Simplified Eigenvalues Distributions of $2 \times 2$ Complex Noncentral Wishart

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Abstract—This paper derives new simplified analytical cumulative density functions for the eigenvalues of complex noncentral Wishart matrix of size $2 \times 2$. Such distributions are often encountered in Multiple Input Multiple Output (MIMO) Ricean channels. The results derived herein are general for any arbitrary non-centrality matrix, and they account the cases having identical or non-identical eigenvalues of the underlying non-centrality matrix. When compared to the generalized distribution recently found in literature which only treated the case of non-identical eigenvalue of the non-centrality matrix, the expressions presented in this paper exhibit a larger parameters range to which the numerical calculation remains computationally viable.

I. INTRODUCTION

A particular case where complex noncentral Wishart is encountered in wireless communication is when the transmitter sends its information over a Ricean channel with a certain Ricean factor $K$. For the case of a Ricean $2 \times 2$ multiple input multiple output channel, there exists a channel mean attributed mainly to a line of sight (LOS) component, and a variable channel component attributed to fading and scattering in the environment. A complex Wishart matrix $W$ is formed by the multiplication of a channel $H$ of independent complex Gaussian distributed entries with its conjugate transpose, specifically, $W = HH^H$. The non-centrality of the Wishart matrix ensues from the presence of the non-zero mean component in the channel $H$. The Ricean channel presents such an example. The need to characterize the eigenvalues distributions of $W$ is of importance in the evaluation of metrics such ergodic capacity of the communication link, or in performing spatial link adaptation techniques such as between diversity Alamouti Coding or Spatial Multiplexing [1]–[3]. The condition number, and subsequently the rank, of the underlying non-centrality matrix plays an important role in the space-time coding techniques used to optimize a specific communication metric. Recent results in [4] describe the behavior of the eigenvalues of $W$ when the non-centrality matrix eigenvalues are non-identical. The expressions in [4] are general for any size of channel $H$. However the aim in this paper is to provide a more tractable simplified form of the eigenvalues distribution of $W$ for the $2 \times 2$ channel, and to additionally provide the novel solution to the previously untreated case of identical non-centrality matrix eigenvalues.

Simplified cumulative distribution functions (CDF) of the maximum and minimum eigenvalues of the complex noncentral Wishart of size $2 \times 2$ are derived. The system model is explained in section II. Section III examines the minimum eigenvalue case, starting from the generalized distribution result discussed in [4] and then deriving the simplified CDFs outlining the two cases of non-identical and identical eigenvalues of the non-centrality matrix. Subsequently, the maximum eigenvalue case is addressed in section IV. Numerical results are presented in section V, and the conclusion follows in section VI.

II. SYSTEM MODEL

The system model is presented in this section for the general case of $M_R \times M_T$ channel $H$, where $M_R$ and $M_T$ are the number of receive and transmit antennas respectively. The $2 \times 2$ channel treatment follows subsequently, Fig. 1.

Fig. 1. $2 \times 2$ system model

A MIMO $M_R \times M_T$ Ricean channel model is depicted as follows

$$H = \frac{\sqrt{K}}{\sqrt{1 + K}} H + \frac{1}{\sqrt{1 + K}} H_w$$

where $H$ is a deterministic arbitrary fixed channel subject to the trace constraint $\text{Tr}(H H^H) = M_T M_R$. $H_w$ is the $M_R \times M_T$ random channel with i.i.d. $CN(0,1)$ entries. The mean of the channel $H$, denoted as $M$, is therefore

$$M = \mathcal{E}(H) = \frac{\sqrt{K}}{\sqrt{K + 1}} \bar{H},$$

and the correlation matrix of $H$, denoted as $\Sigma$, follows as

$$\Sigma = \mathcal{E}((H - M)(H - M)^H) = \frac{1}{K + 1} I_{M_R}.$$  

The complex noncentral Wishart matrix $W$ is formed as

$$W = HH^H.$$  

As denoted in [4] for the general case of a $M_R \times M_T$ channel, let $s = \min(M_T, M_R)$, and $t = \max(M_T, M_R)$, then $W$ follows a noncentral Wishart distribution expressed as

$$W \sim W_s(t, \Sigma, \Omega)$$
where $\Omega$ is the non-centrality matrix defined as
\[ \Omega = \Sigma^{-1}MM^H = KHH^H. \] (6)

Define $\lambda_1, \cdots, \lambda_L$ as the non-zero ordered eigenvalues of the non-centrality matrix $\Omega$, with $\lambda_1$ being the maximum eigenvalue and $\lambda_L$ being the minimum non-zero eigenvalue; effectively,
\[ \lambda_1, \cdots, \lambda_L = \text{eig}(\Omega) = \text{eig} \left( KHH^H \right). \] (7)

Note that the eigenvalues in (7) are defined such as to have an explicit dependence on the Ricean factor $K$. The interest of this paper is to study the behavior of the non-central Wishart, specifically the eigenvalues of $\Sigma^{-1}W$ denoted as $\phi_1, \cdots, \phi_s$. Mainly,
\[ \phi_1, \cdots, \phi_s = \text{eig}(\Sigma^{-1}W) = \text{eig} \left( (\sqrt{K}H + H_w)(\sqrt{K}H + H_w)^H \right). \] (8)

Note that the eigenvalues of $W$ instead of $\Sigma^{-1}W$ just follows by multiplying $\phi_1, \cdots, \phi_s$ with the factor $K + 1$, as $H_w$ is assumed to have the identity matrix as its correlation (3).

Treating the $2 \times 2$ MIMO channel, the analysis focuses on the CDFs of $\phi_1$ and $\phi_2$ in function of $\lambda_1$, $\lambda_2$, and the Ricean factor $K$.

III. SIMPLIFIED CDF OF THE MINIMUM EIGENVALUE

A. Generic CDF of the minimum eigenvalue $\phi_2$

This subsection revisits the generic theorem form stated in [4], where the CDF of the minimum eigenvalue for complex noncentral Wishart is given by
\[ F_{\phi_2}(x) = 1 - |\Psi(x)||\Psi(0)|, \] (9)

where $\Psi(x)$ is a $s \times s$ matrix function with $x \in (0, \infty)$ with entries given by
\[ \{ \Psi(x) \}_{i,j} = \begin{cases} 2^{(2i-s-t)/2}Q_{s+t-2i+1,t-s}(\sqrt{2\lambda_j}, \sqrt{2x}) & j = 1, \cdots, L \\ \Gamma(t + s - i - j + 1, x) & j = L + 1, \cdots, s \end{cases} \] (10)

with $L$ being the number of nonzero eigenvalues of $\Omega$. $Q_{p,q}(a,b)$ is the Nuttall $Q$-function, defined in [5] as
\[ Q_{p,q}(a,b) = \int_b^\infty x^p \exp \left( \frac{-x^2 + a^2}{2} \right) I_q(ax)dx, \] (11)

$I_q(\cdot)$ is the $q^{th}$ order modified Bessel function of the first kind, and $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function, defined as
\[ \Gamma(p, x) = \int_x^\infty t^{p-1}e^{-t}dt \] (12)

\[ = (p-1)!e^{-x} \sum_{k=0}^{p-1} \frac{x^k}{k!}, \quad p = 1, 2, \ldots \] (13)

The CDF of the eigenvalue $\phi_s$, or $\phi_2$ in the $2 \times 2$ case, is not straightforward and is hard to perceive. Additionally for the case of identical non-centrality matrix eigenvalues ($\lambda_1 = \lambda_2$), the current expression in (9) becomes undefined.

B. Simplified CDF of the minimum eigenvalue $\phi_2$

We assume $L = s$, i.e. $\lambda_2 \neq 0$ and hence simplifying and dropping the case of $\Gamma(\cdot, \cdot)$ in (10); it is seen later that the derivation adopted also encompasses the case of $\lambda_2 = 0$. $\Psi(x)$ for the $2 \times 2$ case is expanded to obtain:
\[ \Psi(x) = \begin{bmatrix} 2^{-1}Q_{3,0}(\sqrt{2\lambda_1}, \sqrt{2x}) & 2^{-1}Q_{3,0}(\sqrt{2\lambda_2}, \sqrt{2x}) \\ Q_{1,0}(\sqrt{2\lambda_1}, \sqrt{2x}) & Q_{1,0}(\sqrt{2\lambda_2}, \sqrt{2x}) \end{bmatrix} \] (14)

The focus first is to find a closed form for the Nuttall $Q$ term $Q_{1,0}(\sqrt{2\lambda_1}, \sqrt{2x})$:
\[ Q_{1,0}(\sqrt{2\lambda_1}, \sqrt{2x}) = \int_{\sqrt{2x}}^\infty t \exp \left( -\frac{t^2 + 2\lambda_1}{2} \right) I_0(\sqrt{2\lambda_1}t)dt \] (15)

We recall the definition of the $q^{th}$ order modified Bessel function of the first kind, as
\[ I_q(x) = \sum_{k=0}^\infty \frac{(x/2)^k}{k!\Gamma(k + 1)} \] (16)

$\Gamma(\cdot)$ is the Gamma function. For the $0^{th}$ order modified Bessel function, (16) becomes
\[ I_0(x) = \sum_{k=0}^\infty \frac{(x/2)^k}{k!\Gamma(k + 1)} \] (17)

Noting a property of the Gamma function for integer argument, $\Gamma(k + 1) = k!$. Therefore (17) becomes
\[ I_0(x) = \sum_{k=0}^\infty \frac{(x/2)^k}{k!} \] (18)

Referring back to (15), it can now be written as
\[ Q_{1,0}(\sqrt{2\lambda_1}, \sqrt{2x}) = \sum_{k=0}^\infty \int_{\sqrt{2x}}^\infty t \exp \left( -\frac{t^2 + 2\lambda_1}{2} \right) \frac{(2\lambda_1)^k}{(k!)^2} \frac{2^{2k}}{2^{2k}} dt \]
\[ = \sum_{k=0}^\infty \frac{1}{(k!)^2} \frac{\lambda_1^k}{2^k} \int_{\sqrt{2x}}^\infty t^{2k+1} \exp(-\frac{t^2 + 2\lambda_1}{2})dt \] (19)

The integral expression inside (19) can be abstracted for brevity as
\[ \int_{\sqrt{2x}}^\infty t^{2k+1} \exp(-t^2/2)dt \] (20)

Applying recursive integration by parts on (20), it becomes
\[ \exp(-x^2/2) \sum_{i=0}^{k} x^{2(k-i)} \frac{2^i k!}{(k-i)!} \] (21)

Therefore using the result in (21), the integral expression inside(19) equals
\[ \int_{\sqrt{2x}}^\infty t^{2k+1} \exp \left( -\frac{t^2 + 2\lambda_1}{2} \right) dt = e^{-x^2-\lambda_1} \sum_{i=0}^{k} \frac{x^{2(k-i)}}{(k-i)!} \int_{\sqrt{2x}}^\infty t^{2k+1} \exp(-t^2/2)dt \]
(22)
Plugging the result of (22) in (19) and simplifying,
\[
Q_{1.0}(\sqrt{2\lambda_1}, \sqrt{2\lambda_2}) = e^{-x - \lambda_1} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} \sum_{i=0}^{k} \frac{x^i}{i!}
\]  
(23)

Similarly, the same derivation is applied for \(Q_{3.0}(\sqrt{2\lambda_1}, \sqrt{2\lambda_2})\),
\[
Q_{3.0}(\sqrt{2\lambda_1}, \sqrt{2\lambda_2}) = e^{-x - \lambda_1} \sum_{k=0}^{\infty} \frac{2(k+1)\lambda_1^k}{k!} \sum_{i=0}^{k+1} \frac{x^i}{i!}
\]  
(24)

Hence, it is possible now to arrive to the expression of \(|\Psi(x)|\).

Next, evaluating \(|\Psi(x)|\) at 0, it simplifies to
\[
|\Psi(0)| = \lambda_1 - \lambda_2
\]  
(25)

For the case of identical eigenvalues of the non-centrality matrix, \(\lambda_1 = \lambda_2\), \(F_{\phi_1}(x)\) becomes undefined. Therefore for the special case of \(\lambda_1 = \lambda_2 = \lambda\), the ratio of \(|\Psi(x)|\) will need to be evaluated using l’Hospital rule. The solution stems from taking the derivative with respect to \(\lambda_1\) conditioning on \(\lambda_2\),
\[
\lim_{\lambda_1 \to \lambda_2} \frac{\partial |\Psi(x)|}{\partial \lambda_1} = \lim_{\lambda_1 \to \lambda_2} \frac{\partial |\Psi(x)|}{\partial \lambda_1} = \lim_{\lambda_1 \to \lambda_2} \partial (\lambda_1 - \lambda_2) / \partial \lambda_1
\]  
(26)

The CDF of the minimum eigenvalue of \(2 \times 2\) complex non-central Wishart is shown in (27). As aforementioned regarding the case of \(\lambda_2 = 0\), the expression in (27) remains valid.

IV. SIMPLIFIED CDF OF THE MAXIMUM EIGENVALUE
A. Generic CDF of the maximum eigenvalue \(\phi_1\)

The generic CDF of the maximum eigenvalue \(\phi_1\), from [4], is given by
\[
F_{\phi_1}(x) = |\Xi(x)| / |\Psi(0)|,
\]  
(28)

with \(\Xi(x)\) a \(s \times s\) matrix function with \(x \in (0, \infty)\) with entries given by
\[
\{\Xi(x)\}_{i,j} = \begin{cases} 
2^{(2t-s)/2} (Q_{s+t-2j+1,t-s} (\sqrt{2\lambda_j}, 0) - 
Q_{s+t-2j+1,t-s} (\sqrt{2\lambda_j}, \sqrt{2x}) ) & j = 1, \ldots, L 
\gamma(t + s - i - j + 1, x) & j = L + 1, \ldots, s
\end{cases}
\]  
(29)

\(\gamma(\cdot, \cdot)\) is the lower incomplete gamma function, defined as
\[
\gamma(p, x) = \int_0^x t^{p-1} e^{-t} dt
\]  
(30)
\[
= (p-1)! e^{-x} \sum_{k=p}^{\infty} \frac{x^k}{k!} \quad p = 1, 2, \ldots
\]  
(31)

The CDF of the maximum eigenvalue \(\phi_1\) is still complicated, and for the case of identical eigenvalues of the non-centrality matrix, the expression in \(F_{\phi_1}(x)\) also becomes undefined.

B. Simplified CDF of the maximum eigenvalue \(\phi_1\)

This section follows the same methodology line adopted in III. (29) is expanded assuming \(\lambda_2 \neq 0\), and it is verified that there is no loss of generality. The results in (23) and (24) are used, and when evaluated at zero the equations simplify to,
\[
Q_{1.0} (\sqrt{2\lambda_1}, 0) = 1, \quad Q_{3.0} (\sqrt{2\lambda_1}, 0) = 2(1 + \lambda_1)
\]

Therefore after further simplification the CDF of maximum eigenvalue \(\phi_1\) is obtained in (32). The case corresponding to identical non-centrality matrix eigenvalues is also included in (32).

V. NUMERICAL RESULTS

The analytical result for the CDFs of minimum and maximum eigenvalues of complex non-central Wishart, (27) and (32), are plotted versus the generic results stated in [4] which need to be computed via the use of numerical integration in MATLAB. The CDFs results are also validated statistically.

In Fig. 2, we generate an arbitrary complex channel \(\vec{H}\) satisfying the trace constraint \(\text{Tr}(\vec{H}\vec{H}^H) = M_T M_K = 4\). In this particular case, the eigenvalues of \(\vec{H}\vec{H}^H\) are 3.24, and 0.76. The numerical integration approach fails for \(K = 15\), while the results of this paper remain robust for a larger parameters domain range.

In Fig. 3, \(\vec{H} = [2.0; 0, 0]\), with \(\text{Tr}(\vec{H}\vec{H}^H) = 4\). We notice that
Eigenvectors can instruct a rigorous approach for spatial rate. Furthermore, being able to characterize the distribution of the cross polarized antennas at the transmitter and at the receiver. Well conditioned channel mean. This can be achieved using error probability (PEP) performance, it is best to have a of the minimum eigenvalue, which dominates a pairwise for instance from having vertical polarization of the antennas. Therefore for ill-conditioned channel mean, which can result for instance from having vertical polarization of the antennas at the transmitter and at the receiver in presence of LOS, the value of the maximum eigenvalue can fluctuate over a wide range.

Finally in Fig. 4, we select a rank 2 channel mean $\mathbf{H} = [\sqrt{2}, 0; 0, \sqrt{2}]$. We observe that the CDF behavior is sensitive to the K-factor in both minimum and maximum eigenvalues distributions. Both eigenvalues see reasonable improvement with an increasing K-factor. However for the behavior of the maximum eigenvalue, it is less affected than in the ill-conditioned channel mean case (see Fig. 3). Thus for communication strategy that rely on the characterization of the minimum eigenvalue, which dominates a pairwise error probability (PEP) performance, it is best to have a well conditioned channel mean. This can be achieved using cross polarized antennas at the transmitter and at the receiver. Furthermore, being able to characterize the distribution of the eigenvalues can instruct a rigorous approach for spatial rate adaptation in MIMO communication. This can extend the work in [1], [2] to non-zero mean channels.

VI. CONCLUSION

This paper developed a simplified analytical solution for the CDFs of the minimum and maximum eigenvalues of a $2 \times 2$ noncentral complex Wishart, which is encountered in the MIMO Ricean channel. In addition the paper presents novel results whenever the underlying non-centrality matrix exhibits identical eigenvalues. The expressions proved to be more robust than the expression in [4] where we had to resort to numerical integration. And depending on how the channel mean is conditioned, we illustrated the effect of the K-factor on the behavior of the system performance.

REFERENCES