Sequential Geometric Programming for $2 \times 2$ Interference Channel Power Control

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Abstract—We analyze the performance of sequential geometric programming (SGP) in solving the nonconvex power control problem of maximizing the sum capacity of the interference channel with no multi-user decoding. We focus on the $2 \times 2$ interference channel, for which we know the optimal power allocation tuples. We validate the SGP approaches of single and double condensation methods and we analyze the accuracy and convergence performance of each.

I. INTRODUCTION

We consider the problem of maximizing the sum capacity of the interference channel with no multi-user decoding, subject to individual maximum power constraints and with no cooperation at the transmit nor at the receive side. The capacity region for the interference channel is still an open problem, but achievable rate regions are found in [1] that encompass earlier results by [2]–[4], Geometric Programming (GP) optimization technique is gaining ground in solving certain nonconvex communication problems. However for the problems that can not be formulated into a GP form, SGP comes to offer a viable and appealing solving paradigm approach. This paper aims to evaluate such approach pertaining to the generic example presented herein.

The SGP consists of sequentially solving the original problem at each iteration through an updated approximation of the underlying objective function to a GP-solvable function. The process is repeated until convergence.

The sum capacity maximization objective of a general $n \times n$ interference channel is a nonconvex power control problem, and can not be handled by GP. SGP can be employed to solve such problems, however we need to answer three questions: a) how often does SGP lead to the correct solution, i.e. the global optimum, b) how fast does it take to converge, and c) how close does it get in value to the global optimum when it fails.

To answer such questions, we focus on the $2 \times 2$ case, for which we know what are the possible values that the global optimum can take. We use the results in [5] regarding the two-cell sum capacity maximization and apply them to the interference channel context. For a $2 \times 2$ interference channel, the optimal power control allocation is one out of three possible optimum transmit power tuples. This will allow us to validate the SGP approach and characterize its accuracy and convergence performance.

In section II we present the system model and formulate the problem. In section III we introduce the optimal power control tuples for the $2 \times 2$ interference channel. We explain in section IV the single and double condensation approaches that we used in the SGP technique. We analyze the results in section V, and conclude in section VI.

II. SYSTEM SETUP

We consider two transmitters and two receivers interference channel with no cooperation possible at the transmit nor at the receive side. In Fig. 1 transmit signals are $x_1$ and $x_2$ with individual maximum power constraints of $P_{\text{max}}$. Received signals are $y_1$ and $y_2$, with the additive noises $n_1$ and $n_2$, respectively. The noise is independent complex Gaussian circular symmetric with variance $\sigma_n^2$. We denote $g_{ij}$ the channel power gain at the $i^{th}$ receiver from the $j^{th}$ transmitter, and we denote $P$ the power transmit vector which we are interested in obtaining.

We want to maximize the sum capacity where no multiuser decoding is performed at the receiver, and interference is henceforth handled as additional noise. Therefore, the sum capacity maximization problem in generic form is:

$$\text{maximize } \sum_i C_i(P) \quad \text{subject to } P \leq P_{\text{max}}$$

The optimization objective function can be expressed as:

$$\sum_i C_i(P) = \sum_i \log_2 (1 + \text{SINR}_i)$$

$$= \log_2 \prod_i \left( \frac{\sigma_n^2 + \sum_j g_{ij} P_j}{\sigma_n^2 + \sum_{j \neq i} g_{ij} P_j} \right)$$

$$= -\log_2 \prod_i \left( \frac{\sigma_n^2 + \sum_j g_{ij} P_j}{\sigma_n^2 + \sum_j g_{ij} P_j} \right)$$

Fig. 1. $2 \times 2$ interference channel model

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As the logarithmic function is monotonically increasing, the sum capacity maximization problem can therefore be formulated as:

$$\begin{align*}
\text{minimize} & \quad \prod_i \left( \frac{\sigma_i^2 + \sum_{j \neq i} g_{ij} P_j}{\sigma_i^2 + \sum_{j} g_{ij} P_j} \right) \\
\text{subject to} & \quad \mathbf{P} \leq P_{\text{max}}
\end{align*}$$

(3)

where $\text{SINR}_i$ is the signal to interference plus noise ratio at the $i^{th}$ receiver. The function in (3) is non-convex in $\mathbf{P}$, and it is not in a GP form. In section IV we introduce how we can transform (3) in a tractable SGP formulation.

III. OPTIMUM POWER TUPLES FOR A 2 $\times$ 2 SYSTEM

From [5] in order to maximize the sum capacity of a 2-cell cellular system expressed as:

$$C_1 + C_2 = \log_2 \left( 1 + \frac{P_1 g_{11}}{\sigma_n^2 + P_2 g_{12}} \right) + \log_2 \left( 1 + \frac{P_2 g_{22}}{\sigma_n^2 + P_1 g_{21}} \right)$$

subject to $(P_1 \leq P_{\text{max}}, P_2 \leq P_{\text{max}})$, the optimal power tuple $(P_1^*, P_2^*)$ found is one of the possible following tuples: $(0, P_{\text{max}}), (P_{\text{max}}, 0)$, or $(P_{\text{max}}, P_{\text{max}})$. Each of the tuples can correspond to a local optimum or to the global optimum depending on the channel conditions. For the noise dominated case, corresponding to high noise to interference power ratio, the $(P_{\text{max}}, P_{\text{max}})$ tuple will be the optimal choice. Whereas for the interference dominated case, corresponding to low noise to interference power ratio, the solution shifts to one of the transmitter transmitting at full power and the other one being silent. We will use the result of optimal power tuples in analyzing the behavior of SGP in converging to the correct solution.

IV. SEQUENTIAL GEOMETRIC PROGRAMMING

First we introduce the generic standard GP form [6]–[8] for a function $f(\cdot): \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ as

$$\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 1 \quad i = 1, \ldots, m, \\
& \quad h_l(x) = 1 \quad l = 1, \ldots, M
\end{align*}$$

(5)

where $f_i(\cdot), \ i = 0,1, \ldots, m$, are posynomials of the form:

$$f_i(x) = \sum_{k=1}^{K_i} c_{ik} x_1^{\alpha_{ik}(1)} x_2^{\alpha_{ik}(2)} \cdots x_n^{\alpha_{ik}(n)},$$

(6)

with $c_{ik} \geq 0$ and $\alpha_{ik}^{(j)} \in \mathbb{R}$. The equality constraints allowed $h_l(\cdot), l = 1, \ldots, M$ are monomials, where a monomial is defined as a posynomial with a single term, i.e. $K_i = 1$ in (6). We will next describe two condensation methods used to solve the underlying non-convex problem in (3). The condensation method jargon goes back to the Operation Research literature, and both single and double condensation methods are mentioned in [8]. We note that what we forth introduce in this section is general for any arbitrary $n \times n$ interference channel.

A. Single Condensation Method

We revisit the problem in (3), and we focus on the objective function:

$$\prod_i \left( \frac{\sigma_i^2 + \sum_{j \neq i} g_{ij} P_j}{\sigma_i^2 + \sum_{j} g_{ij} P_j} \right)$$

(7)

We note that by approximating the denominator to a monomial we obtain a posynomial expression, and the problem can be formulated as a GP in standard form. Effectively we need to approximate:

$$f(\mathbf{P}) = \prod_i \left( \frac{\sigma_i^2 + \sum_{j \neq i} g_{ij} P_j}{\sigma_i^2 + \sum_{j} g_{ij} P_j} \right) \approx c P_1^{\alpha_1} \cdots P_n^{\alpha_n}$$

(8)

c, $\alpha_1, \ldots, \alpha_n$ are found by first order fitting of the function $f(\mathbf{P})$ to a monomial. We iteratively compute this fitting with the updated power tuple values obtained by solving the GP problem at each iteration. We denote the $k^{th}$ iteration parameters with the superscript $(k)$, and we denote the initial conditions power vector $\mathbf{P}^{(0)}$. In particular, for the $k^{th}$ iteration we arrive to:

$$\begin{aligned}
\alpha_i^{(k)} &= \frac{P_i^{(k-1)}}{f(\mathbf{P}^{(k-1)})} \frac{\partial f(\mathbf{P}^{(k-1)})}{\partial P_i^{(k-1)}}, \quad i = 1, \ldots, n \\
c^{(k)} &= \frac{f(\mathbf{P}^{(k-1)})}{\prod_{j=1}^{n} P_j^{(k-1)\alpha_j^{(k)}}} \quad \prod_{j=1}^{n} P_j^{(k-1)\alpha_j^{(k)}}
\end{aligned}$$

(9)

For each iteration a newly updated GP problem is solved, hence the nomenclature sequential GP. Therefore for the single condensation approach the problem (3) can be solved as outlined in Algorithm 1.

Algorithm 1 Single Condensation method for SGP

Initialize $\mathbf{P}^{(0)}$

repeat

For every iteration $k$, compute $\alpha_i^{(k)}$ and $c^{(k)}$ from (9), Solve the elementary GP:

$$\begin{align*}
\text{minimize} & \quad \prod_{i} \left( \frac{\sigma_i^2 + \sum_{j \neq i} g_{ij} P_j^{(k-1)}}{\sigma_i^2 + \sum_{j} g_{ij} P_j^{(k-1)}} \right) \\
\text{subject to} & \quad \mathbf{P}^{(k)} \leq P_{\text{max}} \leq 1
\end{align*}$$

until convergence

B. Double Condensation Method

In the double condensation method, the product expression in (3) is directly approximated to a monomial. Thus effectively we have:

$$g(\mathbf{P}) = \prod_i \left( \frac{\sigma_i^2 + \sum_{j \neq i} g_{ij} P_j}{\sigma_i^2 + \sum_{j} g_{ij} P_j} \right) \approx d P_1^{\beta_1} \cdots P_n^{\beta_n}$$

(11)
Similarly, the $d_i, \beta_1, \ldots, \beta_n$ are found by first order fitting of the function $g(P)$ introduced in (11). Hence we obtain:

$$
\begin{align*}
\beta_i^{(k)} & = \frac{P_i^{(k-1)}}{g(P^{(k-1)})} \frac{\partial g(P^{(k-1)})}{\partial P_i^{(k-1)}}, \quad i = 1, \ldots, n \\
\delta^{(k)} & = \frac{P_i^{(k-1)} \beta_i^{(k)} \cdot \cdots \cdot P_n^{(k-1)} \beta_n^{(k)}}{g(P^{(k-1)})}
\end{align*}
$$

We describe the double condensation method in Algorithm 2.

**Algorithm 2 Double Condensation method for SGP**

```
Initialize $P^{(0)}$
repeat
    For every iteration $k$, compute $\beta_i^{(k)}$ and $d^{(k)}$ from (12),
    Solve the elementary GP:

    $$
    \min_{P^{(k)}} \left( d^{(k)} P_1^{(k-1)} \beta_1^{(k)} \cdot \cdots \cdot P_n^{(k-1)} \beta_n^{(k)} \right)
    $$
    subject to $P^{(k)}/P_{\text{max}} \leq 1$
until convergence
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Specifically with the double condensation method, the objective function in (3) becomes an elementary GP problem (11) and exhibits an analytical solution. Hence, if $\beta_i$ is positive, $P_i$ will have the value of $P_{\text{max}}$, or $P_i$ will have the value of 0 otherwise [6]. Therefore the sequential solutions in SGP converge to either $P_{\text{max}}$ or 0 and no other intermediate values in between, effectively leading to the fast convergence observed in V.

V. NUMERICAL RESULTS

In this section, we discuss three different scenarios, A) a noise-dominated case, B) a moderate interference case, and C) an interference-dominated case. For each scenario, we refer to (4) where we present 20 independent realizations of the interference channel, in which the noise is complex Gaussian circular symmetric of variance $\sigma_n^2$, and the channel power gain are exponentially distributed with $\lambda = 1$ corresponding to an underlying complex Gaussian circular symmetric channel gains of variance 1. For each case we present two figures. The first one will give an indication about the optimality of the power tuples if they hit the global maximum. The second one will give an indication of the convergence performance.

A. Noise-dominated case

In this scenario, the noise power is 10 times higher than the average interference power. In the asymptotic case of very high noise power compared to the interference power, the maximum sum capacity is trivially reached by transmitting at full power at each transmitter. Therefore, referring to Fig. 2, the optimal strategy for a noise dominated system will tend to transmit with the power tuple $(P_{\text{max}}, P_{\text{max}})$ most of the time. We know from section III that the optimal strategy for the $2 \times 2$ system is always one of following tuples $(0, P_{\text{max}}), (P_{\text{max}}, 0)$, or $(P_{\text{max}}, P_{\text{max}})$, thus we plot their corresponding sum capacity in Fig. 2 in order to validate the solutions that we reach by the SGP methods. We see that the single condensation and double condensation methods do converge to the global optimum value with a very high probability and perform extremely well. We note that as the noise to average interference power ratio is 10, i.e. finite, we might still have cases where $(P_{\text{max}}, P_{\text{max}})$ is not global optimum, and either $(0, P_{\text{max}})$ or $(P_{\text{max}}, 0)$ would achieve the global solution. We note however that in the event where the SGP would miss, the solution reached is very close in value to the global one. In Fig. 3 we plot the behavior of convergence for each of the SGP methods. We notice that the single condensation method is slow and takes a long time to converge. Whereas the double condensation method converges in at most 2 iterations, which is conform to our discussion at the end of subsection IV-B.
B. Moderate interference case

For this scenario, we set the noise to average interference power ratio to be 1. In Fig. 4 we start to notice that both methods can sometimes miss the global solution, but still nevertheless very close in value. In all the cases the SGP approach will converge to a local or a global optimum, which is in the $2 \times 2$ system confined to three choices. In Fig. 5 we observe the convergence behavior, and the double condensation method still outperforms the single condensation method and it usually converges in at most 2 iterations.

C. Interference-dominated case

We set the noise to average interference power ratio to $1/100$. In the asymptotic case of interference dominated case, the optimal strategy is one transmitter using full power and the other one silent. We notice this trend in Fig. 6. We note however that the SGP approaches start to miss at a higher probability than in the previous cases. However when the algorithm misses, it is still close in value to the global optimum and does not deviate to the lowest local optimum.

VI. CONCLUSIONS

We formulated and studied the performance of SGP, using two methods of single condensation and double condensation, in solving the non-convex problem of optimal power control to maximize the sum capacity of interference channel with individual maximum power constraints and no multiuser decoding, specifically for a $2 \times 2$ system. We presented
the SGP formulation for any arbitrary $n \times n$ interference channel, and we examined its accuracy and convergence performance for the $2 \times 2$ system, of which we know the optimal power allocation tuples [5]. We saw that SGP approach works extremely well in converging to the global optimum. However in the case of a miss, it will converge to a local optimum that is close in value to the global one, thus being suboptimal but remaining close enough. We also concluded that double condensation method is superior in leading faster convergence of the algorithm in usually 1 to 2 iterations. Hence, we evaluated and demonstrated the usefulness of SGP as a powerful technique to solve the non-convex problem of maximizing sum capacity of $2 \times 2$ interference channels with no multiuser decoding. Validating SGP performance for the arbitrary $n \times n$ interference channel remains for future work.

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