Rate region frontiers for $n$–user interference channel with interference as noise

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Introduction: the big picture

Treating the interference as additive noise, what rates could be achieved?
Introduction: why the interference channel?

- Transmitter nodes are closer to each other.
- System performance is more interference-limited.

Cellular

$n$-nodes ad hoc network
# Outline

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Outline

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2. $n$–user Achievable Rate Region
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2. $n$–user Achievable Rate Region

3. Characteristics of 2-user Rate Region Frontiers
Outline

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Introduction: literature review

- The capacity region of a 2-user interference channel has been an open problem for about 30 years [Sato77,78].
- Information-theoretic bounds through achievable rate regions have been proposed, most famously with the Han-Kobayashi approach [HK81].
- The capacity of the Gaussian interference channel under strong interference has been found in [Sato 81].
- Recent results on the 2-user interference channel to within one bit of capacity have been shown in [EtkinTse07].

- Our work treats the $n$-user case with interference as noise + dimension orthogonalization introduced at a certain point.

- **Cases when interference is treated as noise:**
  - low complexity transceivers
  - low power, low cost
  - mostly assumed in cellular and ad hoc networks
Introduction: assumptions and objectives

Assumptions:
- no cooperation at the transmit nor at the receive side
- SISO flat channel

Objectives:
- Characterize the achievable rate region when treating interference as noise for the $n$-user case
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2-user Interference Channel

System setup

\[ a = \frac{g_{1,1}}{\sigma_n^2}, \quad b = \frac{g_{1,2}}{\sigma_n^2}, \quad d = \frac{g_{2,1}}{\sigma_n^2}, \quad c = \frac{g_{2,2}}{\sigma_n^2}. \]

- No assumptions on \( a, b, c \) or \( d \)
- \( P_i \leq P_{\text{max}}, \ i = 1, 2 \)

Achievable rates:

\[
C_1(P_1, P_2) = \log_2 \left( 1 + \frac{aP_1}{1+bP_2} \right),
\]
\[
C_2(P_1, P_2) = \log_2 \left( 1 + \frac{cP_2}{1+dP_1} \right).
\]
Set $C_1 = R$, then $C_1(P_1, P_2) = R = \log_2 \left( 1 + \frac{aP_1}{1 + bP_2} \right)$.

Establish a relation between $P_1$ & $P_2$ such as constant rate $C_1 = R$

$P_1 = \frac{1}{a} (1 + bP_2)(2^R - 1)$. → Express $C_2(P_1, P_2)$ in function of $P_2$ only

$$C_2(P_2) = \log_2 \left( 1 + \frac{cP_2}{1 + \frac{d}{a} (1 + bP_2)(2^R - 1)} \right).$$
Rate region frontier formulation

\[ C_2(P_2) = \log_2 \left( 1 + \frac{cP_2}{1 + \frac{d}{a}(1+bP_2)(2^R-1)} \right) \]
monotonically increasing in \( P_2 \)

**Corollary:**

- Direct implication of \( C_2(P_2) \) monotonicity in \( P_2 \) for a certain \( C_1 = R \), is that there is a *unique* \( P_2^* \) that achieves \( C_2(P_2) = C_2^* \).
- Once \( P_2^* \) is found, there is a *unique* \( P_1^* \) that achieves \( C_1 = R \).
- \( \Rightarrow \) for a unique rate tuple, there is a unique power tuple.

Introduce the *Potential Line* notation \( \Phi(\cdot, P_{\max}) \) to denote \( P_1 \) sweeps its full range and \( P_2 \) is held at \( P_{\max} \).

Based on uniqueness property, potential lines along a certain dimension are non-touching.

i.e. \( \Phi(\cdot, P_2) \) & \( \Phi(\cdot, P_2') \) do not intersect if \( P_2 \neq P_2' \)
Rate region frontier formulation

Rate region frontier formulated as:

\[ \arg \max_{P_2} C_2(P_2) \]

subject to

\[ C_1(P_1, P_2) = R \]

\[ P_i \leq P_{\text{max}} \quad i = 1, 2. \]

\( R \) is swept over the full range of \( C_1 \), i.e.

\[ 0 \leq R \leq C_1(P_{\text{max}}, 0) \]

\[ 0 \leq R \leq C_1(P_{\text{max}}, P_{\text{max}}) \]

\[ C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0) \]

Rate region frontiers for \( n \)-user interference channel.
Rate region frontier formulation: \( 0 \leq R \leq C_1(P_{\text{max}}, P_{\text{max}}) \)

As \( C_2(P_2) \) is monotonically increasing in \( P_2 \), then \( P_2 = P_{\text{max}} \) is attainable for any of the range \( R \leq C_1(P_{\text{max}}, P_{\text{max}}) \).

Therefore for \( 0 \leq R \leq C_1(P_{\text{max}}, P_{\text{max}}) \):
\[
\arg \max_{P_2} C_2(P_2) = P_{\text{max}}.
\]

Hence expressing \( C_2 \) in function of \( C_1 = R \) with \( P_2 = P_{\text{max}} \), we obtain the log-defined frontier equation \( \Phi(:, P_{\text{max}}) \), denoted \( \mathcal{F}_2 \):
\[
C_2(C_1) = \log_2 \left( 1 + \frac{c P_{\text{max}}}{1 + \frac{d}{a}(1 + b P_{\text{max}})(2^{C_1} - 1)} \right).
\]
2-user Channel

**Rate region formulation:** \((2)\) \(C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)\)

By symmetry of previous result, for a constant \(C_2 = \tilde{R}\), we find that the frontier for that range of \(C_1\) and \(\tilde{R}\) is achieved when \(P_1 = P_{\text{max}}\).

- Therefore for \(C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)\), the values of \(C_1\) at the frontier are: \(C_1(P_{\text{max}}, P_2) = \log_2 \left( 1 + \frac{aP_{\text{max}}}{1 + bP_2} \right) = R\). Then, \(P_2 = \frac{1}{b} \left( \frac{aP_{\text{max}}}{2^R - 1} - 1 \right)\). \(\rightarrow\) \(\arg\max_{P_2} C_2(P_2) = \frac{1}{b} \left( \frac{aP_{\text{max}}}{2^R - 1} - 1 \right)\).

- Hence we obtain a log-defined frontier equation \(\Phi(P_{\text{max}}, :)\), \(\mathcal{F}_1\):

\[
C_2(C_1) = \log_2 \left( 1 + \frac{c}{b} \frac{aP_{\text{max}} - (2^{C_1} - 1)}{(2^{C_1} - 1)(1 + dP_{\text{max}})} \right)
\]
Rate region formulation: Summary

The rate region for the 2-user interference channel is:

\[ \mathcal{F} = \text{Convex Hull}\{\mathcal{F}_1 \cup \mathcal{F}_2\} \]
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3. Characteristics of 2-user Rate Region Frontiers
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3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$

- $P_3 = 0$ the same results of 2-user case apply. The frontier is the red potential line $\Phi(:, P_{\text{max}}, 0)$ denoted $\Phi_{AB}$
- when $P_3$ increases, how is the effect traced in the rate region?
3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$

For $C_1$ & $C_2$, $P_3$ dimension effect is lumped as additional noise

Potential lines $\Phi(\cdot, P_{\text{max}}, P_3)$ are monotonically increasing with $P_3$ in the $C_3$ dimension, forming a surface frontier $\mathcal{F}_2 = \Phi(\cdot, P_{\text{max}}, \cdot)$

By symmetry, we obtain the rate region frontier for the 3-user case:

$$\mathcal{F} = \text{Convex Hull}\{\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3\}$$

with $\mathcal{F}_1 = \Phi(P_{\text{max}}, \cdot, \cdot)$, $\mathcal{F}_2 = \Phi(\cdot, P_{\text{max}}, \cdot)$, and $\mathcal{F}_3 = \Phi(\cdot, \cdot, P_{\text{max}})$. 

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$n$-user generalization

- The additional power effect of $P_n$ can be lumped into the noise term of the other $(n - 1)$ dimensions. Thus the results for $C_1, \ldots, C_{n-1}$ hold and carry through.
- The frontier on $C_n$ is monotonically increasing in $P_n$.
- Invoking symmetry we can generalize over all rate ranges.

**Theorem**

The achievable rate region frontier for the $n$–user interference channel is:

$$
\mathcal{F} = \text{Convex Hull}\{\bigcup_{i=1}^{n} \mathcal{F}_i\}
$$

where $\mathcal{F}_i$ is a hyper-surface of $n - 1$ dimensions, characterized by holding the $i^{th}$ transmit power at the maximum power $P_{\max}$.

*Remark*: Results also hold for different thermal noise levels, or different maximum power levels.
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Recall the frontiers equation for the 2-user case:

- $\Phi(\cdot, P_{max}), \mathcal{F}_2$:
  \[
  C_2(C_1) = \log_2 \left( 1 + \frac{c P_{max}}{1 + \frac{d}{a}(1 + b P_{max})(2^{c_1} - 1)} \right).
  \]

- $\Phi(P_{max}, \cdot), \mathcal{F}_1$:
  \[
  C_2(C_1) = \log_2 \left( 1 + \frac{\frac{c}{b} \left( a P_{max} - (2^{c_1} - 1) \right)}{(2^{c_1} - 1)(1 + d P_{max})} \right).
  \]

- When would the frontiers be convex or concave?
- Can they be neither, i.e. exhibiting a non-stationary inflection point?
Convexity or Concavity of the Frontiers

We focus on one frontier by symmetry, $\mathcal{F}_2$, we study its second derivative.

We introduce a quantity $Q_1$ defined as:

$$Q_1 = \Re\left(\sqrt{(a - \theta)(a - \theta + acP_{\text{max}})}\right) - \theta$$

where $\theta = d + dbP_{\text{max}}$

→ where it suffices to study $\text{sign}(P_1 - Q_1)$, as $Q_1$ follows such that

$$\text{sign}\left(\frac{\partial^2 \mathcal{F}_2}{\partial c_1^2}\right) = \text{sign}(P_1 - Q_1)$$
Convexity or Concavity of the Frontiers

- $Q_1 \leq 0$: $F_2$ is convex, as $(P_1 - Q_1) \geq 0$ for all range of $P_1$
- $Q_1 \geq P_{\text{max}}$: $F_2$ is concave, as $(P_1 - Q_1) \leq 0$ for all range of $P_1$
- $0 \leq Q_1 \leq P_{\text{max}}$: $F_2$ has a non-stationary inflection point when $P_1 = Q_1$
  - it follows that the convexity of the frontier follows from the point $\Phi(P_{\text{max}}, P_{\text{max}})$ onwards.
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Optimality of Time-Sharing

Analyzing the $F_2$ frontier:

- $Q_1 \leq 0$: $F_2$ is convex, then the time sharing options are:
  1. between A & B
  2. between A & C
  3. between A & inflection point on $F_1$, E
  4. between A & point on concave section of $F_1$

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Optimality of Time-Sharing

- $Q_1 \geq 0$: $\mathcal{F}_2$ is concave, $\Phi(\cdot, P_{\text{max}})$ is optimal and no time sharing is employed.
- $0 \leq Q_1 \leq P_{\text{max}}$: use the concave segment from A to $\Phi(Q_1, P_{\text{max}})$, and the time-sharing candidates that were mentioned for the $Q_1 \leq 0$ case.
Optimality of Time-Sharing

When to operate with one user at a time?

Discounting case when $\mathcal{F}_1$ or $\mathcal{F}_2$ have inflection points for simplicity

Focus when time-sharing between A & C is better than going through intermediate point B.

Operating with one transmitter active at a certain time (i.e. along $A \leftrightarrow C$) is optimal when:

$$\left( \frac{1+cP_{\text{max}}}{1+cP_{\text{max}}+dP_{\text{max}}} \right)^\gamma \geq \left( \frac{1+aP_{\text{max}}+bP_{\text{max}}}{1+bP_{\text{max}}} \right)^\gamma$$

with $\gamma = \log_2 \left( 1 + cP_{\text{max}} \right) / \log_2 \left( 1 + aP_{\text{max}} \right)$. 

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2–user symmetric channel

for the symmetric case, the time-sharing optimality condition simplifies and leads to the following theorem

**Theorem**

*Time-sharing through operating with one user at a time is optimal when*

\[ b \geq \sqrt{1 + \frac{aP_{\text{max}}}{P_{\text{max}}}} \]

*Remark:* proved that this condition is sufficient to guarantee \( Q_1 \) is always \( \leq 0 \), the assumption we started with.
Summary

- Found the achievable rate region frontiers for \( n \)-user interference channel when receivers treat interference as noise.
- Characterized the convexity and concavity conditions for the 2-user interference channel.
- Found the sufficient condition when time-sharing is optimal between one transmitter being active at a certain time for the case of 2-user symmetric channel.
Comparing with recent IT results [EtkinTse07]

The switching point, using high SIR assumption:

\[ b \geq \sqrt{\frac{aP_{\text{max}}}{P_{\text{max}}}} \]

From our result, the switching point is:

\[ b \geq \sqrt{1 + \frac{aP_{\text{max}}}{P_{\text{max}}}} \]