Reliable Carrier Phase Positioning

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Integrity Requirements

- CAT III landing – Integrity

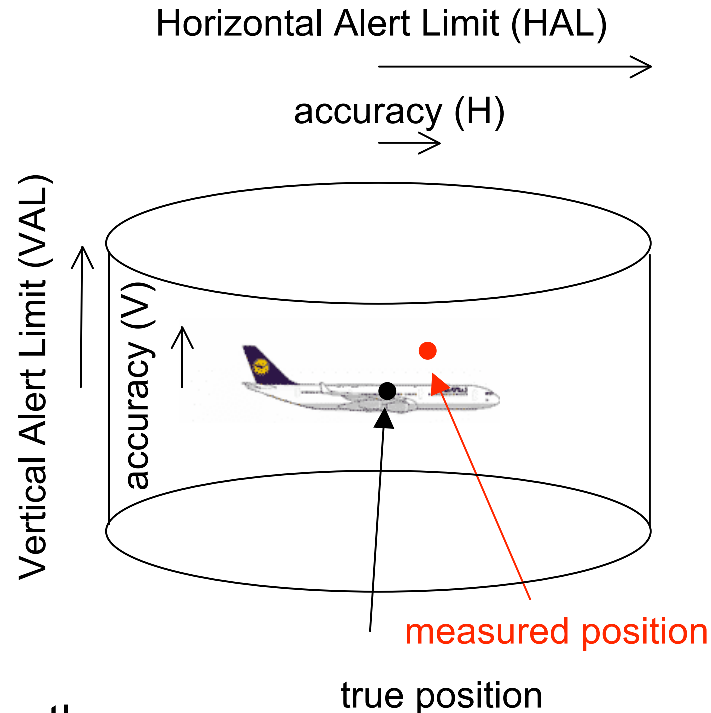
FAA, ICAO	VAL 5.3m … 10 m
EuroCAE ED-144	VAL >2.6 m

- Ground Guidance

Schuster et al. 07	HAL 1.4 m

- Harbor Maritime Navigation

IMO	HAL 1 m

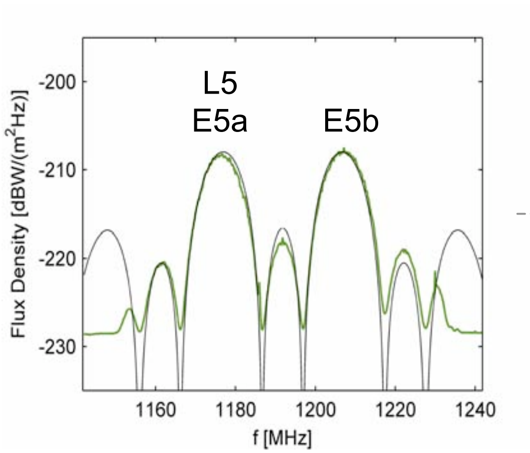


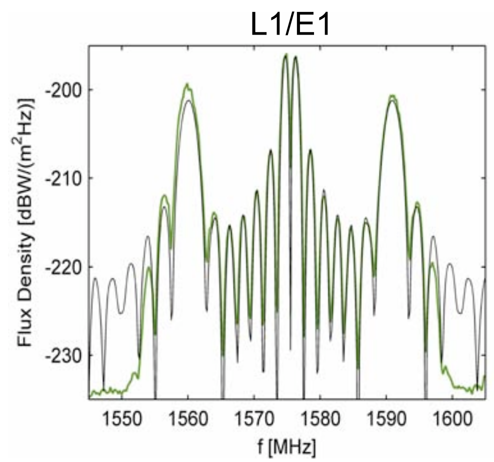
- Inflation factor for the statistics and multipath

  Statistics ca. x 6 (p_{HMI}~10^{-9})

  Multipath ca. x 2-3

Signals and Cramer-Rao Bounds





PLL Phase Noise
typ. 1 mm

$$\sigma_\phi = \sqrt{\frac{B_\phi}{\frac{\mathcal{P}_c}{\mathcal{N}_0}}\left(1+\frac{1}{2\frac{\mathcal{E}_i}{\mathcal{N}_0}}\right)}$$

Cramer – Rao Bound, Galileo Signals

	Modulation	Bandwidth (MHz)	CRB (cm)
E1	BOC(1,1)	4	20
E5a	BPSK(10)	24	5
E5b	BPSK(10)	24	5
E5	BOC(15,10)	51	1

Multipath (-14.2 dB for aeronautical), inflation: x2 (Phase), x3 (Code)

Positioning - Basic Equations

- Code measurements, k satellites, r frequencies

$$P = \begin{pmatrix} \rho_1^1 \\ \vdots \\ \rho_r^k \end{pmatrix} = G\xi + \begin{pmatrix} \eta_1^1 \\ \vdots \\ \eta_r^k \end{pmatrix}$$

noise

- Phase measurements

$$\Phi = \begin{pmatrix} \lambda_1 \varphi_1^1 \\ \vdots \\ \lambda_r \varphi_r^k \end{pmatrix} = GJ\xi + A_\Phi \begin{pmatrix} N_1^1 \\ \vdots \\ N_r^k \end{pmatrix} + \begin{pmatrix} \epsilon_1^1 \\ \vdots \\ \epsilon_r^k \end{pmatrix}$$

integer ambiguity

- Joint least-square problem $\Psi = (P, \Phi)^T$

$$\min_{\xi \in \mathbb{R}^m, N \in \mathbb{Z}^k} \|\Psi - H\xi - AN\|_{Q_\Psi^{-1}}$$

position, clock offset, troposphere, ionosphere

$$\xi = \begin{pmatrix} \vec{x} \\ c\delta \\ T_z \\ I^1 \\ \vdots \\ I^k \end{pmatrix}$$

$$J\xi = \begin{pmatrix} \vec{x} \\ c\delta \\ T_z \\ -\mathbf{I} \end{pmatrix}$$

$$H = \begin{pmatrix} G \\ GJ \end{pmatrix}$$

$$A = \begin{pmatrix} 0 \\ A_\Phi \end{pmatrix}$$

Least squares - Problem Separation

$$\min_{\xi\in\mathbb{R}^\kappa, N\in\mathbb{Z}^k} \|\Psi - H\xi - AN\|_{Q_\Psi^{-1}} =$$

$$\min_{\xi\in\mathbb{R}^\kappa, N\in\mathbb{Z}^k} \left(\underbrace{\|P_H(\Psi - AN) - H\xi\|^2_{Q_\Psi^{-1}}}_{\text{position, clock offset, troposphere, ionosphere}} + \underbrace{\|\hat{N} - N\|^2_{Q_{\hat{N}}^{-1}}}_{\text{ambiguity}} + \underbrace{\|P_A^\perp P_H^\perp \Psi\|^2_{Q_\Psi^{-1}}}_{\text{residual error}} \right),$$

RAIM

Step 1: estimate the integer ambiguity

$$\min_{N\in\mathbb{Z}^k} \|\hat{N} - N\|^2_{Q_{\hat{N}}^{-1}}$$

with

$$Q_{\hat{N}}^{-1} = A^T Q_\Psi^{-1} P_H^\perp A$$

$$P_H^\perp = \mathbb{1} - P_H$$

$$P_H = HS = H(H^T Q_\Psi^{-1} H)^{-1} H^T Q_\Psi^{-1}$$

$\hat{N} =$ float solution of the equation:

$$P_H^\perp \Psi = (P_H^\perp A)N + P_H^\perp \eta$$

Measure of Performance

typical error term in position domain

decomposition of phase component: $\Psi = \psi + AN$

$$
\begin{aligned}
&p(|S_x(\Psi - \hat{\Psi})| > e_x) \\
&= \sum_{\Delta N \in \mathbb{Z}^k} p\Big(|S_x(\psi - \hat{\psi} + A(N - \hat{N}))| > e_x \Big| N - \hat{N} = \Delta N\Big) p(\Delta N) \\
&\leq \underbrace{p\Big(|S_x(\psi - \hat{\psi})| > e_x \Big| \hat{N} = N\Big)}_{\text{similar to above without ambiguity}} \underbrace{p(\hat{N} = N)}_{1 - p_{\mathrm{wf}}} + \underbrace{\sum_{\Delta N \in \mathbb{Z}^k} p(\Delta N)}_{p_{\mathrm{wf}}}
\end{aligned}
$$

similar to above
without ambiguity

$1 - p_{\mathrm{wf}}$

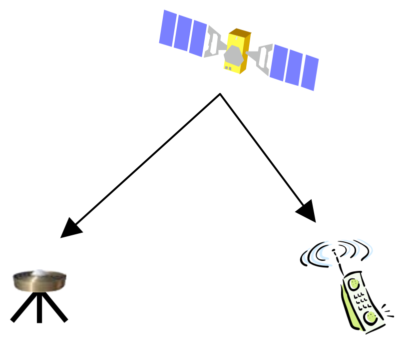
p_{wf}

probability of
wrong fixing

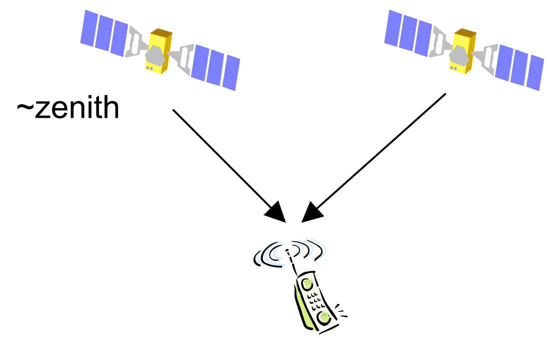
Combination of Measurements

Undifferenced (may be augmented)

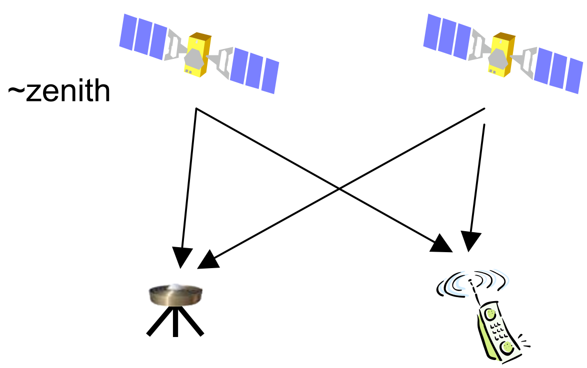
Single differences (GBAS, LAAS)



Single differences (2nd option)



Double differences (geodesy)



single point of failure

Sources of Correlations

- Noise statistics (phase, ditto for code)

$$Q_\Phi = \sigma_\varphi^2 \mathbb{1}$$

undifferenced measurements

$$Q_\Phi = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 2 \end{pmatrix} \sigma_\varphi^2$$

satellite differences
e.g. double differences

$Q_\Psi = \dots$ depends on specific definition of Ψ

- Problem separation (thus correlated even in the undifferenced case)

$$Q_{\hat{N}}^{-1} = A^T Q_\Psi^{-1} P_H^\perp A$$

LAMBDA – Ambiguity Decorrelation

[Teunissen, 93]

<u>Iterative algorithm:</u>
determine the permutation P_n such that the maximum element in D_n is minimum

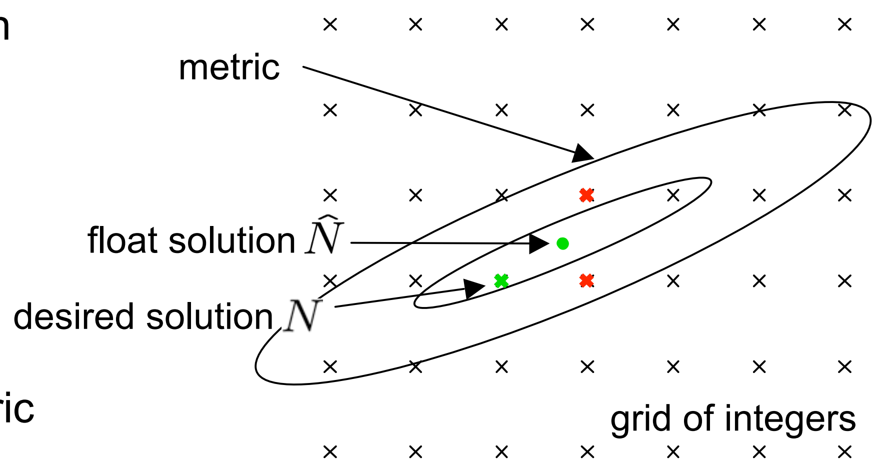
$$D_{n-1} = L_n^T P_n^T Q_{n-1} P_n L_n$$

$$L_n \in \mathbb{R}^{kr} \times \mathbb{R}^{kr}$$

Decorrelate ("diagonalize") the metric using a grid automorphism

$$Z_n \in \mathbb{Z}^{kr} \times \mathbb{Z}^{kr}, \quad \det(Z_n) = 1$$

$$Q_n^{-1} = (Z_n^{-1})^T P_n^T Q_{n-1}^{-1} P_n Z_n^{-1}$$

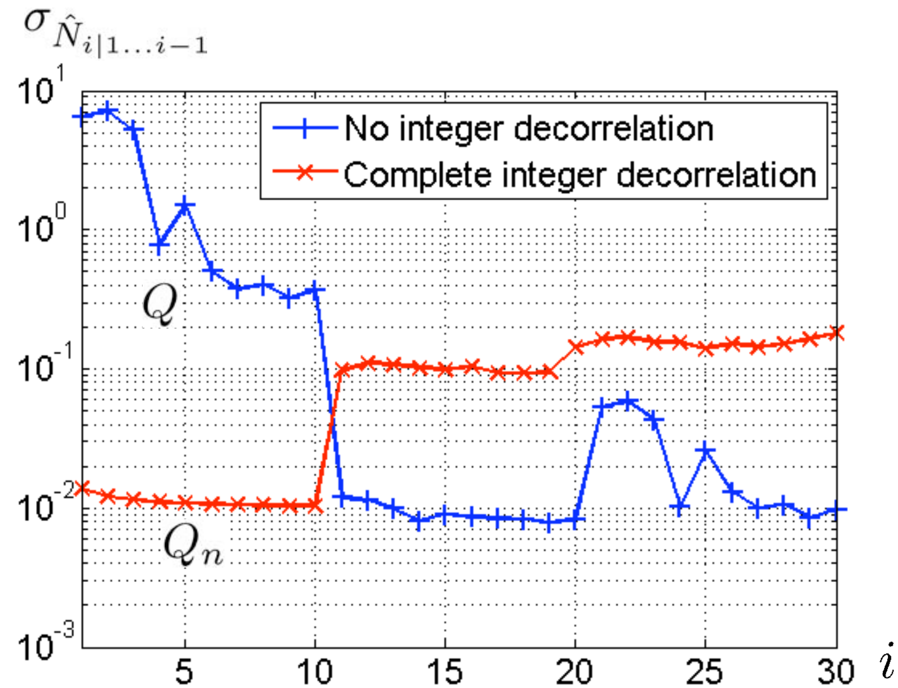


Spectrum of Variances of Conditional Ambiguities

Conditional covariances

[Teunissen, 93]

one cycle →



$$L^T Q L =$$

$$\begin{pmatrix} \sigma^2_{\hat{N}_1} & & \cdots & 0 \\ & \sigma^2_{\hat{N}_{2|1}} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma^2_{\hat{N}_{k|1\ldots k-1}} \end{pmatrix}$$

3 carrier + 3 codes, 10 satellites
Galileo constellation
DD, iono, tropo
5 seconds =
5 measurements

$$Q_n^{-1} = (Z_n^{-1})^T P_n^T \ldots (Z_1^{-1})^T P_1^T Q^{-1} P_1 Z_1^{-1} \ldots P_n Z_n^{-1}.$$

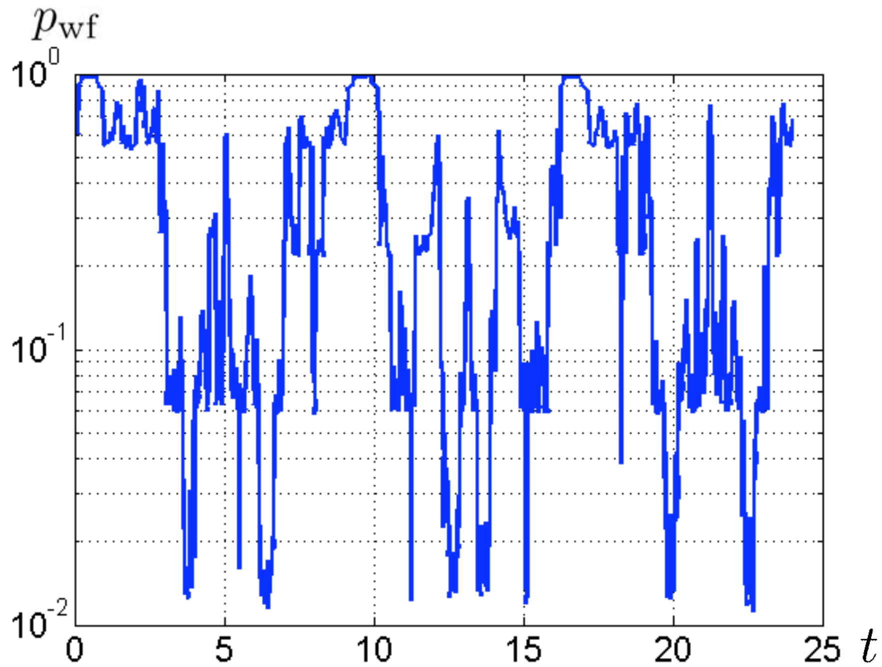
Probability of Wrong Fixing

- Probability of a wrong estimation of the ambiguity
  closed form in special cases (e.g. bootstraping ≡ upper bound)

$$p_{\mathrm{wf}} = 1 - \prod_{i=1}^{n} 2\left[\Phi\left(\frac{1}{2\sigma_{\hat{N}_{i|1\ldots i-1}}}\right) - \frac{1}{2}\right]$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dt\, e^{-t^2/2}$$

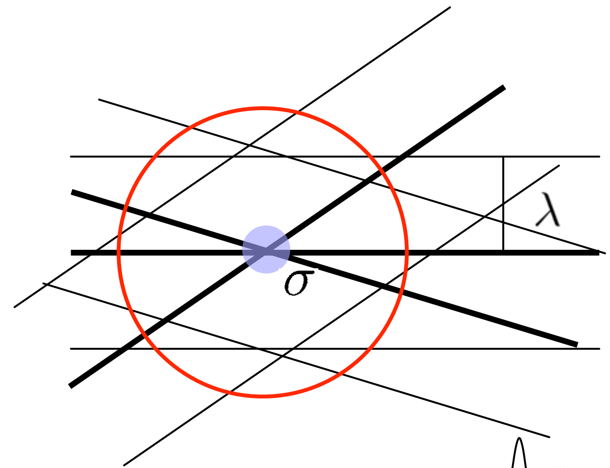
[Teunissen 98]



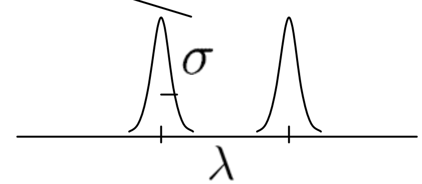
- this was a major obstacle to the use of carrier phase positioning

Reducing the Probability of Wrong Fixing

- Use of linear combinations
  - geometry preserving
  - low/no ionosphere
  - large wavelength
  - low noise



large value of $\lambda/2\sigma$
necessary but not sufficient (DOP)



- Use of smoothed code measurements to further characterize the solution (additional equations)

- Additional measurement epochs provide improved statistics
  - Hatch smoothing, Kalman Filtering

Linear Carrier Combinations

each satellite:

$$\Phi = \sum_i \alpha_i \lambda_i \varphi_i = \underbrace{\sum_i \alpha_i K}_{1} \xi - \underbrace{\sum_i \alpha_i q_i}_{\ll 1} I + \underbrace{\sum_i \alpha_i \lambda_i N_i}_{\lambda N} + \underbrace{\sum_i \alpha_i \epsilon_i}_{}$$

1	$\ll 1$	λN	
geom. preserving pos./clock/tropo.	iono. free	sufficient	minimum noise

$$\frac{\alpha_i \lambda_i}{\lambda} = j_i \in \mathbb{Z}$$

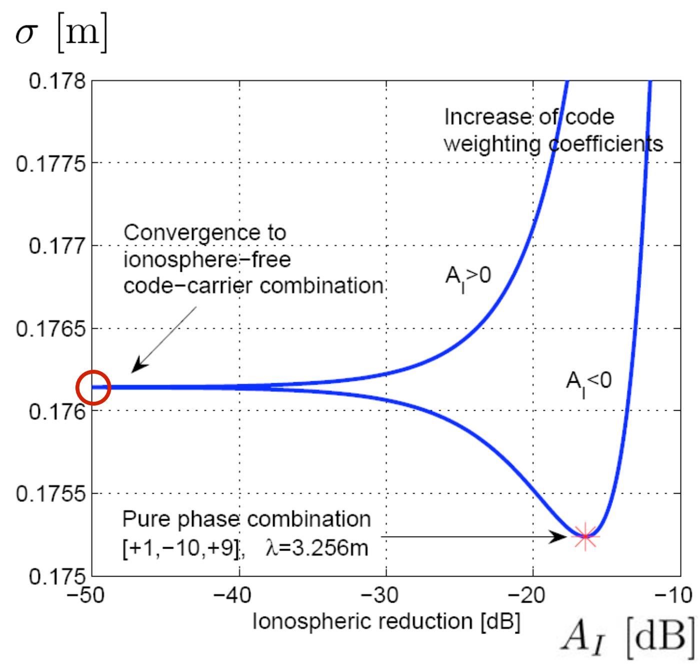
λ[m]	j_{L1}	j_{E5a}	j_{E5b}	$A_{n,\lambda}$[dB]	A_I[dB]
0.0077	57	-17	-26	-2.09	-38
3.256	1	-10	9	10.10	-16.44

→ widelane combinations $(\lambda \geq \max \lambda_i)$ always inflate noise,
no better iono suppression found for widelane combinations

Linear Code and Carrier Combinations

- The addition of a small code component allows to eliminate the ionosphere

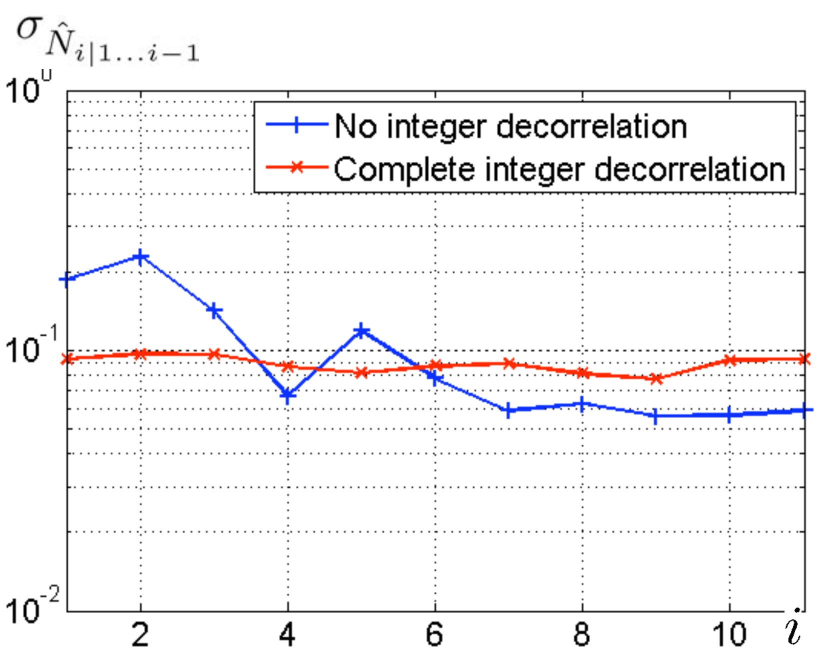
$$\Psi = \sum_i (\alpha_i \lambda_i \varphi_i + a_i \rho_i)$$



λ[m]	j_{L1}	j_{E5a}	j_{E5b}	a_{E5a}	a_{E5B}	σ[m]	R	A_I[dB]
3.214	1	-10	9	0.006	0.007	0.23	7.05	$-\infty$
3.206	1	-2	1	-1.272	-1.339	0.10	15.85	$-\infty$

Linear Code and Carrier Combination and Auxiliary Code-only Combination

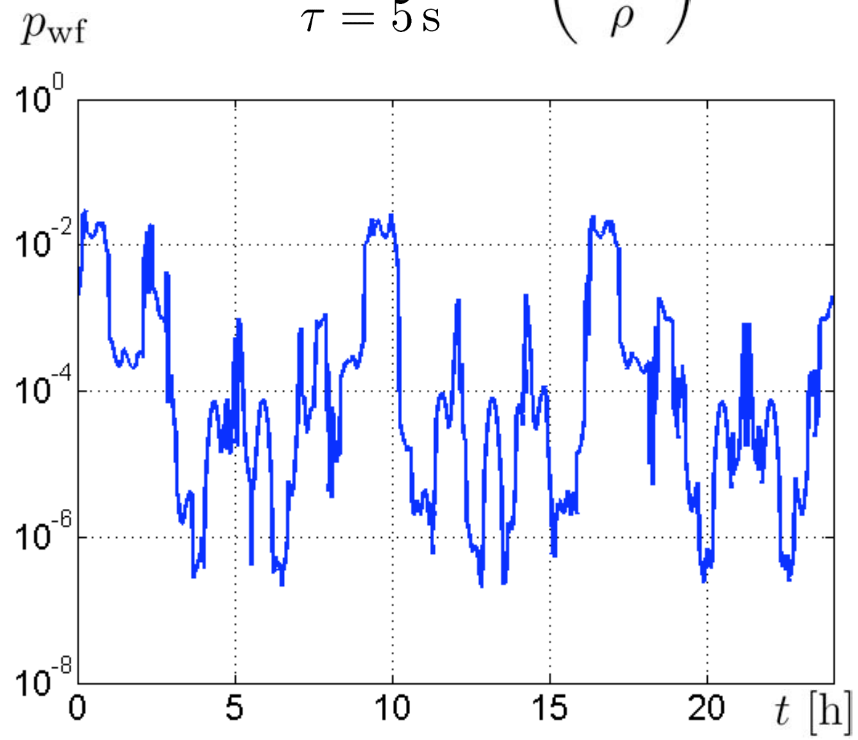
One linear combination with a large wavelength and one code-only combination $\lambda \gg \sigma_\rho$ flattens the spectrum of conditional variances from the start



$\sigma = 23.4\,\mathrm{cm}$

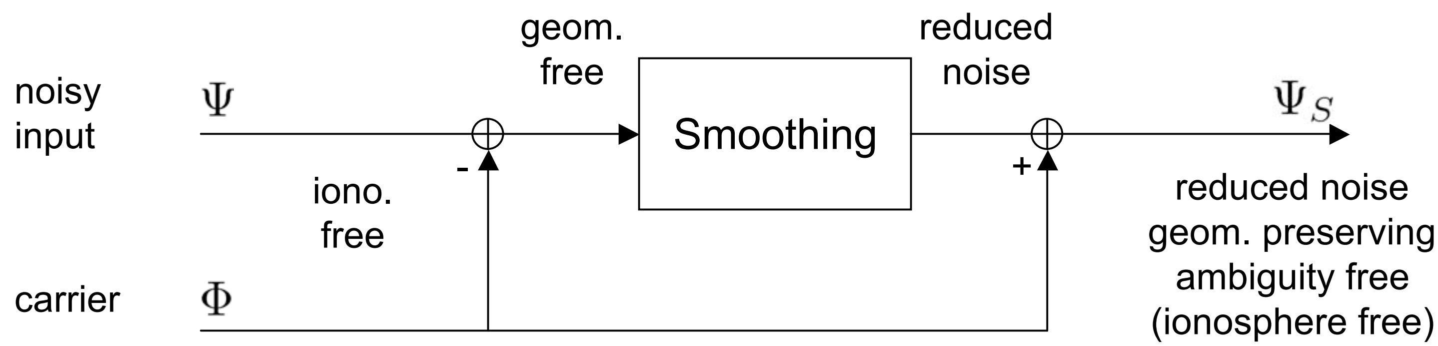
joint solution $\begin{pmatrix} \Psi \\ \rho \end{pmatrix}$

$\tau = 5\,\mathrm{s}$



Non-Recursive Hatch Filter Variant

marginal change in geometry, multiple measurements contribute to the statistics



[Hwang, Mc. Graw, Bader, J. Nav. 1999]

geometry preserving, iono-free, minimum noise code combination

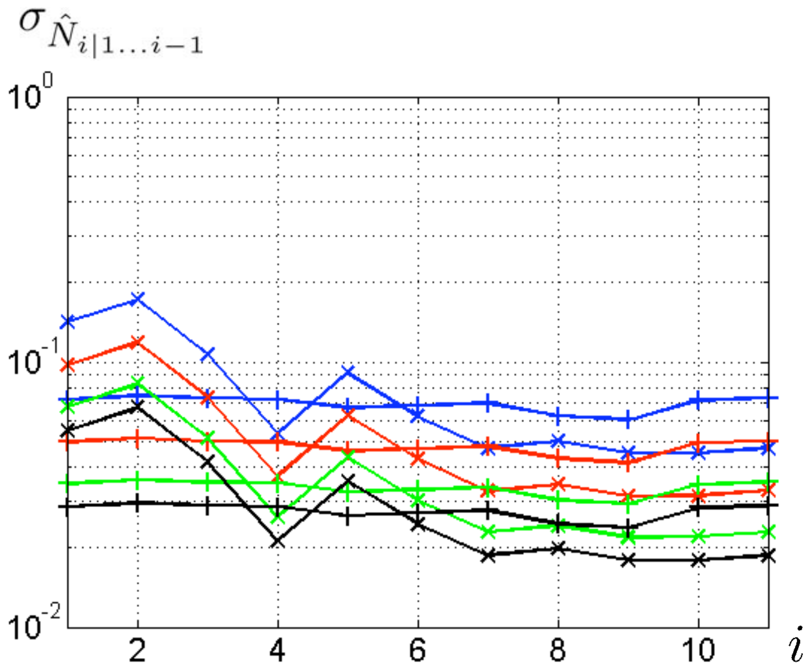
$$\rho = \sum_i a_i \rho_i, \qquad a_{\mathrm{L1}} = 2.09, \quad a_{\mathrm{E5a}} = -2.58, \quad a_{\mathrm{E5b}} = 1.5$$

smoothed by a corresponding carrier combination

$$\Phi = \sum_i \alpha_i \lambda_i \varphi_i, \qquad \alpha_{\mathrm{L1}} = 2.31, \quad \alpha_{\mathrm{E5a}} = -0.84, \quad \alpha_{\mathrm{E5b}} = -0.48$$

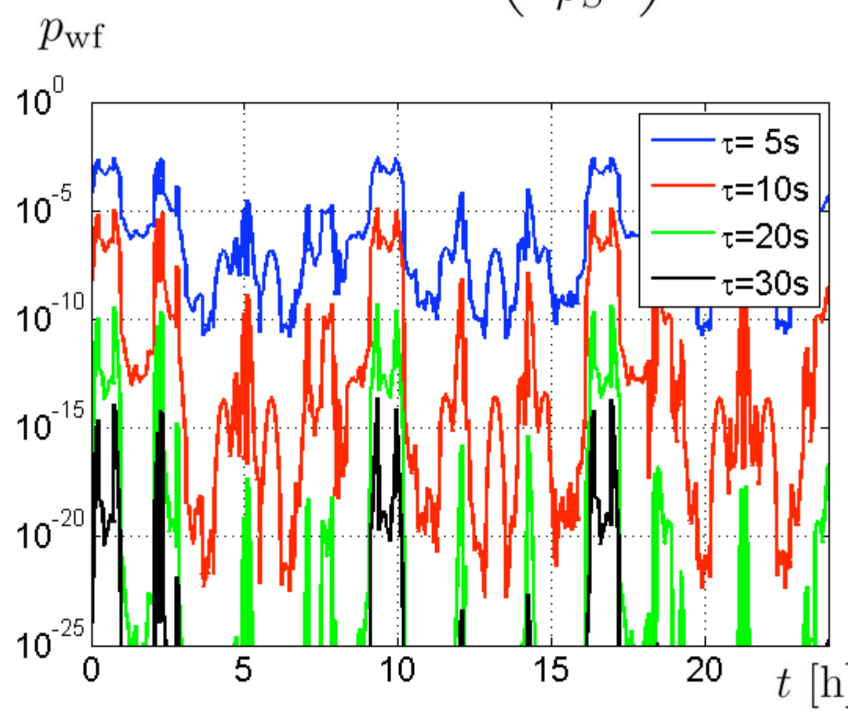
Performance – Three frequencies

Additional measurements do typically not contribute to the geometry (slow change) but can improve the statistics -> Hatch filtering

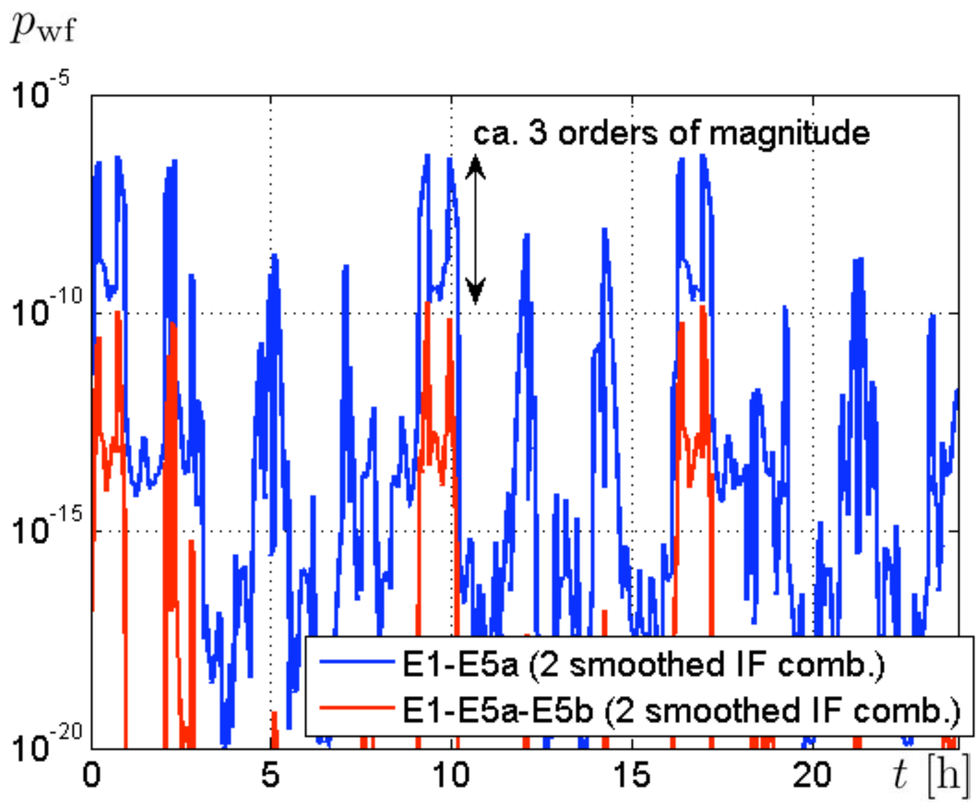


τ [s]	no/5	5	10
σ [cm]	24.3	8.7	6.0

joint solution $\begin{pmatrix} \Psi_S \\ \rho_S \end{pmatrix}$



Performance – Two Frequencies



$\tau = 20\,\mathrm{s}$

increased smoothing interval duration needed

P_{wf} directly from the probability density function

flat spectrum from the start, why decorrelate?
without troposphere (differential and model)

$$\underbrace{\begin{pmatrix} \Psi \\ \rho \end{pmatrix} = \begin{pmatrix} H & A \\ H & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \xi \\ N \end{pmatrix}}_{\Xi} + \underbrace{\begin{pmatrix} \eta_\Psi \\ \eta_\rho \end{pmatrix}}_{\mathrm{cov} = Q}$$

$$p(\Delta\Xi) = \frac{1}{\sqrt{(2\pi)^{k+4} \det(M^T Q^{-1} M)^{-1}}} \exp\left(-\frac{1}{2} \underbrace{\|\Delta\Xi\|^2_{M^T Q^{-1} M}}_{\geq r(q-p^2)\lambda^2 \|\Delta N\|^2}\right)$$

with $r, p, q = f(Q)$

geometry independent bound!

Some Numerical Results

j_{L1}	j_{E5a}	j_{E5b}	τ	Λ [m]	noise $\sqrt{\zeta}$ [cm]	$\Lambda/(2\sqrt{\zeta})$	bias [cm]
1	-5	4	20	3.15	3.8	41	1.5
1	-9	8	100	3.18	2.2	71	1.2

				p_{wf}		
j_{L1}	j_{E5a}	j_{E5b}	τ	no bias	rec. bias	sat. bias
1	-5	4	20	$8 \cdot 10^{-29}$	$6 \cdot 10^{-26}$	$8 \cdot 10^{-20}$
1	-9	8	100	$1 \cdot 10^{-54}$	$1 \cdot 10^{-44}$	$2 \cdot 10^{-40}$

C.G. and P. Henkel, “Reduced-Noise Ionosphere-Free Carrier Smoothed Code”
Trans. AES, accepted Jul. 2008

Conclusions

- Three elements contribute to the control of the probability of wrong fixing
  - linear code carrier combinations with a large wavelength
  - the use of an auxiliary code only linear combinations
  - smoothing the result
- Troposphere and Ionosphere (1st order) addressed
- Sensitivity to biases – number of initial results
- Results are bounds – can potentially be sharpened

- Promising approach to fix carrier phase ambiguities for Safety of Life applications, and Precise Point Positioning