Robustness of TOA and TDOA Positioning under Sub-Optimal Weighting Conditions

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***Abstract*— The performance robustness of time of arrival (TOA) and time difference of arrival (TDOA) is evaluated when measurement variances are not accurately known or are approximated. If the actual variances are applied, then TOA and TDOA are known to have equal performance. However, if sub-optimal weighting matrices are applied either due to inaccurate knowledge of measurement error statistics or for simplified implementations in practice, there are performance losses proportional to the number of ranging sources.**

***Index Terms*— TOA, TDOA, robustness, weighted least square, sub-optimal weighting, covariance estimation error**

I. Introduction

TIME of arrival (TOA) and time difference of arrival (TDOA) are the two most widely used positioning methods—TDOA has been a favorite for land-based positioning systems [4] and TOA has been for space-based positioning systems [5]. Although both of them rely on essentially same measurements—TOA pseudoranges and TDOA pseudoranges can be converted to each other without ambiguity—and are proven to be equivalent [1]–[3], questions remain on their performance in practical situations where imperfect weights are used to calculate position solutions.

Hence, we investigate their dependency on the applied weighting schemes to assess their robustness in practice. The sub-optimality of the weighting schemes is originated either from inaccurate knowledge of measurement error statistics, or from simplified implementations of positioning algorithms. The resulting performance losses are measured in Monte Carlo simulations.

II. Robustness of TOA and TDOA

For a given pseudorange error covariance matrix $\boldsymbol{\Sigma_v}$, the weighting matrices in the weighted least square (WLS) solutions for TOA and TDOA are respectively

$$\boldsymbol{W}_{\text{TOA/WLS}} = \boldsymbol{\Sigma_v}^{-1/2}, \tag{1}$$

$$\boldsymbol{W}_{\text{TDOA/WLS}} = (\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1/2} \tag{2}$$

where $\boldsymbol{D}$ is a differencing matrix for TDOA. Here TDOA pseudoranges are assumed to be generated by subtracting a TOA pseudorange with a lowest variance (called a reference channel) from the rest of TOA pseudoranges so that we can remove a common bias, usually a receiver clock bias. However, the consequence is the error propagation from the selected reference channel to the rest of channels, creating artificial cross-correlation between TDOA pseudoranges—different from intrinsic cross-correlation between TOA measurements—represented by the off-diagonal terms in $\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T$.

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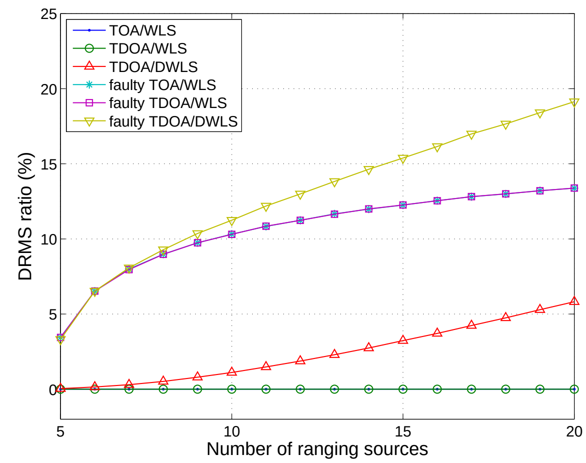

Fig. 1. Performance losses due to inaccurate knowledge of error covariance matrices compared to TOA/WLS based on accurate covariance matrices (n = 5–20, $\sigma^2_{\text{max}}/\sigma^2_{\text{min}}$ = 20 dB, and $\hat{\sigma}^2_i/\sigma^2_i < 5$ dB)

For simpler implementations, these off-diagonal elements in $\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T$ could be ignored and weights can be approximated based only on the diagonal terms of $\boldsymbol{\Sigma_v}$. It is certainly a sub-optimal but often used solution for TDOA, named a diagonal weighted least square (DWLS) method with a diagonal weighting matrix

$$\begin{aligned}\boldsymbol{W}_{\text{TDOA/DWLS}} \\ = diag\left((\sigma_1^2+\sigma_n^2)^{-1/2},\ldots,(\sigma_{n-1}^2+\sigma_n^2)^{-1/2}\right)\end{aligned} \tag{3}$$

where σ_i^2 is the error variance of the ith measurement in a decreasing order such that $\sigma_1^2 = \sigma^2_{\text{max}}$ and $\sigma_n^2 = \sigma^2_{\text{min}}$. Because TDOA/DWLS does not account for the cross-correlation terms (which are dominated by σ_n^2 from the reference channel), its performance critically relies on the quality of the reference measurement, while TDOA/WLS is not affected by the choice of a reference channel since the artificial cross-correlation is treated in the structure of the weighting matrix for TDOA/WLS.

Because all of these three weighting matrices depend on the knowledge of the covariance matrix in (1)–(3), estimation errors on $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{v}}$, an estimate of $\boldsymbol{\Sigma_v}$, generate performance gaps from the ideal cases based on $\boldsymbol{\Sigma_v}$. This unintentional sub-optimality, due to the inaccuracy or perturbation in the covariance matrix, is measured in Monte Carlo simulations, as

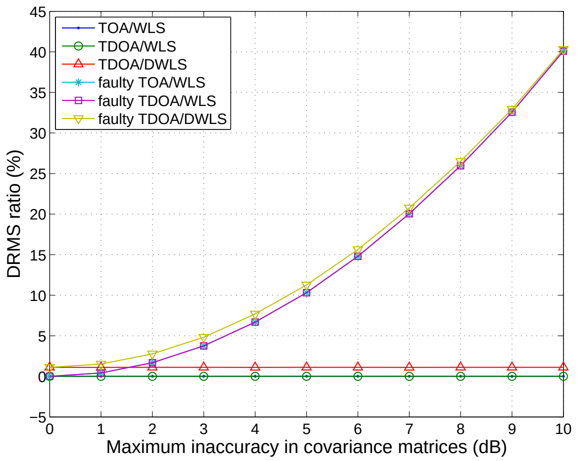

Fig. 2. Inaccuracy on known covariance matrices versus performance losses compared to TOA/WLS based on accurate covariance matrices (n = 10, $\sigma^2_{\max}/\sigma^2_{\min}$ =20 dB, and $\hat{\sigma}^2_i/\sigma^2_i$ < 0–10 dB)

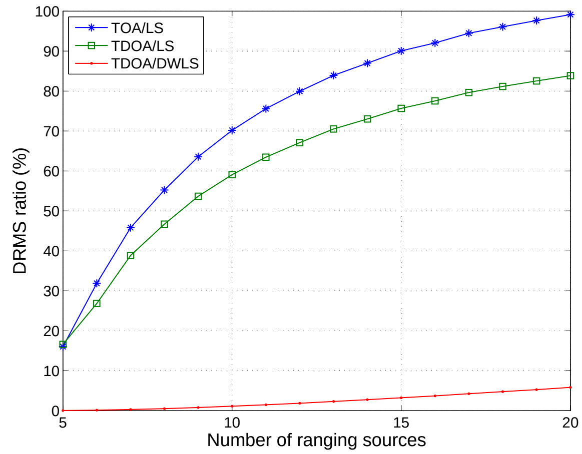

Fig. 3. Performance losses of sub-optimal weighting schemes compared to TOA/WLS (n = 5–20 and $\sigma^2_{\max}/\sigma^2_{\min}$ = 20 dB)

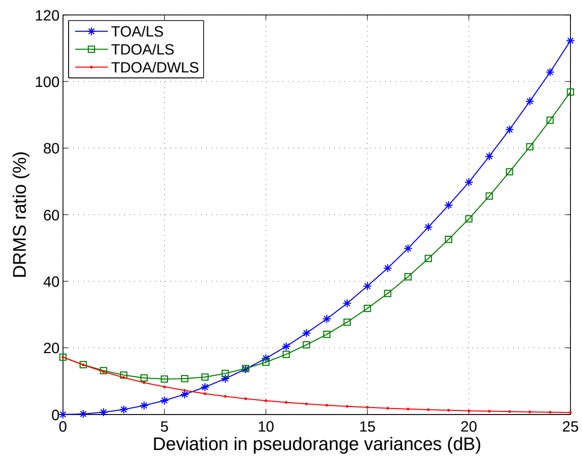

Fig. 4. Deviation in pseudorange variances versus performance losses of sub-optimal weighting schemes compared to TOA/WLS (n = 10 and $\sigma^2_{\max}/\sigma^2_{\min}$ = 0–25 dB)

an assessment of the robustness of TOA and TDOA position solutions. In the simulations, ranging sources (n = 5–20) are randomly located on the surface of a half sphere centered by a user and their range error variances σ_i^2 are randomly generated in the log scale between 0–20 dB while we keep the ratio of $\sigma^2_{\max}/\sigma^2_{\min}$ = 20 dB and assume no intrinsic cross-correlation ($\sigma_{ij} = 0$ for $i \neq j$). The generated error covariance matrices are delivered to the user with the inserted inaccuracy up to 5 dB and then horizontal position errors are evaluated in 10^5 trials.

The performance losses are illustrated in Fig. 1 where each point represents the increase of horizontal positioning errors in DRMS (distance root mean squared) compared to those of TOA/WLS based on $\boldsymbol{\Sigma_v}$, the optimal solution. There are 6 results in comparison: TOA/WLS, TDOA/WLS, and TDOA/DWLS based on the true covariance $\boldsymbol{\Sigma_v}$, noted as the 'healthy' cases; and their correspondents based on the estimated covariance $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{v}}$, noted as the 'faulty' cases. Among them, the equivalence of TOA/WLS and TDOA/WLS is clearly displayed by the completely overlapped curves, for both $\boldsymbol{\Sigma_v}$ and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{v}}$, in agreement with the theoretical proofs [1]–[3]. Within the healthy cases, though the loss in DRMS ratio is proportional to the number of ranging sources, TDOA/DWLS manages a low level of performance losses less than 6%.

When the faulty cases are compared to their healthy correspondents, their gaps start from 4% (n = 5) but gradually build up to 13% (n = 20) for the given 5 dB inaccuracy. As the inaccuracy increases, these losses rapidly develop into substantial losses. With the sweep in the level of the covariance inaccuracy, the robustness of the positioning methods disappears in Fig. 2 where neither of the WLS and DWLS methods remain reliable. Their performance losses reach 40% at 10 dB inaccuracy from 5% at 3 dB inaccuracy. Therefore, for the example of applications with 10% loss tolerance, the positioning methods can be considered to be robust only to covarinace inaccuracy lower than approximately 5 dB, on which 10–12% losses are observed.

From Fig. 1 and Fig. 2, we see that the implementational sub-optimality of DWLS is dwarfed by the sub-optimality due to the inaccurate knowledge of covariance matrices. As a further investigation into this sub-optimality due to simplified implementations, the diagonal weighting method is tested along with the least square methods (LS) where $\boldsymbol{W}_{\text{LS}} = \boldsymbol{I}$. As n increases, TOA/LS and TDOA/LS suffer losses of 70% and 59% for $n = 10$, and 99% and 84% for $n = 20$ respectively, certainly unacceptable to the most applications, while TDOA/DWLS is very close to the optimality with a less than 6% loss (See Fig. 3). But this low loss is subject to changes in the max-to-min variance ratio, $\sigma^2_{\max}/\sigma^2_{\min}$. In Fig. 4 where the variance ratio changes from 0 to 25 dB for n = 10, the loss of TDOA/DWLS varies from 1% to 17%, worst at 0 dB deviation among measurements where the reference channel is no better than the others (i.e. a large σ_n^2). However, for 10–20 dB nominal deviations (meaning a small σ_n^2), the average loss by TDOA/DWLS can be restricted to 5% in a close range from the optimal solutions, while the LS methods remain unreliable.

III. Conclusion

The robustness of TOA and TDOA positioning methods in practical situations has been examined in conjunction with the sub-optimality of applied weighting schemes. First, we have shown that these methods become less reliable when the inaccuracy in the knowledge of measurement covariances exceeds approximately 5 dB, showing rapidly increasing performance losses; only below this threshold TOA and TDOA positioning methods can be considered to be robust. Second, the diagonal weighted least square method (DWLS)—often used for TDOA positioning—has shown a similar level of robustness with the optimal weighting methods (WLS) against perturbations in the covariance matrices, proving itself a valid alternative for low-cost receivers.

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