Rate region frontiers for $n$–user interference channel with interference as noise

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Introduction: the big picture

$n$–user interference channel

Treating the interference as additive noise, what rates could be achieved?

Rate region frontiers for $n$–user interference channel
Introduction: the big picture

$n-$user interference channel

Treating the interference as additive noise, what rates could be achieved?
**Introduction:** why the interference channel?

Cellular

- Transmitter nodes are closer to each other.
- System performance is more interference-limited.

$n$-nodes ad hoc network
Introduction: why the interference channel?

- Transmitter nodes are closer to each others
- System performance is more interference-limited
Introduction

Outline

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1. Introduction
2. $n$–user Achievable Rate Region
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2. $n$-user Achievable Rate Region
3. Characteristics of 2-user Rate Region Frontiers
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3. Characteristics of 2-user Rate Region Frontiers
4. Conclusion
Introduction: literature review

- The capacity region of a 2-user interference channel has been an open problem for about 30 years [Sato77,78].
- Information-theoretic bounds through achievable rate regions have been proposed, most famously with the Han-Kobayashi approach [HK81].
- The capacity of the Gaussian interference channel under strong interference has been found in [Sato 81].
- Recent results on the 2-user interference channel to within one bit of capacity have been shown in [EtkinTse07].
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- Our work treats the $n$-user case with interference as noise + dimension orthogonalization introduced at a certain point.
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Cases when interference is treated as noise:
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- Our work treats the $n$-user case with interference as noise + dimension orthogonalization introduced at a certain point.

- **Cases when interference is treated as noise:**
  - low complexity transceivers
  - low power, low cost
  - mostly assumed in cellular and ad hoc networks
Introduction: assumptions and objectives

Assumptions:
- no cooperation at the transmit nor at the receive side
- SISO flat channel
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- no cooperation at the transmit nor at the receive side
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Objectives:
- Characterize the achievable rate region when treating interference as noise for the $n$-user case
## Outline

1. Introduction

2. $n$–user Achievable Rate Region
   - 2-user Interference Channel Frontiers
   - 3-user Interference Channel Frontiers
   - $n$–user Interference Channel Frontiers

3. Characteristics of 2-user Rate Region Frontiers

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2-user Interference Channel

System setup

\[ a = \frac{g_{1,1}}{\sigma^2_n}, \quad b = \frac{g_{1,2}}{\sigma^2_n}, \]
\[ d = \frac{g_{2,1}}{\sigma^2_n}, \quad c = \frac{g_{2,2}}{\sigma^2_n}. \]
2-user Interference Channel

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\[ a = \frac{g_{1,1}}{\sigma_n^2}, \quad b = \frac{g_{1,2}}{\sigma_n^2}, \]
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- No assumptions on \( a, b, c \) or \( d \)
- \( P_i \leq P_{\text{max}}, \ i = 1, 2 \)
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- No assumptions on a, b, c or d
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Achievable rates:

$$C_1(P_1, P_2) = \log_2 \left( 1 + \frac{aP_1}{1+bP_2} \right),$$
$$C_2(P_1, P_2) = \log_2 \left( 1 + \frac{cP_2}{1+dP_1} \right).$$
Set $C_1 = R$, then $C_1(P_1, P_2) = R = \log_2 \left( 1 + \frac{aP_1}{1+bP_2} \right)$. 
Set \( C_1 = R \), then

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Establish a relation between \( P_1 \) & \( P_2 \) such as constant rate \( C_1 = R \).
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Establish a relation between $P_1$ & $P_2$ such as constant rate $C_1 = R$

$P_1 = \frac{1}{a} (1 + b P_2)(2^R - 1)$. 
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$P_1 = \frac{1}{a}(1 + bP_2)(2^R - 1)$. → Express $C_2(P_1, P_2)$ in function of $P_2$ only
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$P_1 = \frac{1}{a}(1 + bP_2)(2^R - 1)$. → Express $C_2(P_1, P_2)$ in function of $P_2$ only

$$C_2(P_2) = \log_2 \left( 1 + \frac{cP_2}{1 + \frac{d}{a}(1 + bP_2)(2^R - 1)} \right).$$
Rate region frontier formulation

\[ C_2(P_2) = \log_2 \left( 1 + \frac{cP_2}{1 + d \frac{1}{a} (1 + bP_2)(2^R - 1)} \right) \text{ monotonically increasing in } P_2 \]
Rate region frontier formulation

\[ C_2(P_2) = \log_2 \left( 1 + \frac{cP_2}{1 + d \frac{1 + bP_2}{(1+bP_2)(2^R-1)}} \right) \]

monotonically increasing in \( P_2 \)

**Corollary:**

- Direct implication of \( C_2(P_2) \) monotonicity in \( P_2 \) for a certain \( C_1 = R \), is that there is a *unique* \( P_2^* \) that achieves \( C_2(P_2) = C_2^* \).
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- \( \Rightarrow \) for a unique rate tuple, there is a unique power tuple.
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Introduce the *Potential Line* notation \( \Phi(:, P_{\text{max}}) \) to denote \( P_1 \) sweeps its full range and \( P_2 \) is held at \( P_{\text{max}} \).
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Based on uniqueness property, potential lines along a certain dimension are non-touching.
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i.e. \( \Phi(\cdot, P_2) \) & \( \Phi(\cdot, P_2') \) do not intersect if \( P_2 \neq P_2' \)
Rate region frontier formulation

Rate region frontier formulated as:

\[
\arg \max_{P_2} C_2(P_2) \quad \text{subject to} \quad C_1(P_1, P_2) = R \quad P_i \leq P_{\max} \quad i = 1, 2.
\]
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\begin{align*}
\text{arg max}_{P_2} & \quad C_2(P_2) \\
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\(R\) is swept over the full range of \(C_1\), i.e. \(0 \leq R \leq C_1(P_{\text{max}}, 0)\).
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Rate region frontier formulation: 

$0 \leq R \leq C_1(P_{\text{max}}, P_{\text{max}})$

As $C_2(P_2)$ is monotonically increasing in $P_2$, then $P_2 = P_{\text{max}}$ is attainable for any of the range $R \leq C_1(P_{\text{max}}, P_{\text{max}})$. 

\[
C_2(C_1(P_{\text{max}}, P_{\text{max}})) = \log_2 \left( \frac{1}{1 + cP_{\text{max}}} \right)
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Therefore for \( 0 \leq R \leq C_1(P_{\text{max}}, P_{\text{max}}) \):

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Hence expressing \(C_2\) in function of \(C_1 = R\) with \(P_2 = P_{\text{max}}\), we obtain the log-defined frontier equation \(\Phi(:, P_{\text{max}})\), denoted \(\mathcal{F}_2\):
2-user Channel

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Rate region formulation: \( (2) \ C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0) \)

By symmetry of previous result, for a constant \( C_2 = \tilde{R} \), we find that the frontier for that range of \( C_1 \) and \( \tilde{R} \) is achieved when \( P_1 = P_{\text{max}} \).
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Therefore for $C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)$, the values of $C_1$ at the frontier are: $C_1(P_{\text{max}}, P_2) = \log_2 \left( 1 + \frac{aP_{\text{max}}}{1 + bP_2} \right) = R$. Then,

$$P_2 = \frac{1}{b} \left( \frac{aP_{\text{max}}}{2^R - 1} - 1 \right).$$
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Hence we obtain a log-defined frontier equation $\Phi(P_{\text{max}}, :)$, $\mathcal{F}_1$: \[ C_2 \left( \frac{aP_{\text{max}}}{2^R-1} - 1 \right) \]
Rate region formulation: \(2\) \(C_1(P_{\text{max}}, P_{\text{max}}) \leq R \leq C_1(P_{\text{max}}, 0)\)

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\[
P_2 = \frac{1}{b} \left(\frac{aP_{\text{max}}}{2^R - 1} - 1\right).
\]

\(\rightarrow\) \(\arg \max_{P_2} C_2(P_2) = \frac{1}{b} \left(\frac{aP_{\text{max}}}{2^R - 1} - 1\right)\).

Hence we obtain a log-defined frontier equation \(\Phi(P_{\text{max}}, :)\), \(\mathcal{F}_1\):

\[
C_2(C_1) = \log_2 \left(1 + \frac{c}{b} \left(\frac{aP_{\text{max}} - (2^{C_1} - 1)}{(2^{C_1} - 1)(1 + dP_{\text{max}})}\right)\right)
\]
Rate region formulation: Summary

The rate region for the 2-user interference channel is:

$$F = \text{Convex Hull} \{F_1 \cup F_2\}$$
Rate region formulation: Summary

1. $0 \leq C_1 \leq C_1(P_{\text{max}}, P_{\text{max}})$, $\rightarrow \mathcal{F}_2 = \Phi(\cdot, P_{\text{max}})$
2. $C_1(P_{\text{max}}, P_{\text{max}}) \leq C_1 \leq C_1(P_{\text{max}}, 0)$, $\rightarrow \mathcal{F}_1 = \Phi(P_{\text{max}}, \cdot)$
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   - 2-user Interference Channel Frontiers
   - 3-user Interference Channel Frontiers
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3. Characteristics of 2-user Rate Region Frontiers

4. Conclusion
3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$
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- $P_3 = 0$ the same results of 2-user case apply. The frontier is the red potential line $\Phi(\cdot, P_{\text{max}}, 0)$ denoted $\Phi_{AB}$
3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$

- $P_3 = 0$ the same results of 2-user case apply. The frontier is the red potential line $\Phi(:, P_{\text{max}}, 0)$ denoted $\Phi_{AB}$
- when $P_3$ increases, how is the effect traced in the rate region?
3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$
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- For $C_1$ & $C_2$, $P_3$ dimension effect is lumped as additional noise.
- Potential lines $\Phi(:, P_{\text{max}}, P_3)$ are monotonically increasing with $P_3$ in the $C_3$ dimension, forming a surface frontier $\mathcal{F}_2 = \Phi(:, P_{\text{max}}, :)$.
3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$

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By symmetry, we obtain the rate region frontier for the 3-user case:
3-user channel: Effect of increasing $P_3$ from 0 to $P_{\text{max}}$

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$$\mathcal{F} = \text{Convex Hull}\{\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3\}$$

with $\mathcal{F}_1 = \Phi(P_{\text{max}}, :, :)$, $\mathcal{F}_2 = \Phi(:, P_{\text{max}}, :)$, and $\mathcal{F}_3 = \Phi(:, :, P_{\text{max}})$. 

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1 Introduction

2 $n-$user Achievable Rate Region
   - 2-user Interference Channel Frontiers
   - 3-user Interference Channel Frontiers
   - $n-$user Interference Channel Frontiers

3 Characteristics of 2-user Rate Region Frontiers

4 Conclusion
The additional power effect of $P_n$ can be lumped into the noise term of the other $(n-1)$ dimensions. Thus the results for $C_1, \ldots, C_{n-1}$ hold and carry through.
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**n-user generalization**
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Invoking symmetry we can generalize over all rate ranges.
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**Theorem**

The achievable rate region frontier for the $n$–user interference channel is:

$$\mathcal{F} = \text{Convex Hull}\{\bigcup_{i=1}^{n} \mathcal{F}_i\}$$

where $\mathcal{F}_i$ is a hyper-surface of $n–1$ dimensions, characterized by holding the $i^{th}$ transmit power at the maximum power $P_{\max}$. 
The additional power effect of $P_n$ can be lumped into the noise term of the other $(n - 1)$ dimensions. Thus the results for $C_1, \ldots, C_{n-1}$ hold and carry through.

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where $\mathcal{F}_i$ is a hyper-surface of $n - 1$ dimensions, characterized by holding the $i^{th}$ transmit power at the maximum power $P_{\text{max}}$.

**Remark:** Results also hold for different thermal noise levels, or different maximum power levels.
Outline

1. Introduction

2. $n$-user Achievable Rate Region

3. Characteristics of 2-user Rate Region Frontiers
   - Convexity or Concavity of the Frontiers
   - Optimality of Time-Sharing
   - Symmetric 2-user interference channel

4. Conclusion
Convexity or Concavity of the Frontiers

Recall the frontiers equation for the 2–user case:
Convexity or Concavity of the Frontiers

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Recall the frontiers equation for the 2–user case:

\[ \Phi(\cdot, P_{\text{max}}), \mathcal{F}_2: \]
\[ C_2(C_1) = \log_2 \left( 1 + \frac{d}{1 + \frac{cP_{\text{max}}}{a(1 + bP_{\text{max}})(2^{C_1} - 1)}} \right) \cdot \]

\[ \Phi(P_{\text{max}}, \cdot), \mathcal{F}_1: \]
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when would the frontiers be convex or concave?
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- when would the frontiers be convex or concave?
- can they be neither, i.e. exhibiting a non-stationary inflection point?
We focus on one frontier by symmetry, $\mathcal{F}_2$, we study its second derivative.
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We introduce a quantity $Q_1$ defined as:

$$Q_1 = \Re\left(\sqrt{(a - \theta)(a - \theta + acP_{\text{max}})}\right) - \theta$$

where $\theta = d + dbP_{\text{max}}$

→ where it suffices to study $\text{sign}(P_1 - Q_1)$, as $Q_1$ follows such that

$$\text{sign}\left(\frac{\partial^2 \mathcal{F}_2}{\partial c_1^2}\right) = \text{sign}(P_1 - Q_1)$$
Convexity or Concavity of the Frontiers

Convexity or Concavity of the Frontiers

- Convexity or Concavity of the Frontiers
- Frontiers Characteristics
- Conclusion

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Convexity or Concavity of the Frontiers

\[ Q_1 \leq 0: \ F_2 \text{ is convex, as } (P_1 - Q_1) \geq 0 \text{ for all range of } P_1 \]
Convexity or Concavity of the Frontiers

- \( Q_1 \leq 0 \): \( F_2 \) is convex, as \( (P_1 - Q_1) \geq 0 \) for all range of \( P_1 \)
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- \( 0 \leq Q_1 \leq P_{\text{max}} \): \( \mathcal{F}_2 \) has a non-stationary inflection point when \( P_1 = Q_1 \)
Convexity or Concavity of the Frontiers

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- $0 \leq Q_1 \leq P_{\text{max}}$: $\mathcal{F}_2$ has a non-stationary inflection point when $P_1 = Q_1$
  - it follows that the convexity of the frontier follows from the point $\Phi(P_{\text{max}}, P_{\text{max}})$ onwards.
Outline

1 Introduction

2 $n$-user Achievable Rate Region

3 Characteristics of 2-user Rate Region Frontiers
   - Convexity or Concavity of the Frontiers
   - Optimality of Time-Sharing
   - Symmetric 2-user interference channel

4 Conclusion
Optimality of Time-Sharing

Analyzing the $\mathcal{F}_2$ frontier:

Optimality of Time-Sharing
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Optimality of Time-Sharing

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Optimality of Time-Sharing

Analyzing the $F_2$ frontier:

- $Q_1 \leq 0$: $F_2$ is convex, then the time sharing options are:
  1. between A & B
  2. between A & C
  3. between A & inflection point on $F_1$, E
  4. between A & point on concave section of $F_1$
Optimality of Time-Sharing

Q_1 ≥ 0: F_2 is concave, Φ(Q_1, P_{max}) is optimal and no time sharing is employed.

0 ≤ Q_1 ≤ P_{max}: use the concave segment from A to Φ(Q_1, P_{max}), and the time-sharing candidates that were mentioned for the Q_1 ≤ 0 case.
Optimality of Time-Sharing

$Q_1 \geq 0$: $\mathcal{F}_2$ is concave, $\Phi(\cdot, P_{\text{max}})$ is optimal and no time sharing is employed.
Optimality of Time-Sharing

- $Q_1 \geq 0$: $F_2$ is concave, $\Phi(\cdot, P_{\text{max}})$ is optimal and no time sharing is employed.
- $0 \leq Q_1 \leq P_{\text{max}}$: use the concave segment from A to $\Phi(Q_1, P_{\text{max}})$, and the time-sharing candidates that were mentioned for the $Q_1 \leq 0$ case.
When to operate with one user at a time?

Discounting case when $\mathcal{F}_1$ or $\mathcal{F}_2$ have inflection points for simplicity

Operating with one transmitter active at a certain time (i.e. along $A \leftrightarrow C$) is optimal when:

$$\left(1 + c_{P_{\max}}\right) \left(1 + d_{P_{\max}}\right) \geq \left(1 + a_{P_{\max}} + b_{P_{\max}}\right) \gamma$$

with $\gamma = \log_2 \left(1 + c_{P_{\max}}\right) / \log_2 \left(1 + a_{P_{\max}}\right)$.
When to operate with one user at a time?

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Focus when time-sharing between A & C is better than going through intermediate point B.
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$$\frac{(1+cP_{\text{max}})(1+dP_{\text{max}})}{1+cP_{\text{max}}+dP_{\text{max}}} \geq \left( \frac{1+aP_{\text{max}}+bP_{\text{max}}}{1+bP_{\text{max}}} \right)^{\gamma}$$

with $\gamma = \log_2(1 + cP_{\text{max}})/ \log_2(1 + aP_{\text{max}})$. 

\[\text{Rate region frontiers for } n-\text{user interference chan.}.. \text{ Stanford University}\]
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2–user symmetric channel

For the symmetric case, the time-sharing optimality condition simplifies and leads to the following theorem.

Theorem
Time-sharing through operating with one user at a time is optimal when
\[ b \geq \sqrt{1 + \frac{a P_{\text{max}}}{P_{\text{max}}}} \]

Remark: proved that this condition is sufficient to guarantee
\[ Q \] is always \( \leq 0 \), the assumption we started with.
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**Remark:** proved that this condition is sufficient to guarantee $Q_1$ is always $\leq 0$, the assumption we started with.
### Summary

- Found the achievable rate region frontiers for $n$–user interference channel when receivers treat interference as noise.
- Characterized the convexity and concavity conditions for the 2–user interference channel.
- Found the sufficient condition when time-sharing is optimal between one transmitter being active at a certain time for the case of 2–user symmetric channel.
Comparing with recent IT results [EtkinTse07]

The switching point, using high SIR assumption:

\[ b \geq \sqrt{\frac{aP_{\text{max}}}{P_{\text{max}}}} \]

From our result, the switching point is:

\[ b \geq \sqrt{1 + \frac{aP_{\text{max}}}{P_{\text{max}}}} \]