Minimization Using Duality

- Standard Minimization Problem
- Building the Dual Tableau
- Solving Standard Minimization
Standard Minimization Form

Standard minimization form involves a minimization problem where all constraints are $\geq$, for example:

minimize \[ 3x_1 + 4x_2 \]
subject to \[ x_1 + 3x_2 \geq 10 \]
\[ 2x_1 + 2x_2 \geq 8 \]
Standard Minimization Form

If we converted a standard minimization problem straight into a tableau it would look like this:

\[
\begin{bmatrix}
1 & 3 & -1 & 0 & 0 & 0 & 10 \\
2 & 2 & 0 & -1 & 0 & 8 \\
3 & 4 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

“minimize \( f = 3x_1 + 4x_2 \)” becomes “maximize \( f = -3x_1 - 4x_2 \)”, which converts to \( 3x_1 + 4x_2 + f = 0 \)

This tableau has all sorts of things wrong with it, including negative solutions which are illegal.

Constraints with \( \geq \) mean we need to subtract slack variables, not add them.
Duality

Fortunately, a standard minimization problem can be converted into a maximization problem with the same solution. The minimization problem and its corresponding maximization problem are called duals of each other.

The steps for using duality in the simplex method do not make much sense, but the method works.
Duality

The first step in solving a standard minimization problem using duality is to write the information into a matrix, ignoring everything you know about slack variables and objective functions.

minimize \( f = 3x_1 + 4x_2 \)
subject to \( x_1 + 3x_2 \geq 10 \)
\( 2x_1 + 2x_2 \geq 8 \)
Duality

The next step is to create the dual matrix, which starts with the transpose of the matrix we just created. “Transpose” means the first column becomes the first row, and so on:

\[
\begin{bmatrix}
1 & 3 & 10 \\
2 & 2 & 8 \\
3 & 4 & 0
\end{bmatrix}
\]

becomes

\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 4 \\
10 & 8 & 0
\end{bmatrix}
\]
Duality

The next step is where it gets strange. Using the transposed matrix, fill it in as if it were a standard maximization matrix:

\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 0 \\
3 & 2 & 4 & 1 & 0 \\
10 & 8 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Then maximize as you normally would.
Duality

After pivoting twice, the finished matrix looks like this:

\[
\begin{bmatrix}
0 & 4 & 3 & -1 & 0 & 5 \\
12 & 0 & -6 & -6 & 0 & 6 \\
0 & 0 & 12 & 36 & 12 & 180
\end{bmatrix}
\]

The last step is to divide the last row so the “f” column becomes 1. Your new last row is:

\[
\begin{bmatrix}
0 & 0 & 1 & 3 & 1 & 15
\end{bmatrix}
\]

slack variables 1 and 2

From this you can read the solution directly. The minimum of 15 occurs at \(x_1 = 1, x_2 = 3\).
Practice Problem 1

Convert these constraints into a dual matrix in standard maximum form:

Minimize \( 4x_1 + 2x_2 + 5x_3 \)

subject to \( 3x_1 + x_2 + 5x_3 \geq 15 \)
\( 2x_1 + 4x_2 + 2x_3 \geq 20 \)
Practice Problem 2

Using your pivoting program for standard maximum simplex, complete the steps to solve your tableau from problem 1.

2a. Write the final last row.

2b. Identify the values of: the slack variables; $x_1$, $x_2$, and $x_3$; and the minimized objective function.
An office manager is equipping a new workspace with storage units. Traditional shelves can hold 9 cubic feet of material and provide 3 square feet of work area on top. Deep file cabinets can hold 12 cubic feet of material and provide 6 square feet of work area on top. The manager needs at least 50 cubic feet of storage and 36 square feet of work area. File cabinets cost $100 each and shelves $70. Minimize cost. Write a full answer in context.