A General Framework for Systemic Risk

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Joint work with Chen Chen and Garud Iyengar.
Systemic Risk

‘systen’ \equiv \text{collection of ‘entities’}
Systemic Risk

‘system’ ≡ collection of ‘entities’

Examples:

- firms in an economy
- business units in a company
- suppliers, sub-contractors, etc. in a supply chain network
- generating stations, transmission facilities, etc. in a power network
- flood walls, pumping stations, etc. in a levee system
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Systemic risk refers to the risk of catastrophic collapse of the entire system. Involves:

• the simultaneous analysis of outcomes across all entities in a system
• the possibility of complex interactions between components
Joint Distribution of Outcomes

- 3 firms in 3 future scenarios (equally likely)

- Loss matrix:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Firm 1</th>
<th>Firm 2</th>
<th>Firm 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>+50</td>
<td>-40</td>
<td>+20</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-40</td>
<td>+50</td>
<td>-40</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>+20</td>
<td>+20</td>
<td>+50</td>
</tr>
</tbody>
</table>

$(+ \text{‘loss’}; - \text{‘profit’})$
Complex Interactions

Complex interactions between entities can create contagion, or cascades of failures.

In financial markets, structural mechanisms for contagion include:
- Interbank loans
- Interbank derivatives exposures (e.g., AIG)
- Transmission of illiquidity, "bank runs" (e.g., Lehman)
- Fire sales, asset price contagion (e.g., CDOs)
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Contributions

• A general, axiomatic framework for coherent systemic risk analyzes joint distribution of outcomes allows for some endogenous mechanisms of contagion subsumes many recently proposed systemic risk measures

• A structural decomposition of systemic risk

• A dual representation for systemic risk measures ‘shadow price of risk’

• A mechanism for systemic risk attribution & decentralization

• Methodology extends to a much broader class of risk functions
Literature Review

- Axiomatic theory of single-firm risk measures: 
  Artzner et al., (2000); see survey of Schied (2006)

- Systemic risk measures: portfolio approach 
  Gauthier et al., (2010); Tarashev et al., (2010); Acharya et al., (2010); Brownlees & Engle (2010); Adrian & Brunnermeier (2009)

- Systemic risk measures: deposit insurance / credit approach 
  Lehar (2005); Huang et al., (2009); Giesecke & Kim (2011)

- Structural models of contagion & systemic risk: 
  Acharya et al., (2010); Staum (2011); Liu & Staum (2010); Cont et al., (2011); Bimpikis & Tahbaz-Salehi (2012)

- Portfolio attribution: 
  Denault (2001); Buch & Dorfleitner (2008)
Single-Firm Risk Measures

\[ \Omega = \text{set of scenarios} \]

\[ x \in \mathbb{R}^\Omega \]

\[ x_\omega = \text{loss in scenario } \omega \]
Coherent Risk Measures

Definition. A coherent single-firm risk measure is a function $\rho : \mathbb{R}^\Omega \to \mathbb{R}$ that satisfies, for all $x, y \in \mathbb{R}^\Omega$:

(i) **Monotonicity**: if $x \geq y$, then $\rho(x) \geq \rho(y)$

(ii) **Positive homogeneity**: for all $\alpha \geq 0$, $\rho(\alpha x) = \alpha \rho(x)$

(iii) **Convexity**: for all $0 \leq \alpha \leq 1$,

$$\rho(\alpha x + (1 - \alpha)y) \leq \alpha \rho(x) + (1 - \alpha)\rho(y)$$

(iv) **Cash-invariance**: for all $\alpha \in \mathbb{R}$,

$$\rho(x + \alpha 1_\Omega) = \rho(x) + \alpha$$

[Artzner et al., 2000]
Systemic Risk Measures

\[ \Omega = \text{set of scenarios} \]
\[ \mathcal{F} = \text{set of firms (entities in the system)} \]
\[ X_i,\omega = \text{loss for firm } i \text{ in scenario } \omega \]
Systemic Risk Measures: Definition

- $\Omega =$ set of scenarios, $\mathcal{F} =$ set of entities in the system, $X \in \mathbb{R}^{\Omega \times \mathcal{F}}$
- $X_{i,\omega} =$ loss for firm $i$ in scenario $\omega$, $X_\omega =$ loss vector in scenario $\omega$
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**Definition.** A systemic risk measure is a function $\rho : \mathbb{R}^{\Omega \times \mathcal{F}} \rightarrow \mathbb{R}$ that satisfies, for all economies $X, Y, Z \in \mathbb{R}^{\Omega \times \mathcal{F}}$:

(i) *Monotonicity*: if $X \geq Y$, then

$$\rho(X) \geq \rho(Y)$$

(ii) *Positive homogeneity*: for all $\alpha \geq 0$,

$$\rho(\alpha X) = \alpha \rho(X)$$

(iii) *Normalization*: $\rho(1_\mathcal{E}) = |\mathcal{F}|$
Systemic Risk Measures: Definition

Definition. (con’t.) Given $x, y \in \mathbb{R}^F$, define the ordering $x \succeq_{\rho} y$ by

$$x \succeq_{\rho} y \iff \rho(x, \ldots, x) \geq \rho(y, \ldots, y)$$
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(iv) **Preference consistency:** if \( X_\omega \succeq_\rho Y_\omega \) for all scenarios \( \omega \), then

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\rho(X) \geq \rho(Y)
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| Scenario | Firm 1 | \( \omega_1 \) | \ldots | \( \omega \) | \ldots | \( \omega|\Omega| \) |
|----------|--------|-------------|--------|-------------|--------|-----------------|
| Firm 1   | \( X_{1,\omega_1} \) | \ldots | \( X_{1,\omega} \) | \ldots | \( X_{1,\omega|\Omega|} \) |
| \vdots   | \vdots | \ddots | \vdots | \ddots | \vdots |
| Firm \( |F| \) | \( X_{|F|,\omega_1} \) | \ldots | \( X_{|F|,\omega} \) | \ldots | \( X_{|F|,|\Omega|} \) |

\[
X_\omega \succeq_\rho Y_\omega \quad \forall \omega \quad \Rightarrow \quad \rho(X) \geq \rho(Y)
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\]
Definition. (con’t.)

(v) **Convexity:** for all $0 \leq \alpha \leq 1$, $\bar{\alpha} = 1 - \alpha$

(a) **Outcome convexity:** if

$$Z = \alpha X + \bar{\alpha} Y$$

then, \( \rho(Z) \leq \alpha \rho(X) + \bar{\alpha} \rho(Y) \)
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then, \( \rho(Z) \leq \alpha \rho(X) + \bar{\alpha} \rho(Y) \)

(b) Risk convexity: if for all scenarios \( \omega \in \Omega \),

\[ \rho(Z_\omega, \ldots, Z_\omega) = \alpha \rho(X_\omega, \ldots, X_\omega) + \bar{\alpha} \rho(Y_\omega, \ldots, Y_\omega) \]

then, \( \rho(Z) \leq \alpha \rho(X) + \bar{\alpha} \rho(Y) \)
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Two different notions of diversity
Systemic Risk Measures: Definition

**Definition.** (con’t.)

1. Outcome convexity: Increasing diversification reduces risk

\[
\begin{align*}
X_\omega & \xrightarrow{\alpha} Z_\omega \\
Y_\omega & \xrightarrow{\bar{\alpha}} Z_\omega
\end{align*}
\Rightarrow \rho(Z) \leq \alpha \rho(X) + \bar{\alpha} \rho(Y)
\]

2. Risk convexity: Removing randomness reduces risk

\[
\begin{align*}
\rho(Z_\omega 1_{\omega}^T) & \xrightarrow{\alpha} \rho(X_\omega 1_{\omega}^T) \\
\rho(Z_\omega 1_{\omega}^T) & \xrightarrow{\bar{\alpha}} \rho(Y_\omega 1_{\omega}^T)
\end{align*}
\Rightarrow \rho(Z) \leq \alpha \rho(X) + \bar{\alpha} \rho(Y)
\]
**Structural Decomposition**

**Definition.** An **aggregation function** is a function \( \Lambda : \mathbb{R}^\mathcal{F} \rightarrow \mathbb{R} \) that is monotonic, positively homogeneous, convex, and normalized so that \( \Lambda(1_\mathcal{F}) = |\mathcal{F}| \).

Aggregation function: aggregates risk **across firms in a given scenario**
**Structural Decomposition**

**Definition.** An aggregation function is a function $\Lambda: \mathbb{R}^\mathcal{F} \rightarrow \mathbb{R}$ that is monotonic, positively homogeneous, convex, and normalized so that $\Lambda(\mathbf{1}_\mathcal{F}) = |\mathcal{F}|$.

Aggregation function: aggregates risk across firms in a given scenario.

**Theorem.** A function $\rho: \mathbb{R}^{\Omega \times \mathcal{F}} \rightarrow \mathbb{R}$ is a systemic risk measure with $\rho(-\mathbf{1}_\mathcal{E}) < 0$ iff there exists

- an aggregation function $\Lambda$
- coherent single-firm base risk measure $\rho_0$ such that

$$\rho(X) = (\rho_0 \circ \Lambda)(X) \triangleq \rho_0 \left( \Lambda(X_1), \Lambda(X_2), \ldots, \Lambda(X_{|\Omega|}) \right)$$
Example: Economic Systemic Risk Measures

- $\mathcal{F} =$ firms in the economy
- $X_{i,\omega} =$ loss of a firm $i$ in scenario $\omega$
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**Example.** (Systemic Expected Shortfall)

$$
\Lambda_{\text{total}}(x) \triangleq \sum_{i \in \mathcal{F}} x_i, \quad \rho_{\text{SES}}(X) \triangleq (\text{CVaR}_\alpha \circ \Lambda_{\text{total}})(X)
$$

[Acharya et al., 2010; Brownlees, Engle 2010]
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[Acharya et al., 2010; Brownlees, Engle 2010]

**Example.** (Deposit Insurance)

$$
\Lambda_{\text{loss}}(x) \triangleq \sum_{i \in \mathcal{F}} x_i^+, \quad \rho_{\text{DI}}(X) \triangleq \mathbb{E} [\Lambda_{\text{loss}}(X_\omega)] = \mathbb{E} \left[ \sum_{i \in \mathcal{F}} X_{i,\omega}^+ \right]
$$

[e.g., Lehar, 2005; Huang et al., 2009]
Example: Investing with Performance Fees

- $\mathcal{F}$ = a collection of hedge funds or portfolio managers
- $X_{i,\omega}$ = loss of hedge fund $i$ in scenario $\omega$
- $\gamma_i \in [0, 1]$ is the performance fee of fund $i$
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**Example.** (Hedge Fund Investor)

$$\Lambda_{HF}(x) \triangleq \sum_{i \in \mathcal{F}} (x_i + \gamma_i x_i^-)$$
Example: Investing with Performance Fees

- $F = \text{a collection of hedge funds or portfolio managers}$
- $X_{i,\omega} = \text{loss of hedge fund } i \text{ in scenario } \omega$
- $\gamma_i \in [0, 1] \text{ is the performance fee of fund } i$

Example. (Hedge Fund Investor)

$$\Lambda_{HF}(x) \triangleq \sum_{i \in F} (x_i + \gamma_i x_i^-)$$

- $\gamma_{FoF} \in [0, 1] \text{ is the performance fee of the fund-of-funds manager}$

Example. (Fund-of-Funds Investor)

$$\Lambda_{FoF}(x) \triangleq \sum_{i \in F} (x_i + \gamma_i x_i^-) + \gamma_{FoF} \left( \sum_{i \in F} (x_i + \gamma_i x_i^-) \right)^-$$
Example: Resource Allocation

- $\mathcal{A} = \text{a set of activities}$
- $\mathcal{F} = \text{a set of capacitated resources}$
- $x_i = \text{shortage of resource } i$
Example: Resource Allocation

• $\mathcal{A} = \text{a set of activities}$
• $\mathcal{F} = \text{a set of capacitated resources}$
• $x_i = \text{shortage of resource } i$

Consider the aggregation function:

$$\Lambda_{RA}(x) \triangleq \min_{u} \sum_{a \in \mathcal{A}} c_a u_a$$
subject to

$$\sum_{a \in \mathcal{A}} b_{ia} u_a \geq x_i, \quad \forall i \in \mathcal{F}$$

where

• $u_a = \text{reduction in level of activity } a$ (decision variable)
• $c_a = \text{per-unit cost of reductions in activity } a$
• $b_{ia} = \text{per-unit consumption of resource } i \text{ by activity } a$
Example: Interbank Contagion Model

- $\mathcal{F} = \text{firms, who have assets and obligations to each other}$
- $\Pi_{ij} = \text{fraction of the debt of firm } i \text{ owed to firm } j$
- $x_i = \text{losses in excess of obligations of firm } i$
Example: Interbank Contagion Model

- $\mathcal{F} =$ firms, who have assets and obligations to each other
- $\Pi_{ij} =$ fraction of the debt of firm $i$ owed to firm $j$
- $x_i =$ losses in excess of obligations of firm $i$

Consider the aggregation function:

$$\Lambda_{CM}(x) \triangleq \min_{y \in \mathbb{R}^\mathcal{F}_+, b \in \mathbb{R}^\mathcal{F}_+} \sum_{i \in \mathcal{F}} y_i + \gamma \sum_{i \in \mathcal{F}} b_i$$

subject to

$$b_i + y_i \geq x_i + \sum_{j \in \mathcal{F}} \Pi_{ji} y_j, \quad \forall i \in \mathcal{F}$$

where

- loss $x_i$ is covered by firm $i$ reducing the payments by an amount $y_i$, or relying on an injection from the regulator in the amount $b_i$
- reminiscent of Eisenberg & Noe (2001)
“General” Aggregation Function

Given:

• $c \in \mathbb{R}_+^N$, $A \in \mathbb{R}_+^{K \times F}$, $B \in \mathbb{R}^{K \times N}$

• $\mathcal{K} \subset \mathbb{R}^N$ a convex cone, such that $\exists \bar{y} \in \mathcal{K}$ with $B\bar{y} > 0$

Define:

$$\Lambda_{\text{OPT}}(x) \triangleq \minimize_y \quad c^\top y$$

subject to

$$Ax \leq By$$

$$y \in \mathcal{K}$$

• $\Lambda_{\text{OPT}}$ is monotonic, positively homogeneous, and convex

• if $\Lambda_{\text{OPT}}(1_F) > 0$, it can also be normalized

• allows for general endogenous mechanisms for ‘co-operative’ contagion
Structural Decomposition: Proof Sketch

‘If’ part is not hard. ‘Only if’ part:

- Define $\Lambda(x) \triangleq \rho(x^\top 1_{\Omega})$, $\forall x \in \mathbb{R}^F$

- Define $\rho_0(z) \triangleq \rho(X)$, $\forall z \in Q^\Omega$, for some $X: \Lambda(X_\omega) = z_\omega$, $\forall \omega$
Structural Decomposition: Proof Sketch

‘If’ part is not hard. ‘Only if’ part:

- Define $\Lambda(x) \triangleq \rho(x 1^T_{\Omega})$, $\forall x \in \mathbb{R}^F$

- Define $\rho_0(z) \triangleq \rho(X)$, $\forall z \in \mathcal{Q}^\Omega$, for some $X$: $\Lambda(X_\omega) = z_\omega$, $\forall \omega$

- **Step 1**: $\rho_0$ is well-defined. Suppose $X$, $Y$ have $\Lambda(X_\omega) = \Lambda(Y_\omega)$, $\forall \omega \in \Omega$. Preference consistency of $\rho$ implies
  
  $\Lambda(X_\omega) \geq \Lambda(Y_\omega)$, $\forall \omega \in \Omega \implies \rho(X) \geq \rho(Y)$,
  
  $\Lambda(X_\omega) \leq \Lambda(Y_\omega)$, $\forall \omega \in \Omega \implies \rho(X) \leq \rho(Y)$.

  Thus, $\rho(X) = \rho(Y)$
Structural Decomposition: Proof Sketch

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\]
\[
\Lambda(X_\omega) \leq \Lambda(Y_\omega), \forall \omega \in \Omega \implies \rho(X) \leq \rho(Y).
\]

Thus, $\rho(X) = \rho(Y)$

**Step 2**: Derive other properties: monotonicity, convexity, homogeneity of $\Lambda$ and $\rho_0$.

\[
\rho = (\rho_0 \circ \Lambda)(X) \triangleq \rho_0 \left( \Lambda(X_1), \Lambda(X_2), \ldots, \Lambda(X_{|\Omega|}) \right)
\]
Theorem. Any systemic risk measure \( \rho = (\rho_0 \circ \Lambda) \) can be expressed as

\[
\rho(X) = \min_{m, \ell} m \\
\text{subject to } (m, \ell) \in A, \\
(\ell_\omega, X_\omega) \in B, \\
\forall \omega \in \Omega,
\]

where acceptance sets \( A \) and \( B \) can be taken as the epigraphs of \( \rho_0 \) and \( \Lambda \), i.e.,

\[
A \triangleq \left\{ (m, z) \in \mathbb{R} \times \mathbb{R}^\Omega : m \geq \rho_0(z) \right\},
\]

\[
B \triangleq \left\{ (\ell, x) \in \mathbb{R} \times \mathbb{R}^\mathcal{F} : \ell \geq \Lambda(x) \right\}.
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**Interpretation:**

- $\ell = \text{regulator's position required to support the cross-sectional profile}$
- $m = \text{cash position required to support } \ell$
Theorem. Any systemic risk measure $\rho = (\rho_0 \circ \Lambda)$ can be expressed as

$\rho(X) = \minimize_{m, \ell} m$

subject to $(m, \ell) \in A,$
$(\ell_\omega, X_\omega) \in B, \quad \forall \omega \in \Omega,$
$m \in \mathbb{R}, \ell \in \mathbb{R}^\Omega.$

where acceptance sets $A$ and $B$ can be taken as the epigraphs of $\rho_0$ and $\Lambda$, i.e.,

$A \triangleq \{(m, z) \in \mathbb{R} \times \mathbb{R}^\Omega : m \geq \rho_0(z)\},$

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Interpretation:

- $\ell =$ regulator’s position required to support the cross-sectional profile
- $m =$ cash position required to support $\ell$
Dual Representation

Theorem. Any systemic risk measure \( \rho \) can be expressed as

\[
\rho(X) = \maximize_{\bar{\pi}, \Xi} \sum_{i \in \mathcal{F}} \sum_{\omega \in \Omega} \Xi_{i,\omega} X_{i,\omega}
\]

subject to

\[
(1, \bar{\pi}) \in \mathcal{A}^* \\
(\bar{\pi}_\omega, \Xi_\omega) \in \mathcal{B}^*, \forall \omega \in \Omega \\
\bar{\pi} \in \mathbb{R}^\Omega, \Xi \in \mathbb{R}^{\mathcal{F} \times \Omega}
\]

where \( \mathcal{A}^* \subset \mathbb{R} \times \mathbb{R}^\Omega, \mathcal{B}^* \subset \mathbb{R} \times \mathbb{R}^\mathcal{F} \) are (up to a sign change) the dual cones to \( \text{epi}(\rho_0), \text{epi}(\Lambda) \).

Further, \( (\bar{\pi}, \Xi) \) must satisfy

\[
\bar{\pi} \geq 0_\Omega, \quad 1_\Omega^T \bar{\pi} \leq 1, \quad \Xi \geq 0_\varepsilon, \quad 1_\mathcal{F}^T \Xi \leq |\mathcal{F}| \bar{\pi}^T
\]

Robust optimization interpretation: \( \rho(X) \) is worst-case expected loss of a rescaled economy over a set of probability distributions and scaling functions
Shadow Price of Risk

\[ \rho(X) = \maximize_{\bar{\pi}, \Xi} \quad \sum_{i \in \mathcal{F}} \sum_{\omega \in \Omega} \Xi_{i,\omega} X_{i,\omega} \]

subject to

\[ (1, \bar{\pi}) \in A^* \]

\[ (\bar{\pi}_\omega, \Xi_\omega) \in B^*, \forall \omega \in \Omega \]

\[ \bar{\pi} \in \mathbb{R}^\Omega, \Xi \in \mathbb{R}^{\mathcal{F} \times \Omega} \]

**Corollary.** If \((\bar{\pi}^*, \Xi^*)\) is an optimal solution to (DUAL), then \(\Xi^*\) is a subgradient of \(\rho\) at \(X\).

\(\Xi^*_{i,\omega}\) is the **shadow price of risk** \(\equiv\) the minimum marginal rate of increase of systemic risk given an increase of losses for firm \(i\) in scenario \(\omega\).
Risk Attribution

Suppose $\rho$ is a systemic risk measure, and $\Xi^*$ is an optimal solution to (DUAL) at $X$. Define the risk attribution of firm $i$ as

$$y_i^* = \sum_{\omega \in \Omega} \Xi_{i,\omega}^* X_{i,\omega}$$

By strong duality,

$$\rho(X) = \sum_{i \in \mathcal{F}} y_i^*$$
Suppose $\rho$ is a systemic risk measure, and $\Xi^*$ is an optimal solution to (DUAL) at $X$. Define the risk attribution of firm $i$ as

$$y^*_i = \sum_{\omega \in \Omega} \Xi^*_i,\omega X_i,\omega$$

By strong duality,

$$\rho(X) = \sum_{i \in \mathcal{F}} y^*_i$$

**Theorem.** (No Undercut) Given $\alpha \in \mathbb{R}^\mathcal{F}_+$, define

$$r(\alpha) \equiv \rho(\alpha_1 x_1; \ldots; \alpha_{|\mathcal{F}|} x_{|\mathcal{F}|})$$

Then,

$$\alpha^\top y^* \leq r(\alpha)$$

Decentralization

- $X^{(i)} = \text{outcomes of firm } i$, $X \triangleq (X^{(1)}; X^{(2)}; \ldots; X^{(|\mathcal{F}|)})$
- $\mathcal{T}_i = \text{set of outcomes of firm } i$, $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \ldots \times \mathcal{T}_{|\mathcal{F}|}$

**Definition.** (Social Optimality)
An economy $\bar{X} \in \mathcal{T}$ is **socially optimal** if it maximizes risk-adjusted welfare

$$\maximize_{X \in \mathcal{T}} \sum_{i \in \mathcal{F}} U_i(X^{(i)}) - \tau \rho(X)$$

Here, $\tau > 0$ captures the systemic risk externality.
Theorem. Suppose that $\bar{X} \in \mathcal{T}$ is a **socially optimal** economy. There exists $\Xi^*$ that is an optimal solution to the dual problem for $\rho(\bar{X})$ so that if we define, for each firm $i$, the tax function

$$t_i(X^{(i)}) \triangleq \tau \sum_{\omega \in \Omega} \Xi^*_{i,\omega} X_{i,\omega}$$

then, $\bar{X}^{(i)}$ is an optimal outcome for firm $i$, i.e.,

$$\bar{X}^{(i)} \in \argmax_{X^{(i)} \in \mathcal{T}^{(i)}} U_i(X^{(i)}) - \tau \sum_{\omega \in \Omega} \Xi^*_{i,\omega} X_{i,\omega}$$

- Decentralized computation of optimal taxes possible
Decentralization

Planner’s problem:

\[
\max_{X \in \mathcal{T}} \sum_{i \in \mathcal{F}} U_i(X^{(i)}) - \rho(X).
\]

Decentralization scheme: apply proximal gradient method

\[
\max_{X \in \mathcal{T}} \sum_{i} U_i(X^{(i)}) - \left( \rho(\bar{X}) + \sum_{i} (X^{(i)} - \bar{X}^{(i)})^\top \frac{\partial \rho(\bar{X})}{\partial X^{(i)}} + \frac{t}{2} \| X - \bar{X} \|_2^2 \right)
\]
Decentralization

Planner’s problem:

$$\max_{X \in \mathcal{T}} \sum_{i \in \mathcal{F}} U_i(X^{(i)}) - \rho(X).$$

Decentralization scheme: apply proximal gradient method

$$\max_{X \in \mathcal{T}} \sum_{i} U_i(X^{(i)}) - \left( \rho(\bar{X}) + \sum_{i} (X^{(i)} - \bar{X}^{(i)}) \top \frac{\partial \rho(\bar{X})}{\partial X^{(i)}} + \frac{t}{2} \|X - \bar{X}\|_2^2 \right)$$

Individual firm’s problem:

$$X^{*}_i = \arg\max_{X^{(i)} \in \mathcal{T}_i} \left\{ U_i(X^{(i)}) - (X^{(i)} - \bar{X}^{(i)}) \top \frac{\partial \rho(\bar{X})}{\partial X^{(i)}} - \frac{t}{2} \|X^{(i)} - \bar{X}^{(i)}\|_2^2 \right\}$$
Decentralization

Planner’s problem:

$$\max_{X \in \mathcal{T}} \sum_{i \in \mathcal{F}} U_i(X^{(i)}) - \rho(X).$$

Decentralization scheme: apply proximal gradient method

$$\max_{X \in \mathcal{T}} \sum_{i} U_i(X^{(i)}) - \left( \rho(\bar{X}) + \sum_{i} (X^{(i)} - \bar{X}^{(i)})^\top \frac{\partial \rho(\bar{X})}{\partial X^{(i)}} + \frac{t}{2} \| X - \bar{X} \|^2_2 \right)$$

Individual firm’s problem:

$$X_i^* = \arg\max_{X^{(i)} \in \mathcal{T}_i} \left\{ U_i(X^{(i)}) - (X^{(i)} - \bar{X}^{(i)})^\top \frac{\partial \rho(\bar{X})}{\partial X^{(i)}} - \frac{t}{2} \| X^{(i)} - \bar{X}^{(i)} \|^2_2 \right\}$$

Information sent by the planner: $\frac{\partial \rho(\bar{X})}{\partial X^{(i)}}$

Information sent by the firm: $\bar{X}^{(i)}$
Decentralization

Communication between the planner and firms

\[
\frac{\partial \rho}{\partial X^{(1)}} \quad \bar{X}^{(1)} \quad \ldots \\
\frac{\partial \rho}{\partial X^{(n)}} \quad \bar{X}^{(n)}
\]
Extensions

Homogeneous Systemic Risk Measures:

- monotone, +vely homogeneous, preference consistent, **not** convex
- structural decomposition exists
  homogeneous single-firm base risk measure
  homogeneous aggregation function

Convex Systemic Risk Measures:

- monotone, convex, preference consistent, **not** +vely homogeneous
- structural decomposition exists
  convex single-firm base risk measure
  convex aggregation function

**Key idea:** Preference consistency allows for the structural decomposition
Conclusions / Future Directions

- A general, axiomatic framework for coherent systemic risk analyzes joint distribution of outcomes
  allows for ‘cooperative’ endogenous forms of contagion
  potential applications in a broad array of engineering & economic systems

- A structural decomposition of systemic risk

- Mechanisms for systemic risk attribution & decentralization
Conclusions / Future Directions

• A general, axiomatic framework for coherent systemic risk analyzes joint distribution of outcomes allows for ‘cooperative’ endogenous forms of contagion potential applications in a broad array of engineering & economic systems

• A structural decomposition of systemic risk

• Mechanisms for systemic risk attribution & decentralization

Future Directions:

• Statistical estimation of systemic risk

• Strategic mechanisms of contagion

• Is network structure important for systemic risk in financial markets?