Maximum Sum Rates via Analysis of 2–User Interference Channel Achievable Rates Region

Mohamad Charafeddine and Arogyaswami Paulraj
Information Systems Laboratory, Department of Electrical Engineering, Stanford University
Packard 225, 350 Serra Mall, Stanford, CA 94305, Email: {mohamad, apaulraj}@stanford.edu

Abstract—Treating the interference as noise, the paper studies the first derivative of the frontiers which trace the achievable rates region of the 2–user interference channel. The achievable rates region in this case was found to be the convex hull of the union of two regions, each is bounded by a log-defined line. Those log-defined lines are characterized by holding one of the transmitters at full power, while the other transmitter sweeps its full power range [1]. Maximizing the sum rates for the 2–user interference channel translates to the study of the first intersection point with lines of slope −1 approaching the rates region from positive infinity. The paper achieves the same result reported in [2], that the maximum sum rates solution is one of three points: one user transmitting with full power while the other user is silent, or both users transmitting at full power simultaneously. The result in [2] is achieved through analysis of the objective function, while the solution presented herein follows from analyzing the first derivative of the rates region frontiers.

I. INTRODUCTION

The capacity region of a 2–user communication channel has been an open problem for about 30 years [3], [4]. Information-theoretic bounds through achievable rates region have been proposed, most famously with the Han-Kobayashi bounds [5]. The capacity of the Gaussian interference channel under strong interference has been found in [6]. Recent results about the 2–user interference channel to within one bit of capacity have been shown in [7]. This paper is interested in the context when the receiver treats the interference as additive thermal noise and does not employ multiuser detection. Such scenario is encountered in cellular communications, ad hoc and sensor networks, where low-complexity transceivers are preferred.

Treating the interference as noise, the achievable rates region for the general n–user interference channel was found in [1] to be the convex hull of the union of n regions; where each region is outer-bounded by a hyper-surface frontier of dimension n–1. Each hyper-surface is characterized by having one of the transmitters transmitting at full power, and the other users sweeping their full range of transmit powers. For the case of 2–user interference channel, the achievable rates region is translated as the union of two regions R₁ and R₂. Each region Rᵢ is outer-bounded by a log-defined line Φᵢ, which is characterized by having Pᵢ = P_max and Pⱼ ≠ i sweeps its full power range from 0 to P_max. Explicitly, the achievable rates region for the 2–user interference channel is:

Convex Hull{R₁ ∪ R₂},

where Rᵢ = R(Φᵢ) is the rates region outer-bounded by Φᵢ. The details of Φᵢ will be presented in the following section.

Fig. 1: 2–user interference channel

The authors in [2] have found that the power tuple (P₁, P₂) which maximizes the sum rates of the 2–user interference channel is confined to the following set: {(0, P_max), (P_max, 0), (P_max, P_max)}. The result followed from analyzing the sum rates objective function. In this work, the same result is reached from a different perspective as a corollary from the theorem describing the achievable rates region of the interference channel in [1]. The analysis is based on the study of the slope of the frontiers tracing the rates region.

II. SYSTEM MODEL

The 2–user interference channel is illustrated in Fig. 1. User i transmits its signal xᵢ with power Pᵢ to the receiver yᵢ. Without loss of generality, the transmitters are assumed to have a maximum transmit power of P_max, and the receivers are assumed to have equal noise variance σ². For notational brevity, a, b, c, and d denote the channel power gains normalized by the noise variance. Explicitly, with gᵢᵢ denoting the flat channel gain at receiver i from transmitter j, we have:

\[ a = |g_{1,1}|^2/\sigma^2, \quad b = |g_{1,2}|^2/\sigma^2, \]

\[ c = |g_{2,1}|^2/\sigma^2, \quad d = |g_{2,2}|^2/\sigma^2. \]

With no cooperation at the transmit side nor at the receive side, and by treating the interference as additive noise, the achievable rates C₁ and C₂, by user 1 and user 2 respectively, are:

\[ C₁(P₁, P₂) = \log_2 \left( 1 + \frac{aP₁}{1 + bP₂} \right), \]

\[ C₂(P₁, P₂) = \log_2 \left( 1 + \frac{cP₂}{1 + dP₁} \right). \] (2)

The following notation Φ(p₁, p₂) is used to denote a point in the rates region with the x-y coordinates C₁(p₁, p₂), and C₂(p₁, p₂), respectively. Similarly, the notation Φ(·, p₂)
designates a potential line in the rates region marked by having $P_1$ sweeping its full power range from 0 to $P_{\text{max}}$ and with having $P_2$ fixed at the value $p_2$. Referring to (1), and therefore assigning $\Phi_1 = \Phi(P_{\text{max}}, \cdot)$ and $\Phi_2 = \Phi(\cdot, P_{\text{max}})$, the equations defining $\Phi_1$ and $\Phi_2$ in the rates region are [1]:

$$\Phi_1 = \log_2 \left( 1 + \frac{c}{a} \left( \frac{aP_{\text{max}} - (2C_1 - 1)}{(2C_1 - 1)(1 + aP_{\text{max}})} \right) \right),$$

$$\Phi_2 = \log_2 \left( 1 + \frac{C_2}{a} \left( \frac{1}{1 + bP_{\text{max}}(2C_1 - 1)} \right) \right).$$

The range for $C_1$ in $\Phi_1$ is from $C_1(P_{\text{max}}, P_{\text{max}}) \leq C_1 \leq C_1(P_{\text{max}}, 0)$, see Eq. (2) and Fig. 2(a). And $C_1$ in $\Phi_2$ takes the complementary range of $0 \leq C_1 \leq C_1(P_{\text{max}}, P_{\text{max}})$. $\Phi_1$ and $\Phi_2$ are referred to as log-defined to emphasize the fact that they are formed from those two log equations in (3).

The convex hull operation in (1) stems from the fact that $\Phi_1$ can be convex, concave, or having a non-stationary inflection point. Focusing on $\Phi_2$ without loss of generality, and by analyzing the second derivative of $\Phi_2$ with respect to $C_1$ (see Eq. (3)), an inflection threshold index $Q_1$ is devised such that

$$\text{sign} \left( \frac{\partial^2 \Phi_2}{\partial C_1^2} \right) = \text{sign}(P_1 - Q_1).$$

The detail of $Q_1$ is given in the Appendix B (Eq. (15)). Hence, the convexity characteristics of $\Phi_2$ are summarized from [1] as follows:

- $\Phi_2$ is concave (enclosing a convex region), if $Q_1 \leq 0$.
- $\Phi_2$ is convex (enclosing a concave region), if $Q_1 \geq P_{\text{max}}$.
- $\Phi_2$ has an inflection point, if $0 < Q_1 < P_{\text{max}}$.

Next, this paper goes further in analyzing the characteristics of the achievable rates region frontiers $\Phi_1$ and $\Phi_2$. Section III studies the slope of $\Phi_1$ which leads to the answer of the optimization problem of maximizing the sum rates.

III. MAXIMIZING SUM RATES

A. Problem Formulation

The maximization sum rates problem is formulated as:

$$\max_{P_1, P_2} \sum_{i=1}^{2} \log_2 \left( 1 + \frac{|g_{i}|^2P_i}{\sigma_n^2 + |g_{i,j} \neq i|^2P_j} \right),$$

subject to $P_i \leq P_{\text{max}}, i = 1, 2.$

The objective function to this power control problem is non-concave in the power vector $P = [P_1, P_2]^T$.

B. Solution through Analysis of Rates Region

Let $v^*$ denotes the maximum sum rates value, then the problem in (4) can be expressed as:

$$v^* = \max_{P_1, P_2} C_1 + C_2,$$

subject to $P_i \leq P_{\text{max}}, i = 1, 2.$

For a line equation of the form $v = C_1 + C_2$, or alternatively $C_2 = v - C_1$, the solution to the problem in (5) can be interpreted as the first intersection point of a straight line with slope value of $-1$ approaching the rates region from positive infinity. This line will intersect both the $C_1$ and $C_2$ axes at the value $v = v^*$. In Fig. 2(a) this is illustrated as the solid red line intersecting the achievable rates region frontiers at Point B in this specific case. Point B is the point $\Phi(P_{\text{max}}, P_{\text{max}})$, signifying that both users are transmitting simultaneously at full power. For completeness, point A is $\Phi(0, P_{\text{max}})$ signifying that user 2 is transmitting at full power while user 1 is silent; and symmetrically, point C is $\Phi(P_{\text{max}}, 0)$.

We focus on one of the frontiers, $\Phi_2$, and by symmetry the results extend for $\Phi_1$. In order to determine where the $-1$-slope lines will first intersect the frontiers, we base on a geometrical approach by comparing the frontiers slope values versus the slope value of $-1$. Then, let $s$ be the value of $\Phi_2$’s first derivative with respect to $C_1$:

$$s(P_1) = \frac{\partial C_2}{\partial C_1}_{\Phi_2} = \frac{\partial \Phi_2}{\partial C_1}_{\Phi(P_1, P_{\text{max}})}, 0 \leq P_1 \leq P_{\text{max}}.$$

Let $s_A$ and $s_B$ denotes the slope values at Point A and Point B, respectively, see Fig. 2(a). Geometrically, if $-1 \leq s_B$ for all the range of $P_1$, then the $-1$-slope line will first intersect $\Phi_2$ always at point B, and consequently the maximum sum rates solution is point B. If $s(p_1) = -1$ for $0 < p_1 < P_{\text{max}}$ then the $-1$-slope line is tangential to $\Phi_2$ at the point $\Phi(p_1, P_{\text{max}})$, which would be the maximum sum rates solution. The case when the $-1$-slope line approaches the rates region frontier in a non-tangential manner is when the slope range between $s_A$ and $s_B$ evidently does not include the value $-1$. Using this concept we examine the values of $s$ for all the range of $P_1$ in the context of the three convexity characteristics of $\Phi_2$: concave, convex, or with an inflection point.

C. Concave Frontier

$\Phi_2$ is concave, it implies that $\frac{\partial^2 \Phi_2}{\partial C_1^2} < 0$. Therefore the first derivative is monotonically decreasing in $C_1$. As a result, the minimum value of the slope is at the maximum value that $C_1$ can attain, which is at Point B with a $C_1(P_{\text{max}}, P_{\text{max}})$ value. Then $s$ is bounded as:

$$s_{\text{min}} \leq s \leq 0$$

with $s_{\text{min}}$ defined as:

$$s_{\text{min}} = s_B = \frac{\partial \Phi_2}{\partial C_1}_{\Phi(P_{\text{max}}, P_{\text{max}})}.$$

Hence the following lemma is reached:

**Lemma 1:** In solving the maximum sum rates problem for a concave frontier, it suffices to study the value of the frontier’s first derivative at the point $\Phi(P_{\text{max}}, P_{\text{max}})$, which represents a lower bound.

At this point, we make a conjecture that the $-1$-slope line will intersect the concave $\Phi_2$ always from the right at point...
B. This is proved if and only if \( s_B = s_\Phi(P_{\text{max}}, P_{\text{max}}) \) satisfies:

\[
-1 \leq s_B
\]

By proving this condition in Appendix A, the following corollary is obtained:

**Corollary 1:** For concave frontiers, the maximum sum rates solution is achieved through \( \Phi(P_{\text{max}}, P_{\text{max}}) \). Users transmit simultaneously at full power.

The following applies and makes sense in noise-limited regimes, for which the rates region exhibits concave frontiers.

**D. Convex Frontier**

Referring to Fig. 2(b), whenever the log-defined frontier \( \Phi_2 \) is convex, therefore it is upper-bounded by the time-sharing line connecting the extremities points A and B. In this case, it suffices to consider these two points which belong to \( \Phi_2 \). Thus the \(-1\)-slope lines will intersect \( \Phi_2 \) first at either point A (\( \Phi(0, P_{\text{max}}) \)) or point B (\( \Phi(P_{\text{max}}, P_{\text{max}}) \)). Generalizing for a convex \( \Phi_1 \) the following corollary is obtained:

**Corollary 2:** For convex frontiers, the maximum sum rates solution is achieved through a point in the set: \( \{\Phi(0, P_{\text{max}}), \Phi(P_{\text{max}}, P_{\text{max}}), \Phi(P_{\text{max}}, 0)\} \).

**E. Frontier with an Inflection Point**

For the \( \Phi_2 \) frontier with a non-stationary inflection point D (see Fig. 2(c)), the same approach follows by conjecturing that the approaching \(-1\)-slope line will first intersect the frontier at the point B (\( \Phi(P_{\text{max}}, P_{\text{max}}) \)), and then by proving that this conjecture holds. The inflection point D happens when \( P_1 \) equals the inflection point threshold index \( Q_1 \). In other words, point D has the coordinates \( \Phi(Q_1, P_{\text{max}}) \). For the conjecture to hold, then two conditions should be satisfied:

1) \(-1 \leq s_D \): which translates that the \(-1\)-slope line will first intersect the log-defined segment \( \Phi_{AD} \) from the right at point D. The logic is similar to the one followed in the aforementioned concave frontier case. This is proved in Appendix B.

2) \(-1 \leq s_DB \): where \( s_DB \) denotes the slope of the straight time-sharing line connecting point D and point B. This condition translates that the approaching \(-1\)-slope line will first intersect the D-B straight line from the right at point B. This is proved in Appendix C.

Putting these two conditions together, for the \( \Phi_{AD} \) segment, the \(-1\)-slope line will intersect at point D first; and for the A-D straight line, the \(-1\)-slope line will intersect at point B first. Hence it follows that the \(-1\)-slope line will intersect \( \Phi_2 \) always at point B first. The following corollary is consequently obtained:

**Corollary 3:** For frontiers with inflection points, the maximum sum rates solution is achieved through \( \Phi(P_{\text{max}}, P_{\text{max}}) \).

**IV. INSIGHTS AND CONCLUSION**

For noise-limited regimes the rates region frontiers are concave, and from Corollary 1 the maximum sum rates solution is \( \Phi(P_{\text{max}}, P_{\text{max}}) \) where every user transmits at full power. This makes sense as the interference is marginal. As the interference gain grows more significant, it forces the frontiers \( \Phi_1 \) and \( \Phi_2 \) more into convexity. As the interference increases, the frontiers transition from concavity, to exhibiting non-stationary inflection points, and then into convexity. This shifts the maximum sum rates solution from being point B (all users transmitting at full power: \( \Phi(P_{\text{max}}, P_{\text{max}}) \)) towards either A or C (only a single user transmitting: \( \Phi(0, P_{\text{max}}) \) or \( \Phi(P_{\text{max}}, 0) \)). Different maximum transmit power for each transmitter or different noise power at each receiver leads to the same outcome. The work in [2] arrives to the result that the maximum sum rates solution is either one of the three points in the set \( \{\Phi(0, P_{\text{max}}), \Phi(P_{\text{max}}, P_{\text{max}}), \Phi(P_{\text{max}}, 0)\} \) through examining the objective function in (4). The work presented herein approaches the problem from the achievable rates region perspective, and it details when each of the solution occurs based on the convexity characteristics of the frontiers. This was achieved by analyzing the first derivative of the frontiers tracing the achievable rates region.
Appendix

A. Concave Frontier: Proof $-1 \leq s_B$

**Proof:** The frontier equation for $\Phi_2$ in function of $C_2$ is

$$C_2(C_1) = \log_2 \left(1 + \frac{aCP_{\text{max}}}{A + dA(C_1^2 - 1)} \right),$$

where $A = 1 + bP_{\text{max}}$ for notational brevity. Then the first derivative equation follows as:

$$\frac{\partial C_2}{\partial C_1} = \frac{-aCP_{\text{max}}dA^2C_1}{(a + dA(C_1^2 - 1))(a + dA(C_1^2 - 1) + acP_{\text{max}})} \tag{7}$$

As $\Phi_2 = \Phi(P_1, P_{\text{max}})$, and thus using the fact that $(2C_1^2 - 1) = aP_1/A$, Eq. (7) can be expressed in function of $P_1$. Hence,

$$s(P_1) = \frac{\partial C_2}{\partial C_1} \Phi(P_1, P_{\text{max}}) = \frac{-acP_{\text{max}}dA^2C_1}{(a + dA(P_1^2 - 1))(a + dA(P_1^2 - 1) + acP_{\text{max}})} \tag{8}$$

From Eq. (8), $s_B$ at $P_{\text{max}}$ becomes:

$$s_B = s(P_1 = P_{\text{max}}) = \frac{-cdP_{\text{max}}(1 + aP_{\text{max}} + bP_{\text{max}})}{a(1 + dP_{\text{max}})(1 + dP_{\text{max}} + cP_{\text{max}})}$$

Therefore the objective in proving $-1 \leq s_B$ translates as:

$$\text{iff } \frac{cdP_{\text{max}}(1 + aP_{\text{max}} + bP_{\text{max}})}{a(1 + dP_{\text{max}})(1 + dP_{\text{max}} + cP_{\text{max}})} \leq 1 \tag{9}$$

Or equivalently, after simplification, iff

$$cdP_{\text{max}}(1 + bP_{\text{max}}) - a(1 + cP_{\text{max}}) \leq 2adP_{\text{max}}(1 + dP_{\text{max}}) \tag{10}$$

It is not clear at this stage if this relation always hold. At this point, we make use of the fact that the frontier is concave. Therefore we draw a relation from the concavity condition, explicitly $\Phi_2$ is concave if $\partial^2 \Phi_2/\partial C_1^2 \leq 0$, if it follows that at $\Phi(P_{\text{max}}, P_{\text{max}})$:

$$\frac{\partial^2 \Phi_2}{\partial C_1^2} \Phi(P_{\text{max}}, P_{\text{max}}) = \frac{(a + dP_{\text{max}})(a + dP_{\text{max}} + acP_{\text{max}})}{(\theta + dP_{\text{max}})^2} - (\alpha - \theta)(\alpha - \theta + acP_{\text{max}}) \leq 0 \tag{11}$$

where $\theta = d(1 + bP_{\text{max}})$. Expanding the relation in Eq. (11) and making further simplifications, we arrive to the following relation: $\Phi_2$ is concave:

$$\text{iff } cdP_{\text{max}}(1 + bP_{\text{max}}) - a(1 + cP_{\text{max}}) \leq -ad^2P_{\text{max}}^2 - 2d - 2d^2P_{\text{max}} - 2dB_{\text{max}} - 2d^2bP_{\text{max}}^2 \tag{12}$$

As $a, b, d,$ and $P_{\text{max}}$ are all positive real numbers, then the right hand side of Eq. (12) is always a negative number. Therefore, we can conclude the following regarding the left hand side of Eq. (12), mainly:

$$cdP_{\text{max}}(1 + bP_{\text{max}}) - a(1 + cP_{\text{max}}) \leq 0 \tag{13}$$

Going back to Eq. (10), using the result in Eq. (13), we see that the relation in Eq. (10) is always satisfied, as the right hand side of Eq. (10) $2adP_{\text{max}}(1 + dP_{\text{max}}) \geq 0$.

B. Frontier with Inflection Point: Proof $-1 \leq s_D$

**Proof:** Recalling the slope expression Eq. (8) in function of $P_1$, and evaluating it at the inflection point $D$ where $P_1 = Q_1$, $s_D = s(Q_1)$ follows as:

$$s(Q_1) = \frac{-cdP_{\text{max}}(1 + aQ_1 + bP_{\text{max}})}{a(1 + dQ_1)(1 + dQ_1 + cP_{\text{max}})} \tag{14}$$

The non-stationary inflection threshold index $Q_1$ is found as

$$Q_1 = \frac{R(\sqrt{(a - \theta)(a - \theta + acP_{\text{max}})} - \theta)}{ad} \tag{15}$$

as we consider the case when the frontier $\Phi_2$ exhibits an inflection point, then for the inflection point to exist, $Q_1$ needs to be between 0 and $P_{\text{max}}$; hence we can drop the real operator. $Q_1$ becomes:

$$Q_1 = \frac{\sqrt{T_1T_2} - \theta}{ad} \tag{16}$$

**Remark:** In Eq. (16) $T_1T_2$ should be positive, this happens in two cases:

- Case i: $T_1 \geq 0$, which implies that $T_2 \geq 0$, as $T_2 = T_1 + acP_{\text{max}}$.
- Case ii: $T_2 \leq 0$, which implies that $T_1 \leq 0$.

Appendix B.1 proves that the Case ii actually never occur; in this regard we only have to focus the analysis on Case i in treating the scenario of a frontier with inflection point. Eq. (16) results that

$$\sqrt{T_1T_2} = \theta + adQ_1 = d(1 + bP_{\text{max}} + aQ_1); \tag{17}$$

Therefore, $s(Q_1)$ in Eq. (14) becomes

$$s(Q_1) = \frac{-cP_{\text{max}}}{(1 + dQ_1)(a + adQ_1 + acP_{\text{max}})}. \tag{18}$$

Furthermore, we can express each term in Eq. (18) in function of $T_1$ and $T_2$. This leads to

$$s(Q_1) = \frac{-T_2 - T_1}{(T_1^2 + T_1)(\sqrt{T_1T_2} + T_1)} \tag{19}$$

where Eq. (19) is explicitly expressed in function of $a, b, c, d,$ and $P_{\text{max}}$ only (due to definition of $T_1$ and $T_2$). The dependence on $Q_1$, as in Eq. (14), has been implicitly accounted for. This renders the analysis much more tractable. Therefore the goal is to prove if $-1 \leq s(Q_1)$. This holds if and only if:

$$\text{iff } \frac{(T_2 - T_1)(\sqrt{T_1T_2})}{(\sqrt{T_1T_2} + T_1)(\sqrt{T_1T_2} + T_1)} \leq 1$$

iff $T_1 \sqrt{T_1T_2} \leq 2T_1T_2 + T_1 \sqrt{T_1T_2}$

iff $0 \leq T_1 + T_1 \sqrt{T_1T_2}$
Which is always the case, given the assumption started with in Case \( i \) \((T_1 \geq 0, \text{ and } T_2 \geq 0)\). Hence for \( s(Q_1) \), or equivalently \( s_D \), the relation \(-1 \leq s_D \) holds.

1) Infeasibility of Case \( ii \):

Proof: Case \( ii \) is considered when \( T_2 \leq 0 \), i.e. \( a + acP_{\text{max}} - \theta \leq 0 \). For Case \( ii \) to exist, \( Q_1 \) should be in the range of 0 and \( P_{\text{max}} \). We prove that this is not the case. As \( Q_1 \) is a power value, it should be positive. Therefore \( Q_1 \geq 0 \) if and only if:

\[
\begin{align*}
\text{iff } & \sqrt{T_1T_2 - \theta} \geq \gamma, \quad 0 \\
\text{iff } & (a - \theta)(a + acP_{\text{max}} - \theta) \geq \gamma, \quad 0 \\
\text{iff } & a + acP_{\text{max}} - \theta \geq \gamma, \quad 0 \\
\text{iff } & T_2 \geq \gamma, \quad 0
\end{align*}
\]

Which negates the assumption we started with that \( T_2 \leq 0 \), therefore \( Q_1 \) never takes a positive value for Case \( ii \). Hence Case \( ii \) is proved infeasible and is discounted.

C. Frontier with Inflection Point: Proof

Proof: The slope \( s_{DB} \) of the straight line \( DB \), is evaluated as

\[
s_{DB} = \frac{C_2|D - C_2|B}{C_1|D - C_1|B} = \log_2 \left( \frac{(1 + dQ_1 + cP_{\text{max}})(1 + dP_{\text{max}})}{(1 + dQ_1)(1 + dP_{\text{max}} + cP_{\text{max}})} \right) = \log_2 \left( \frac{(1 + aQ_1 + bP_{\text{max}})(1 + bP_{\text{max}} + cP_{\text{max}})}{(1 + bP_{\text{max}})(1 + bP_{\text{max}} + cP_{\text{max}})} \right) \geq \gamma - 1
\]

Therefore this condition can be written as

\[
(1 + dQ_1 + cP_{\text{max}})(1 + dP_{\text{max}}) \leq (1 + dQ_1)(1 + dP_{\text{max}} + cP_{\text{max}}) \leq \gamma, \quad 1 + bP_{\text{max}} + aP_{\text{max}}
\]

or equivalently

\[
(1 + dQ_1 + cP_{\text{max}})(1 + bP_{\text{max}} + aQ_1) \leq \gamma, \quad (1 + bP_{\text{max}} + aQ_1)(1 + dP_{\text{max}} + cP_{\text{max}})
\]

\[
\frac{1 + dQ_1}{1 + dP_{\text{max}}} \geq \gamma
\]

Subsequently the left hand side of Eq. (20) is proved to be monotonically increasing in \( Q_1 \) in Appendix C.1. Therefore as \( Q_1 \) is upper bounded by \( P_{\text{max}} \), then Eq. (20) holds.

1) Increasing Monotonicity of Eq. (20)’s LHS:

Proof: Define the function \( y(x) \) as

\[
y(x) = \frac{(1 + dx + m)(1 + ax + n)}{1 + dx}
\]

therefore, evaluating the first derivative, it leads to

\[
\frac{\partial y}{\partial x} = \frac{a + 2adx + am + ad^2 x^2 - md - nd}{(1 + dx)^2}
\]

Replacing \( m = cP_{\text{max}} \) and \( n = bP_{\text{max}} \), Eq. (21) becomes

\[
\frac{\partial y}{\partial x} = \frac{ad^2 x^2 + 2adx + a + cP_{\text{max}}(a - \theta)}{(1 + dx)^2}
\]

And as \( T_1 \geq 0 \), i.e. \( (a - \theta) \geq 0 \), then

\[
\frac{\partial y}{\partial x} \geq 0
\]

Therefore the left hand side of Eq. (20) is monotonically increasing in \( Q_1 \).

REFERENCES


