an approximation, we call it the DG derivative, to the deformation gradient, $\mathbf{F}$. In accordance with [2] and [3] the DG derivative for the nonlinear elasticity problem is

$$ D_{DG}\varphi = \nabla \varphi + R([\varphi - \hat{\varphi}]) + L(\{\varphi - \hat{\varphi}\}) $$

where $\hat{\varphi}$ is an approximation, usually single–valued, of $\varphi$ in $\Gamma$. The quantities $R()$ and $L()$ are linear operators called lifting operators, such that $R() : L^2(\Gamma) \mapsto W^h$ and $L() : L^2(\Gamma) \mapsto W^h$. These are precisely the operators that allow us to avoid looping over element faces, and maintain the traditional computational straightforwardness of the finite element method. Notice that we have introduced the space $W^h$, and thus require $\nabla V^h \subseteq W^h$, so that $D_{DG}\varphi \in W^h$. In a traditional finite element method, $\mathbf{F} = \nabla \varphi \in \nabla V^h$, however, the DG approximation will exist in the much richer space, $W^h$. The operators $[z]$ and $\{z\}$ for $z \in V^h$, in equation 3 are defined as

$$ [z] = z^- - z^+, \quad \{z\} = \frac{1}{2}(z^+ + z^-) $$

which result in arguments that are single valued on $\Gamma$. Thus, the approximation to the gradient of $\varphi$ is completely unambiguous and in [3] we show that its simple construction exploits the block diagonal nature of discontinuous Galerkin mass matrices.

Having defined the DG derivative, we see the deformation gradient $\mathbf{F}$ can be approximated in a space where $\varphi$ is discontinuous. This is simply $\nabla \varphi$ plus the addition from the lifting operators which take into account any discontinuities in $\Gamma$. For those not accustomed to discontinuous Galerkin techniques, we wish to stress that in no way are we trying to minimize discontinuities. The entire reason for using a discontinuous space is to expand the space of admissible configurations when looking for the minimizers of equation 1. The lifting operators are unquestionably the quantities that give any possible advantage over using a continuous $V^h$. 

Figure 1: Quasistatic evolution of a uniaxial tension test with a Neohookean material. $\lambda/\nu=9.0$
We find the minimizers of equation 1 by taking variations with respect to \( \varphi \). However, now we are approximating \( F \) with \( D_{DG} \varphi \). Equating the first variation to zero we get

\[
0 = \langle \delta I_h[\varphi], \delta \varphi \rangle = \sum_{E \in \mathcal{T}_h} \int_E \frac{\partial W}{\partial \mathcal{F}}(X, D_{DG} \varphi) : D_{DG} \delta \varphi \, dV - \int_{\partial_e B_0} T \cdot \delta \varphi \, dS, \tag{5}
\]

where \( \delta \varphi = \partial \varphi / \partial \varepsilon \big|_{\varepsilon=0} \). Solving the nonlinear problem involves taking a second variation and using a Newton–like iterative method. The second variation is

\[
\langle \delta^2 I_h[\varphi], \delta \varphi, u \rangle = \sum_{E \in \mathcal{T}_h} \int_E D_{DG} u : A(X, D_{DG} \varphi) : D_{DG} \delta \varphi \, dV, \tag{6}
\]

The resulting method can now be expressed as

\[
0 = \langle \delta^2 I_h[\varphi], \delta \varphi, u \rangle + \langle \delta I_h[\varphi], \delta \varphi \rangle \tag{7}
\]

where \( u \) is another admissible variation that must be computed. Here, \( u \), can be regarded as a displacement field with respect to the current configuration \( \varphi \).

4 LIFTING OPERATORS AND THE NUMERICAL FLUXES

The lifting operators are defined in equation 8. Taking a moment to look at their definition, we see that in a subspace of \( W^h \) and \( V^h \) where functions are strictly continuous, we quickly notice that \( \mathbf{L} \) in equation 8 drops out immediately, and from equation 3, \( \mathbf{R} \) drops out leaving \( D_{DG} \varphi = \nabla \varphi \), which is exactly a conforming method. In other words, the conforming finite element solution will exist in a subspace of the admissible configurations for the discontinuous Galerkin method.

\[
\int_{B_0} \mathbf{R}(v) \cdot \mathbf{z} \, dV = -\int_{\Gamma} v \otimes \mathbf{N} \cdot \{ \mathbf{z} \} \, dS, \quad \int_{B_0} \mathbf{L}(v) \cdot \mathbf{z} \, dV = -\int_{\Gamma} v \otimes \mathbf{N} \cdot [\mathbf{z}] \, dS, \tag{8}
\]

for all \( \mathbf{z} \in W_h \) and all \( v \in V^d_h \).

The defining property of a discontinuous Galerkin implementation will be the choice of numerical fluxes, \( \hat{\varphi} \). The choice of fluxes determines all the resulting characteristics of the method such as computational cost, accuracy, consistency, and stability. In this paper we implement the fluxes of Bassi and Rebay [1] which are often a starting point for DG since they are a rather natural selection. For these fluxes, \( \hat{\varphi} \) is \( \{ \varphi \} \) on interior element boundaries, \( \bar{\varphi} \) on Dirichlet boundaries, and \( \varphi \) on Neumann boundaries. Here we take note that the Dirichlet boundary conditions, \( \bar{\varphi} \), are weakly imposed through the lifting operator. As opposed to more traditional finite element methods where weakly imposed boundary conditions often are incorporated in the form of either a penalty method or with Lagrange multipliers, discontinuous Galerkin does not need any additional constraints. Not only does discontinuous Galerkin maintain its computational simplicity with weakly imposed boundary conditions, there is also no need to loop over the element faces on Dirichlet boundaries, since the boundary conditions are embedded in the lifting operator which are integrated over element interiors when assembling the stiffness matrix. For cases where weakly imposing boundary conditions are highly desirable, discontinuous Galerkin could prove to be quite advantageous.
5 STABILIZATION

Although this choice for a numerical flux has several nice features, the linearized elasticity problem in equation 7 can be unstable. This is true even when the problem is stable if \( V^h \) is restricted to a continuous space. However, there is a rather simple and common method of stabilization that can be applied to this problem. We modify the energy functional with

\[
I^\beta_h[\varphi] = I_h[\varphi] + \frac{\beta}{h} \int_{\Gamma} [\varphi] - \varphi(\varphi) \cdot [\varphi] \cdot \varphi(\varphi) \, dS, \tag{9}
\]

where \( \beta \) is some positive number and \( h \) is a measure of the mesh fineness. This simply translates to a penalization of discontinuities on element boundaries. To provide an estimate for the amount of stabilization needed we begin in [3] by first proving the existence of a Korn’s inequality of the form

\[
k_{1h} \|u\|_F^2 \leq \|u\|_{F,s}^2 \leq \|u\|_F^2, \tag{10}
\]

where \( k_{1h} > 0 \). The two norms, \( \|u\|_F \), and \( \|u\|_{F,s} \) are norms that give a measure of \( D_{DG}\varphi \). A very general statement of stability can be made, assuming

\[
g : a(X, F(X)) : g > C_1 g_s : g_s - C_2 g : g \tag{11}
\]

where \( C_1 > 0, C_2 \) is some constant, \( g \) is any tensor in \( \mathbb{R}^{d\times d} \), \( g^e = (g + g^T)/2 \), and \( a \) is a spatial elasticity tensor [3]. The necessary and sufficient conditions on \( a \) to guarantee coercivity is still an open question. The relation given in equation 11 is an effort to capture the largest space of admissible configurations in which the nonlinear elasticity problem is linearly stable. For the case where \( k_{1h}C_1 - C_2 > 0 \), with \( k_{1h} > 0 \), there exists \( C_h > 0 \) such that

\[
\langle \beta^2 I^\beta_h[\varphi], u, u \rangle > C_h \|u\|_F^2 \tag{12}
\]

for all \( u \in V^h \), provided that

\[
\beta > \left( \alpha - 1 + \frac{1}{1 - \alpha} \right) \frac{C_1}{k_4}, \tag{13}
\]

where \( 0 \leq \alpha < 1 \) and \( k_4 > 0 \). It can also be shown in [3] that the minimum amount of stabilization necessary scales with the largest compressive principle stress.

6 NUMERICAL EXAMPLES

The numerical examples presented in the paper compare and contrast the results generated by the DG method and that of a conforming finite element method. For each test the material is a Neo-Hookean material extended to the compressible range. The material properties are set by the parameter \( \nu = 1/2(\mu/\lambda + 1) \) which represents the Poisson ratio. The stabilization parameter \( \beta \) was linearly incremented at each quasistatic loading step. Each test only considers linear interpolation functions, so all 2–D examples use 3–node triangles and all 3–D examples use 4–node tetrahedra.

One of the properties first noticed about DG, was its robustness when simulating problems with large deformations. A model problem consisting of a square block undergoing vertical compression was simulated with both DG and conforming finite elements, see figure 2(a). It was found that due to mesh based kinematic constraints, the conforming solution always produced an asymmetric result, despite completely symmetric loading conditions. The problem could not
be corrected with any level of mesh refinement. The DG method, on the other hand, was free of this constraint and was able to produce symmetric deformations.

The lack of constraints also proved to be highly advantageous when simulating materials that are nearly incompressible. In the simulation of a hollow cylinder undergoing radial expansion it was found that the DG method did not lock in the large deformation regime. A cross section of the deformed configuration is shown in figure 3. Locking was identified by evaluating the resulting tractions measured on the inner wall of the cylinder. When the tractions increased to unreasonable values for this particular deformation, locking has occurred. This is the inverse approach to the traditional technique of identifying locking by applying a load and then looking for unreasonably small displacements.

![Fig 2 and 3](image)

Figure 2: Compression of a block using 1000 steps. \( \nu=0.45, \Delta/\beta = 10^{-3} \text{ Pa/Step.} \)

Figure 3: A cross section of a cylinder where the inner radius expanded 10 percent of its original radius. (a) shows both the discontinuous Galerkin solution and the conforming solution. A magnified view of the far right side of the cylinder is shown in (b). Visual evidence of locking can be seen for the conforming solution in (b).

Figure 4 shows that even when the Poisson ratio is pushed to 0.4999, the DG solution maintains a reasonable approximation to the exact solution. However, the tractions (i.e. pressure) needed to expand the cylinder are several orders of magnitude larger than expected when a conforming method is used. This is a case where the conforming method locks and the DG method
does not even when low–order elements are used.

![Figure 4](image-url)

Figure 4: Traction measured on inner wall of expanding cylinder as a function of the ratio of the deformed to reference radius using 10,000 steps. E=1 Pa, $\Delta \beta = 10^{-4}$ Pa/Step.

In addition to situations where DG can provide solutions that the conforming method cannot, DG is also competitive with regards to computational cost. Figure 5(b) shows the simulation of a square block undergoing expansion coupled with shear, we found to our surprise that this test did not require any stabilization (i.e. $\beta = 0$). The block is a Neohookean material extended to the compressible range with $\nu = 0.45$. When the $L^2$ norm of the error in displacements were compared with the total computational cost it was found that the DG method provided the same accuracy per computational cost as that of a conforming method. This can be seen in figure 5(a). The computational cost was measured in terms of total number of degrees of freedom. Also the exact solution was found with a conforming solution consisting of 79,202 linear triangles.
We also conducted a test where stabilization was critical. Figure 6(b) shows several snapshots from the quasistatic evolution of a square block undergoing an applied shear force on the upper surface. The block is a Neo-Hookean material extended to the compressible range with \( \nu = 0.15 \). For this case several different increment schemes were used for the stabilization parameter \( \beta \). We found that either too much or too little stabilization can make the method quite costly. However, as shown in figure 6(a), for optimal values of \( \beta \) the discontinuous Galerkin method can in fact result in an improved computational performance over traditional conforming methods. Where computational performance is measured as the computational cost for a given numerical accuracy, just as in the example with the expanding block. For this example, the exact solution was also computed with a conforming finite element method where the mesh consisted of 79,202 linear triangles.
Since it is our full intent to study DG for the purpose of analyzing real physical phenomena, we present in figure 7, one case simply to show the behavior of DG when modeling a three dimensional hollow cylinder. The following example simulates a hollow cylinder with a shear force applied to the upper surface. For this particular case, the stabilization parameter, $\beta$, was set to the value of the largest compressive principle stress in the hollow cylinder at each quasistatic loading increment.

7 SUMMARY

We have found the DG method to be highly effective in its application to nonlinear elasticity problems. The concise and simple one-field formulation readily presents itself to both analysis and implementation. The distinguishing quantity of the DG method, the lifting operator, greatly enhances accuracy without greatly adding to the computational cost. Despite the method being unstable, we have provided a detailed estimate of the amount of stabilization necessary for the general nonlinear elasticity problem. Both two dimensional and three dimensional model problems have shown specific cases where the DG method is very robust and computationally economical. Not only does it provide very accurate approximations to materials that are nearly incompressible, but it can provide a solution when other methods may not be able to at all.
Figure 7: Cylinder being compressed using 1000 steps. \( \nu=0.45, \beta=\text{maximum compressive stress} \)

REFERENCES

