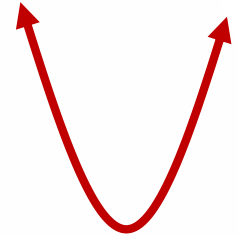# Introduction to Constrained Optimization

- Overview
- Graphical Optimization

# Constrained Optimization

In the previous unit, most of the functions we examined were unconstrained, meaning they either had no boundaries, or the boundaries were soft.
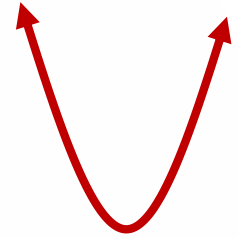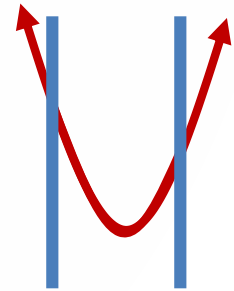
# Constrained Optimization

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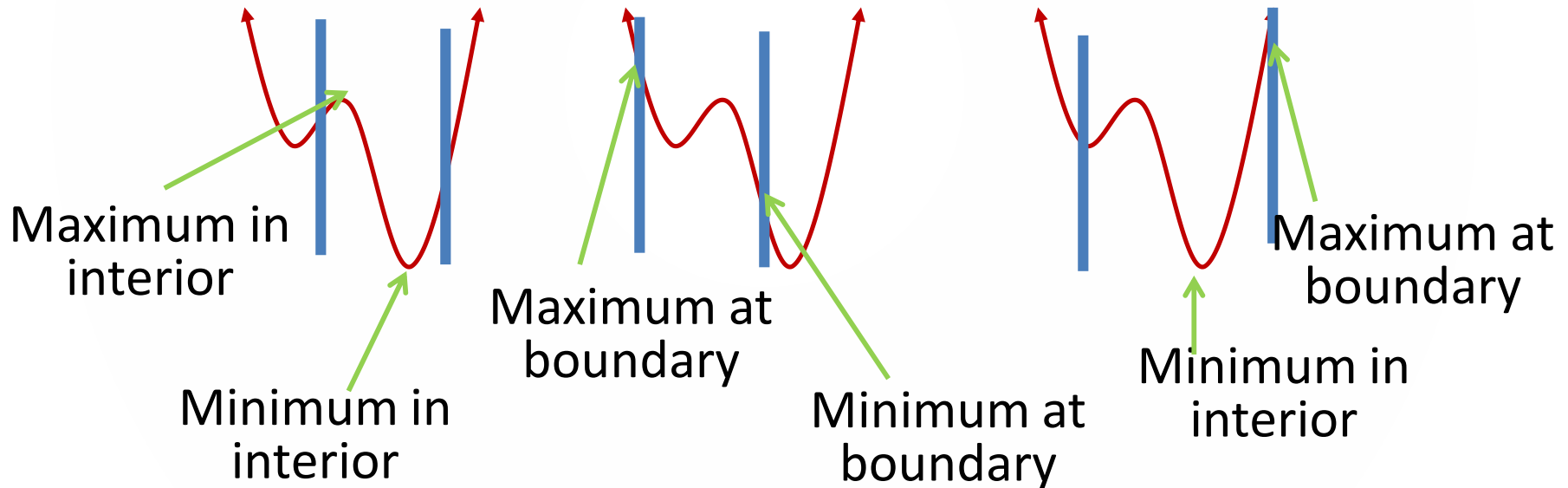
In this unit, we will be examining situations that involve *constraints*.

A constraint is a hard limit placed on the value of a variable, which prevents us from going forever in certain directions.

# Constrained Optimization

With nonlinear functions, the optimum values can either occur at the boundaries or between them.

Maximum in interior

Minimum in interior

Maximum at boundary

Minimum at boundary

Minimum in interior

Maximum at boundary

# Constrained Optimization

With linear functions, the optimum values can only occur at the boundaries.

Minimum at boundary

Maximum at boundary
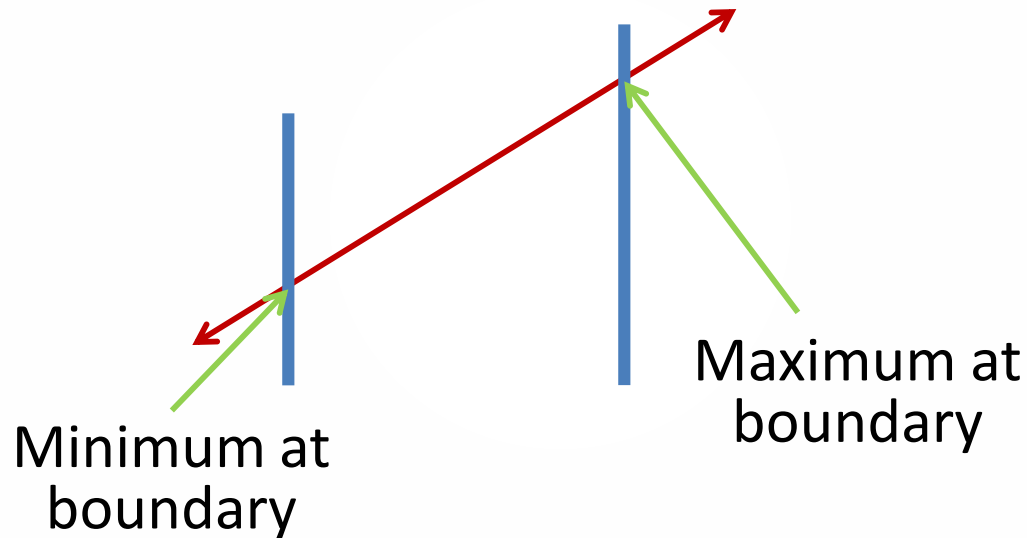
In this unit, we will mostly be working with linear functions.

# Examples of Constraints

If you are attempting to maximize the objective function, typical constraints might involve time, money, and resources. The amounts of these things are limited, and these limits also place limits on the best possible value of the objective function.

If you are attempting to minimize, the constraints are more particular to the situation.

# Writing Constraints

Recall from Algebra that if a single gizmo costs $4, then two gizmos cost $8, five gizmos cost $20, and g gizmos cost 4g.

If you buy g gizmos at $4 and s sprockets at $2, then your total cost will be 4g + 2s.

If you only have $70 to spend at the gizmo-and-sprocket store, then your total cost must be

$$4g + 2s \le 70.$$

Linear inequality

Boundary

# Writing Constraints

Recall from Algebra that if a single gizmo costs $4, then two gizmos cost $8, five gizmos cost $20, and g gizmos cost 4g.

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$$4g + 2s \leq 70.$$

Linear inequality  +  Boundary

=  Linear Constraint

# Practice Problem 1

1. Write constraints for each of the following:

a) A batch of cookies requires 3 cups of flour, and a cake requires 4. Write a constraint limiting the amount of cookies and cakes that can be made with 24 cups of flour.

b) Box type 1 can hold 20 books and box type 2 can hold 12. Write a constraint for the number of boxes needed in order to box up 100 books.

c) If it takes you 4 minutes to bike a mile, 9 minutes to run a mile and 14 minutes to walk a mile, write a constraint that limits how many miles of each type of exercise you can get in a 45-minute lunch break.
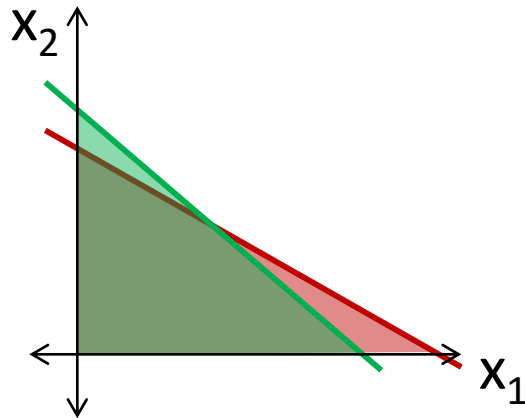
# Some Algebra Review

Suppose you have two constraints as follows:
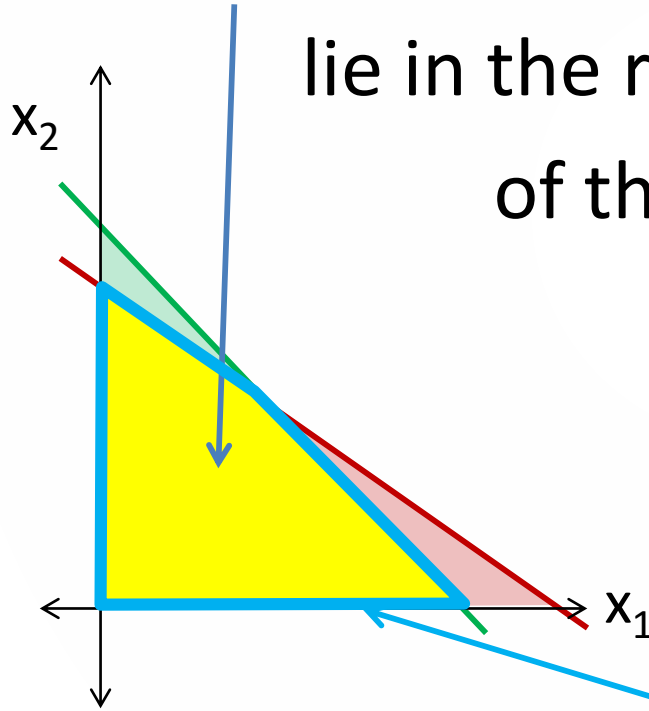
$$2x_1 + 3x_2 \leq 34$$

$$3x_1 + 5x_2 \leq 54$$

Also assume that $x_1$ and $x_2$ are objects and must be $\geq 0$.

You can graph these inequalities...

# Graphical Optimization

The overlap of these graphs is known as the *feasible region*. A solution to the problem must lie in the region in order to obey both of the constraints.
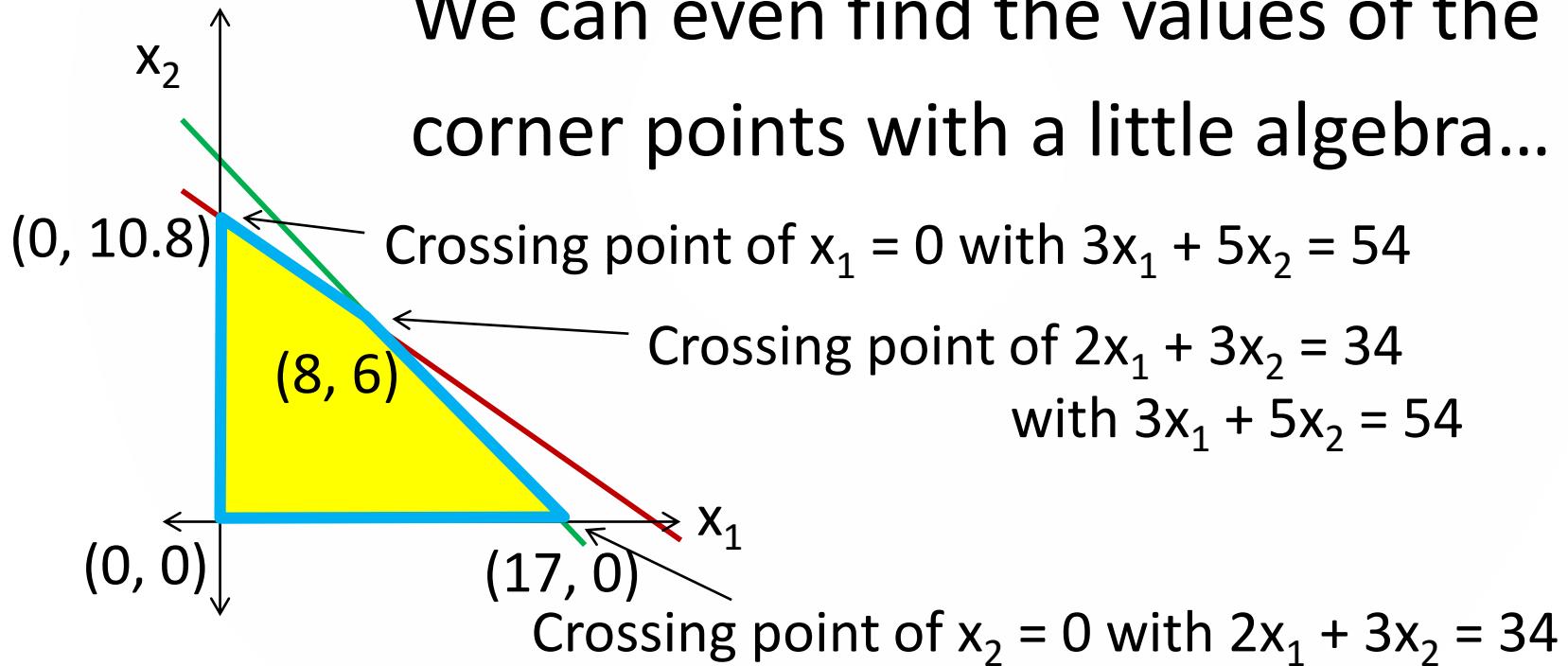
And, because the constraints are linear, the maximum and minimum must lie on the boundary.

# Graphical Optimization

In fact, it is most likely that the optimum occurs at one of the corner points.

We can even find the values of the corner points with a little algebra…

$x_2$

(0, 10.8)   Crossing point of $x_1 = 0$ with $3x_1 + 5x_2 = 54$

Crossing point of $2x_1 + 3x_2 = 34$ with $3x_1 + 5x_2 = 54$

(8, 6)

$x_1$

(0, 0)

(17, 0)

Crossing point of $x_2 = 0$ with $2x_1 + 3x_2 = 34$

# Practice Problem 2

Suppose a company hires both experienced and inexperienced workers.

Experienced workers are paid $15/hour and inexperienced workers are paid $10/hour. The company can spend $1200/hour on labor.

Experienced workers require an average of 1 minute an hour of contact with a supervisor; inexperienced workers require 2. There are two supervisors who can provide 120 minutes in an hour.

2. Convert both of these into inequality constraints. Graph them, find the feasible region, and find all four corner points.

# Enter the Objective Function

After you have the feasible region and the corner points, it's time to consider the objective function.

The simplest way to optimize is to find the value of the objective function by plugging in each point, then choose the best one.

$x_2$

(0, 10.8)

(8, 6)

(0, 0)

(17, 0)

$x_1$
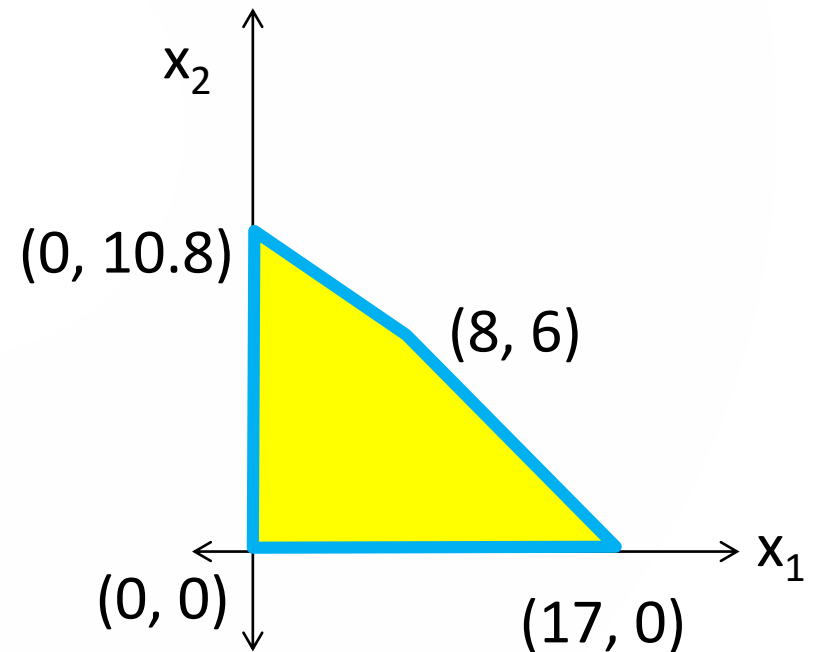
# Enter the Objective Function

Suppose your objective was to maximize

$f = 10x_1 + 8.4x_2$. If you plug in the $x_1$ and $x_2$ at each corner, you would get…

…a maximum of 170 when $x_1 = 17$ and $x_2 = 0$.

$x_2$

(0, 10.8)

90.72

(8, 6)

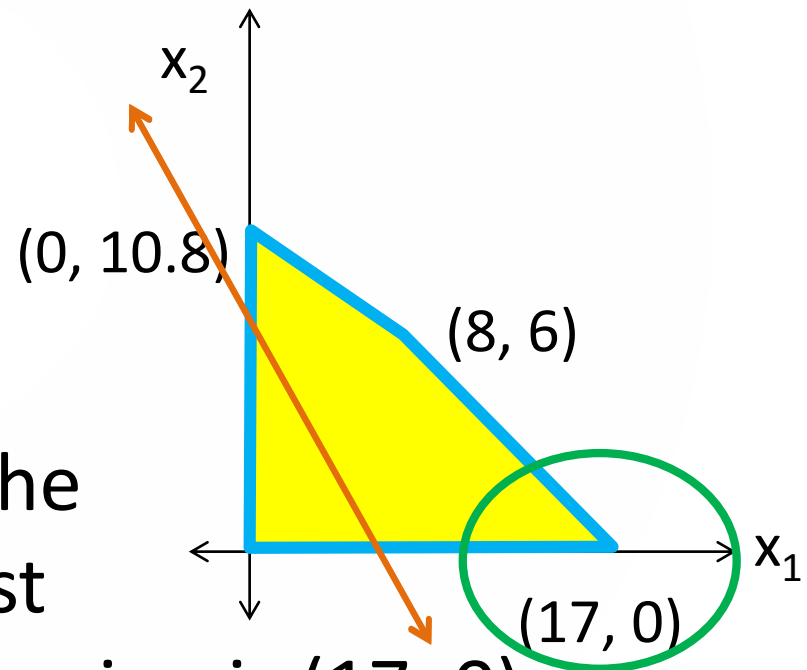130.4

(0, 0)

(17, 0)

170

$x_1$

0
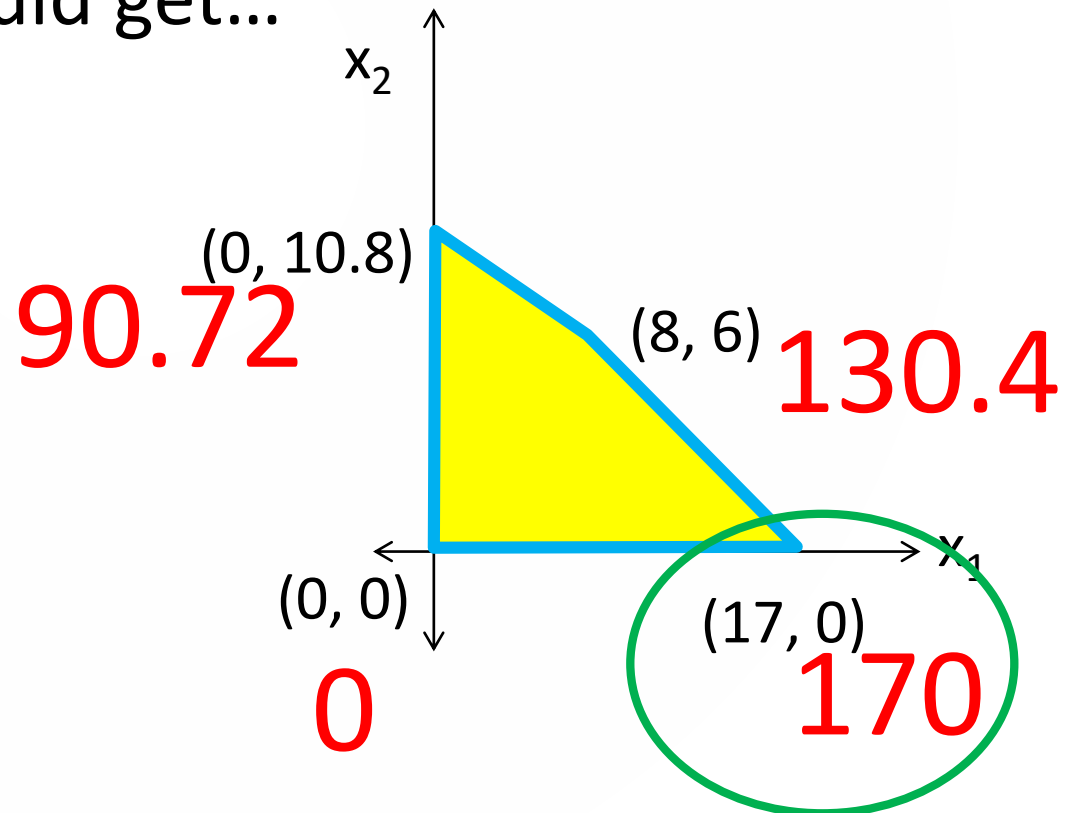
# Enter the Objective Function

A way to find the optimum without plugging in points is to sketch the slope of the objective function on the graph.

$f = 10x_1 + 8.4x_2$ has slope $x_2/x_1 = -10/8.4$, a little steeper than -1.

$x_2$

(0, 10.8)

(8, 6)

$x_1$

(17, 0)

If you drag the slope line to the right, you can see that the last place it touches the feasible region is (17, 0). This will be the best point.

# Practice Problem 3

Write and optimize each objective function using your graph and points from problem 2. First plug in all the points to find the maximum, then use the slope of the objective function to verify your answer.

3a. The company finds that experienced workers complete 10 tasks per minute, while inexperienced workers only complete 9. Maximize task completion.

3b. The company finds that experienced workers make higher quality products, generating 3 new customers per worker per year. Inexperienced workers generate 2. Maximize customer gain.

# Active and Inactive Constraints

An optimal solution that lies at the intersection point of two constraints causes both of those constraints to be considered *active*.

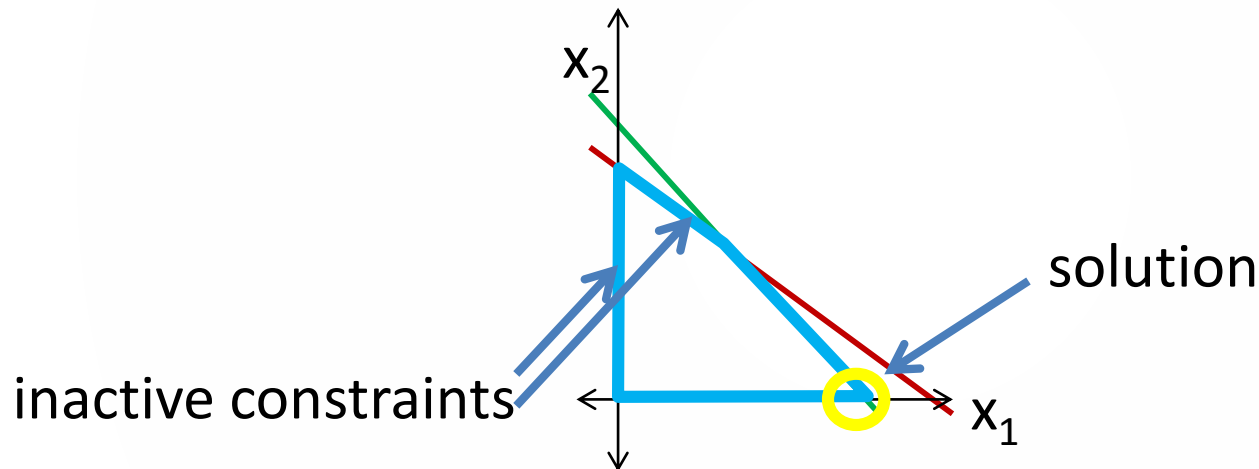active constraints

$x_2$

$x_1$
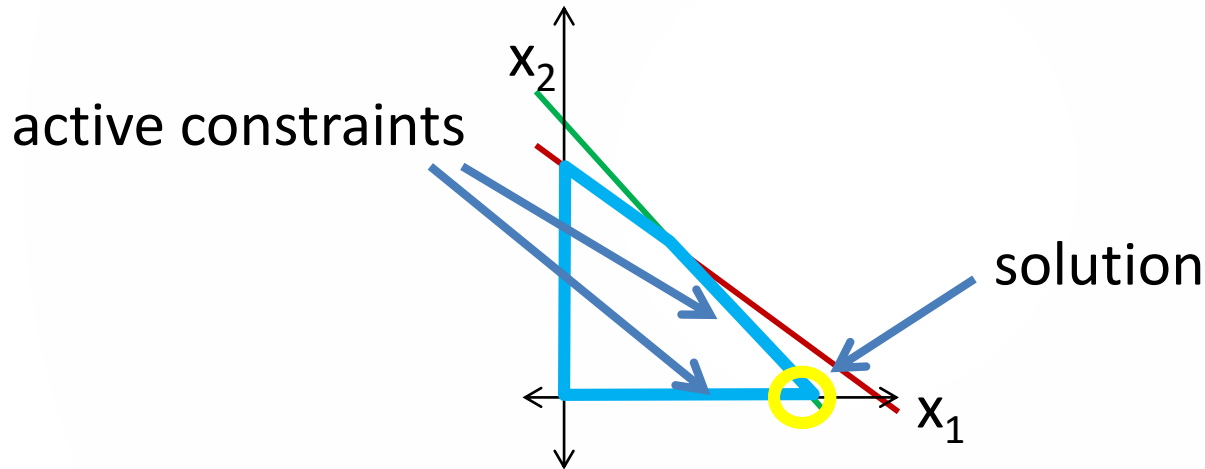
solution

# Active and Inactive Constraints

An optimal solution that lies at the intersection point of two constraints causes both of those constraints to be considered *active*.



If any of the constraint lines do not pass through the optimal point, those constraints are called *inactive*.

# Active and Inactive Constraints

In general, we ignore the constraints at 0 and focus on the constraints generated by limits on resources.

An *active constraint* means that this factor is causing the limitation on the objective function. If an active constraint was *amount of flour*, then by increasing the flour available you could improve your objective. If all your constraints are active, that is good news – you are using all your resources.

An inactive constraint means that this factor is not causing the limitation. If *amount of flour* was an inactive constraint, then you will have flour left over; perhaps you could use it for something else or sell it.

# Practice Problems 4 and 5

4. For your answers to both 3a and 3b, identify the active and inactive constraints (do not include the 0 boundaries).

5. Discuss the implications of each of the following (both active and inactive constraints):

- money is active, supervision inactive
- supervision active, money inactive
- both active