# Analyzing the Hessian

- Premise
- Determinants
- Eigenvalues
- Meaning

# The Problem

In 1-variable calculus, you can just look at the second derivative at a point and tell what is happening with the concavity of a function: positive implies concave up, negative implies concave down.

But because the Hessian (which is equivalent to the second derivative) is a matrix of values rather than a single value, there is extra work to be done.

This lesson forms the background you will need to do that work.

# Finding a Determinant

Given a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant, symbolized $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is equal to a·d - b·c. So, the determinant of

$\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ is... 6 - -4 = 10

The determinant has applications in many fields. For us, it's just a useful concept.

Determinants of larger matrices are possible to find, but more difficult and beyond the scope of this class.

# Practice Problem 1

Find the determinant. Check your work using `det(A)` in Julia.

a. $\begin{vmatrix} 3 & 1 \\ -2 & 0 \end{vmatrix}$

b. $\begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix}$

c. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

# Eigenvectors and Eigenvalues

One of the biggest applications of matrices is in performing geometric transformations like rotation, translation, reflection, and dilation.

In Julia, type in the vector `X = [3; -1]`

Then, multiply `[2 0; 0 2]*X`

You should get [6; -2], which is a multiplication of X by a factor of 2, in other words a dilation.

Next, try $\begin{bmatrix} \mathtt{cos(\frac{pi}{6})} & \mathtt{-sin(\frac{pi}{6})} \\ \mathtt{sin(\frac{pi}{6})} & \mathtt{cos(\frac{pi}{6})} \end{bmatrix} \mathtt{*X}$

Although this one isn’t immediately clear, you have accomplished a rotation of vector X by $\pi/6$ radians.

# Eigenvectors and Eigenvalues

In the last slide, we were looking at a constant X and a changing A, but you can also get interesting results for a constant A and a changing X.

For example, the matrix $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ doesn't look very special, and it doesn't do anything special for most values of X.

But if you multiply it by $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, you get $\begin{bmatrix} 21 \\ 35 \end{bmatrix}$, which is a scalar multiplication by 7.

# Eigenvectors and Eigenvalues

When a random matrix A acts as a scalar multiplier on a vector X, then that vector is called an *eigenvector* of X.

The value of the multiplier is known as an *eigenvalue*.

For the purpose of analyzing Hessians, the eigenvectors are not important, but the eigenvalues are.

# Finding Eigenvalues

The simplest way to find eigenvalues is to open Julia and type in:

```
eig(A)
```

This will give you the eigenvalue(s) of A as well as a matrix composed of the associated eigenvectors.

However, it’s also useful to know how to do it by hand.

# Finding Eigenvalues

To find eigenvalues by hand, you will be solving this equation…

determinant symbol

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\right| = 0$$

original matrix

variable matrix, will solve for x

…which turns into the following determinant:

$$\begin{vmatrix} a - x & b \\ c & d - x \end{vmatrix} = 0$$

# Finding Eigenvalues

So, if you were trying to find the eigenvalues for the matrix $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$, you would need to solve the determinant $\begin{vmatrix} 2-x & 3 \\ 5 & 4-x \end{vmatrix} = 0$.

Cross-multiplying, you would get

$(2 – x)(4 – x) – 15 = 0$

$8 – 6x + x^2 – 15 = 0$

$x^2 – 6x – 7 = 0$

$(x – 7)(x + 1) = 0$ so $x = 7$ or $-1$.

eigenvalues!

# Practice Problem 2

Find the eigenvalues of the following matrices by hand, then check using Julia:

a. $\begin{bmatrix} 3 & 8 \\ 4 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$

Find the eigenvalues using Julia:

c. $\begin{bmatrix} 2 & 1 & -4 \\ -2 & 3 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

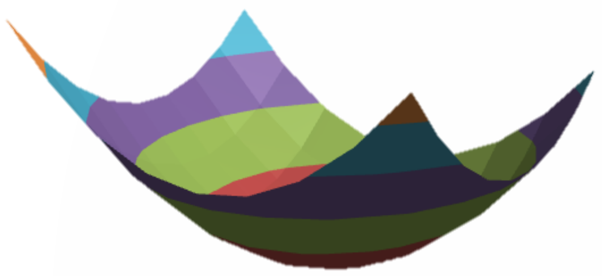
# Meaning of Eigenvalues

Because the Hessian of an equation is a square matrix, its eigenvalues can be found (by hand or with computers – we'll be using computers from here on out).

Because Hessians are also symmetric (the original and the transpose are the same), they have a special property that their eigenvalues will always be real numbers.

So the only thing of concern is whether the eigenvalues are *positive* or *negative*.

# Meaning of Eigenvalues



If the Hessian at a given point has all positive eigenvalues, it is said to be a positive-definite matrix. This is the multivariable equivalent of "concave up".

If all of the eigenvalues are negative, it is said to be a negative-definite matrix. This is like "concave down".

# Meaning of Eigenvalues

If either eigenvalue is 0, then you will need more information (possibly a graph or table) to see what is going on.

And, if the eigenvalues are mixed (one positive, one negative), you have a saddle point:

Here, the graph is concave up in one direction and concave down in the other.

# Practice Problem 3

Use Julia to find the eigenvalues of the given Hessian at the given point. Tell whether the function at the point is concave up, concave down, or at a saddle point, or whether the evidence is inconclusive.

a. $\begin{bmatrix} 12x^2 & -1 \\ -1 & 2 \end{bmatrix}$ at (3, 1)

b. $\begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$ at (-1, -2)

c. $\begin{bmatrix} -2y^2 & -4xy \\ -4xy & -2x^2 \end{bmatrix}$ at (1, -1) and (1, 0)

# Practice Problem 4

Determine the concavity of

$$f(x, y) = x^3 + 2y^3 - xy$$

at the following points:

a) (0, 0)
b) (3, 3)
c) (3, -3)
d) (-3, 3)
e) (-3, -3)

# Practice Problem 5

For $f(x, y) = 4x + 2y - x^2 - 3y^2$

a) Find the gradient. Use that to find a critical point (x, y) that makes the gradient 0.
b) Use the eigenvalues of the Hessian at that point to determine whether the critical point in a) is a maximum, minimum, or neither.

# Practice Problem 6

For $f(x, y) = x^4 + y^2 - xy$,

a) Find the critical point(s)
b) Test the critical point(s) to see if they are maxima or minima.