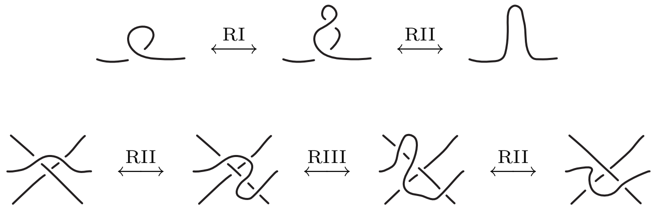
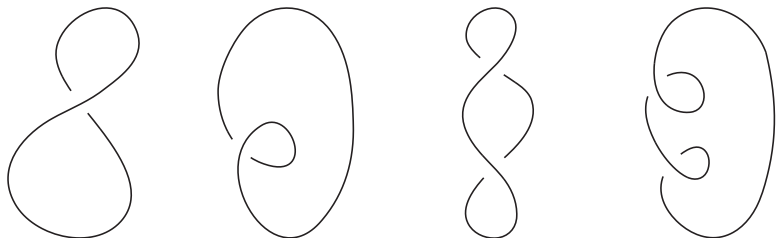
1.



2. The basic one- and two-crossing knots are shown below; all others are the same as these with crossing of the kinks reversed. They can all be transformed into the unknot using only RI.



3.

4. (a) The crossing number is a minimum over all manipulations, and if two knots are equivalent, they can be transformed into each other, and therefore into all the same knots. They must therefore have the same crossing number.

   (b) The R-moves can never change the number of components, since they do not cut and rejoin arcs. Thus the number of components for two equivalent links must be the same.

5. (a) The Hopf link has two positive crossings, giving it linking number 2. The Borromean rings have three each positive and negative crossings, so they have linking number 0.

   (b) The linking number of a knot is by definition 0, so it is a knot invariant.

   (c) Since two knots are equivalent if and only if they are connected by a sequence of R-moves, we must simply show that R-moves do not change the linking number. The part of a knot outside the effects of the R-move does not change, so we just consider how the crossings changed by the R-move affect the linking number. Clearly R0 moves do not affect it. RI moves remove/add a crossing, but it is always a self-crossing, so it never contributes to the linking number. For RII moves, we can similarly ignore the self-crossing case; if the two crossings belong two different components, they always have opposite sign, so contribution to linking number is always zero. Finally, RIII moves have three crossings. One is unchanged by the move, and the other two keep the same sign, so the linking number is again unchanged. Since none of the R-moves change linking number, it is a link invariant.

6. The three arcs of the trefoil may be colored all the same (3 possibilities) or all different (6 possibilities), so $\tau = 9$

7. We again consider the R-moves, with R0 obviously not affecting τ. We would like to show that every coloring of the knot before the move corresponds to exactly one coloring of the knot after the move.

   For an RI move, any coloring of the initial or final picture must be all the same color c, so the colorings are essentially equivalent.

For RII, the initial picture must have the top two ends the same color a, and from the constraint, the bottom two ends must be the same color b (if $a = b$ the middle arc must also be that color, and if $a \neq b$ the middle arc is the third color c). The final picture must also have the top two ends the same color and the bottom two ends the same color, so a coloring of either defines a coloring of the other.

For RIII, there are five cases, based on the colors of the three left-hand ends. They could be all the same, all different, or two the same and one different (giving three cases for the odd one out). It can quickly be seen that each case determines the colors of the three right-hand ends, and matching end colors to the final picture creates a unique coloring.

Therefore, all R-moves preserve τ, so it is a link invariant.

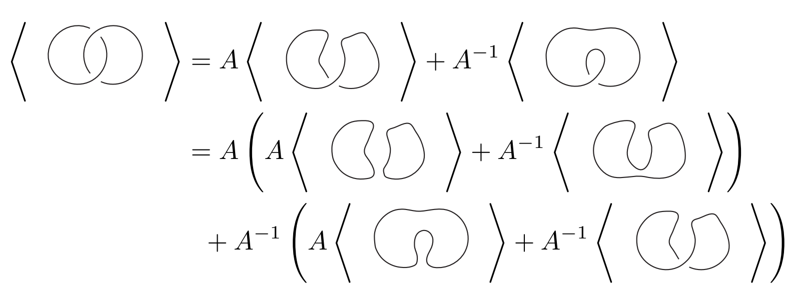
8. The splitting of each knot to form the connect sum breaks apart an arc which was all the same color. A valid coloring of the sum can have any valid coloring of each summand, so long as they agree on the joining segments. Suppose that the colors are a, b, c, and without loss of generality that the first summand has color a on the split arc. Since a coloring of the second summand is still a valid coloring with two colors switched, $\frac{1}{3}$ of the colorings will have a on the split segment, $\frac{1}{3}$ will have b, and $\frac{1}{3}$ will have c. Multiplying the number of choices for each knot gives the formula:

$$\tau(K_1 \# K_2) = \frac{1}{3}\tau(K_1)\tau(K_2)$$

9. We calculated that τ of the trefoil knot is 9. Therefore, τ of a connect sum of n trefoils is $\left(\frac{1}{3}\right)^{n-1} 9^n = 9 \cdot 3^{n-1}$. This takes a different value for each n, so the connect sums of trefoils are never equivalent to each other, and there are infinitely many distinct knots.

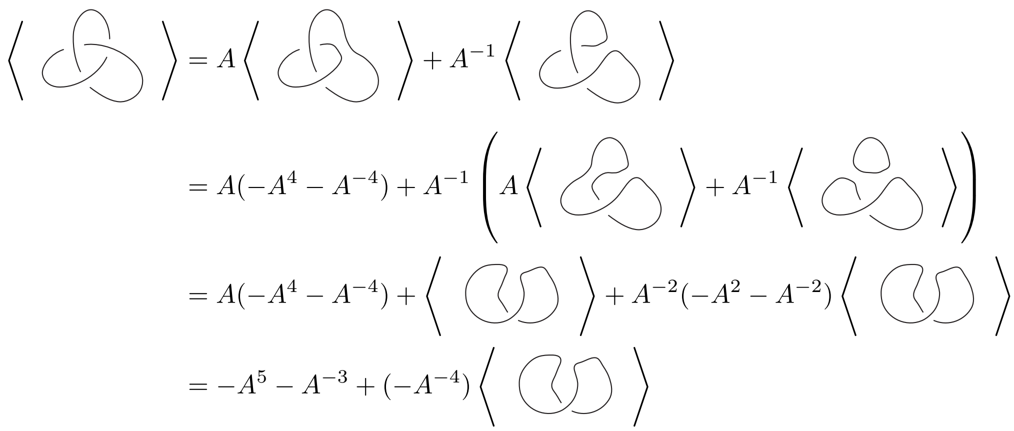
10. Before turning the cylinder into a torus, number the points at the left end of the segments $x_1, x_2, \ldots, x_p$. To find a component, start at a point x_{i_1} and follow it around the torus once, so that we are at a point x_{i_2} which is rotated by $2\pi\frac{q}{p}$ from x_{i_1}. Repeat this until the component closes up, that is, until $x_{i_n} = x_{i_1}$. We must have come an integer number of full rotations from x_{i_1}, so we must have $2\pi\frac{q}{p}n$ be a multiple of 2π. The first time this happens is when $n = \frac{p}{\gcd(p,q)}$. Each component therefore uses up $\frac{p}{\gcd(p,q)}$ of the points, so the number of components is $\gcd pq$. The special case of a knot has one component, so torus knots must have $\gcd pq = 1$.

11.



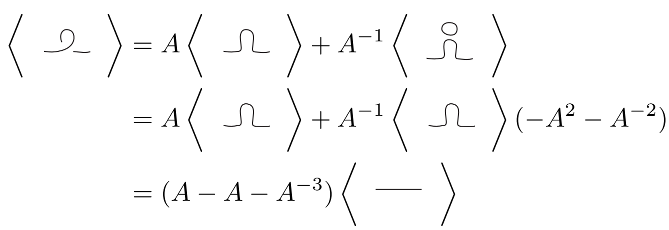
$$= A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2}(-A^2 - A^{-2})$$
$$= -A^4 - A^{-4}$$

In finding the Kauffman bracket of the trefoil, we take advantage of the previous calculation for the Hopf link, and the calculation for the one-crossing knot included therein.



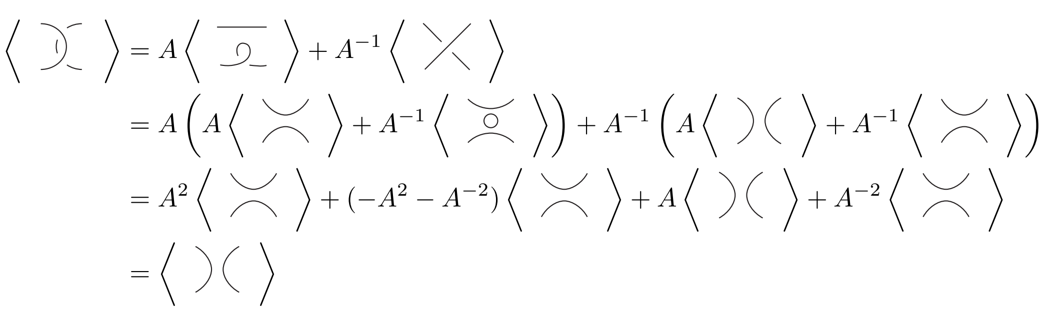
$$= -A^5 - A^{-3} + (-A^{-4})(-A^3)$$
$$= -A^5 - A^{-3} + A^{-1}$$

12. The bracket is not RI invariant:

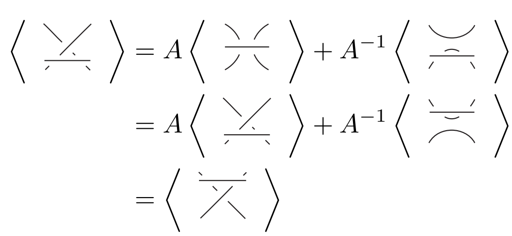


That is, an RI move changes the bracket by a factor of $-A^{-3}$. Similarly, a reverse RI move changes it by a factor of $-A^3$.

The bracket is RII invariant:



The bracket is RIII invariant (using RII invariance in the calculation):



13. Clearly the writhe is invariant under R0. Our proofs for RII and RIII invariance of linking number hold here as well. For RI, the crossing in the kink has sign $+1$ (and -1 for RI_2) and is removed by RI, so the writhe changes by -1 under RI (and $+1$ under RI_2).

14. Since both writhe and Kauffman brackets are invariant under RII and RIII, the Jones polynomial is as well. Under an application of RI, the bracket changes by a factor of $(-A^3)$, which is made up for by a loss of a factor of $-A^3$ when the writhe in the exponent decreases by 1. We have shown that the Jones polynomial is constant under all R-moves, and is therefore a link invariant.

15. As we've seen when calculating examples, the Kauffman bracket ultimately expands to powers of A multiplied by $n-1$ factors of $(-A^2-A^{-2})$, where n is the number of disjoint unknots left after applying the recursive relation repeatedly. Suppose we first apply the relation to D_1 in the disjoint union. If D_1 creates n unknots, we must remove them all to begin working on D_2 by itself - this is one more unknot than we would have had to remove when working just with D_1, so all terms have an extra factor of $(-A^2 - A^{-2})$ compared to $\langle D_1 \rangle$. All terms also still have $\langle D_2 \rangle$ multiplied onto them. Factoring these two parts out gives $\langle D_1 \coprod D_2 \rangle = (-A^2 - A^{-2}) \langle D_1 \rangle \langle D_2 \rangle$. The writhe is a sum over all crossings, so it is the sum of the writhes of the individual links D_1 and D_2. The Jones polynomial is therefore

$$f_{D_1 \coprod D_2}(A) = (-A^3)^{-w(D_1+D_2)}(-A^2 - A^{-2}) \langle D_1 \rangle \langle D_2 \rangle = (-A^2 + A^{-2}) f_{D_1}(A) f_{D_2}(A)$$

16. We can make a similar argument to that seen in the disjoint union, being careful to consider what happens to the two arcs connecting D_1 to D_2 as we reduce it using the recursion relation by considering the two crossings in D_1 neighboring those two arcs. If we apply all the transformations for a particular term in the bracket that would reduce D_1 to n disjoint unknots, those two crossings must have been reduced. Since the link must remain connected, the two arcs must eventually be joined, meaning that effectively those two crossings became connected. Without the connect sum, this would have led to an unknot, but now it simply creates an unknot joined to D_2 by a connect sum. We therefore get only $n-1$ disjoint unknots from D_1 before being fully reduced to D_2, which is exactly what would have gone into the calculation of $\langle D_1 \rangle$. As with the previous problem, the writhe is unaffected, so the Jones polynomial is

$$f_{D_1 \# D_1}(A) = (-A^3)^{-w(D_1+D_2)} \langle D_1 \rangle \langle D_2 \rangle = f_{D_1}(A) f_{D_2}(A)$$

17. Mirroring a knot inverts the writhe (positive and negative crossings are switched). The Kauffman bracket switches A and A^{-1} in the recursive part of the definition; everything else is unaffected. In the Jones polynomial, we therefore flip the sign of the exponent of $(-A^3)$, and switch A and A^{-1} in the bracket, therefore switching A and A^{-1} in the entire polynomial. Thus an amphichiral knot has palindromic Jones polynomial (that is, the coefficient of A^k is the same as that of A^{-k}, while a chiral knot does not. As seen in the earlier calculation, the Kauffman bracket of one of the trefoil knots is $-A^5 - A^{-3} + A^{-1}$. This cannot be made palindromic by multiplying by a power of A, so the trefoil knot must be chiral.