1. Let a, b, c, and d be the numbers that show when four fair dice, numbered 1 through 6 are rolled. What is the probability that $|(a-1)(b-2)(c-3)(d-6)| = 1$?

   Answer: $\mathbf{\frac{1}{324}}$

   The conditions implies that $|a-1| = |b-2| = |c-3| = |d-6| = 1$. a can equal 2, b can equal 1 or 3, c can equal 2 or 4, and d can equal 5. So the probability is $\frac{1}{6} * \frac{2}{6} * \frac{2}{6} * \frac{1}{6} = \frac{1}{324}$.

2. Find all possibilities for the second-to-last digit of a number whose square is of the form 1_2_3_4_5_6_7_8_9_0 (each _ is a digit).

   Answer: 3, 7

   Zero is the only digit with square ending in 0. The square of a number ending in zero will therefore end in two zeros. Next digit of the number therefore needs a square ending in 9, so it is 3 or 7.

3. Ten gears are lined up in a single file and meshed against each other such that the i^{th} gear from the left has $5i+2$ teeth. Gear $i = 1$ (counting from the left) is rotated 21 times. How many revolutions does gear 10 make?

   Answer: $\mathbf{\frac{147}{52}}$

   The number of teeth meshed does not vary. Thus, if n is the number of revolutions that gear 10 make, then $(5(1)+21)(21) = (5(10)+2)n \Rightarrow n = \frac{7\times 21}{52} = \frac{147}{52}$.

4. In the game Pokeymawn, players pick a team of 6 different Pokeymawn creatures. There are 25 distinct Pokeymawn creatures, and each one belongs to exactly one of four categories: 7 Pokeymawn are plant-type, 6 Pokeymawn are bug-type, 4 Pokeymawn are rock-type, and 8 Pokeymawn are bovine-type. However, some Pokeymawn do not get along with each other when placed on the same team: bug-type Pokeymawn will eat plant-type Pokeymawn, plant-type Pokeymawn will eat rock-type Pokeymawn, and bovine-type Pokeymawn will eat anything except other Bovines. How many ways are there to form a team of 6 different Pokeymawn such that none of the Pokeymawn on the team want to eat any of the other Pokeymawn?

   Answer: 245

   If we make our team all the same type, then there are $\binom{7}{6} + \binom{6}{6} + \binom{4}{6} + \binom{8}{6} = 7 + 1 + 0 + 28 = 36$ ways to do this. If we make our team partially bug and partially rock type, there are $\binom{6}{2}\binom{4}{4} + \binom{6}{3}\binom{4}{3} + \binom{6}{4}\binom{4}{2} + \binom{6}{5}\binom{4}{1} = 15*1 + 20*4 + 15*6 + 6*4 = 15 + 80 + 90 + 24 = 209$ ways. Any other combination of types will not work. This gives a total of 245 ways.

5. Four cards are drawn from a standard deck (52 cards) with suits indistinguishable (for example, A♠ is the same as A♣). How many distinct hands can one obtain?

   Answer: 1820, or $\mathbf{\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4}}$

   We proceed by casework.
   Case 1 All cards have the same face value. There are $\binom{13}{1}$ ways to choose the face values.
   Case 2 Some cards have face value A; some have face value B. There are $\binom{13}{2}$ ways to choose A and B. One can have the combinations $ABBB$, $AABB$, $AAAB$, so there are $3\binom{13}{2}$ distinct ways for this case.
   Case 3 Some cards have value A, some B, and some C. There are $\binom{13}{3}$ ways to choose the A, B, C. One can have the combinations $ABCC$, $ABBC$, and $AABC$. There are $3\binom{13}{2}$ distinct ways for this case.
   Case 4 The cards are distinct: $ABCD$. There are $\binom{13}{4}$ ways to do this. Since these cases are mutually exclusive, we have $\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4} = 1820$ distinct hands.

6. Find all complex numbers z such that $z^5 = 16\bar{z}$, where if $z = a + bi$, then $\bar{z} = a - bi$.

   Answer: $\mathbf{0, \pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}}$

Clearly 0 is a solution. Now we assume $z \neq 0$. We have $|z^5| = |16\bar{z}|$. By DeMoivre's Theorem, $|z^5| = |z|^5$. The left hand side becomes $|z^5| = 16|\bar{z}| = 16|z|$. Equating the two sides, $16|z| = |z|^5 \Rightarrow |z|^4 = 16 \Rightarrow |z| = 2$.
Multiplying both sides of the given equation by z,

$$z^6 = 16|z|^2 = 64.$$

Let $z = r(\cos\theta + i\sin\theta)$. Then $r^6(\cos(6\theta) + i\sin(6\theta)) = 64$. Thus, $r = 2$ and $6\theta = 360k$, for $k = 0, 1, 2, 3, 4, 5$. So our other solutions are $2, 2\text{cis}(60°), 2\text{cis}(120°), -2, 2\text{cis}(240°), 2\text{cis}(300°)$, which are equal to $\pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}$.

7. Evaluate $\sqrt{\frac{1+\sqrt{3}i}{2}}$

   Answer: $\boldsymbol{e^{\frac{\pi}{6}i}}$**, or** $\boldsymbol{\pm\frac{\sqrt{3}+i}{2}}$

   Let $x = \sqrt{\frac{1+\sqrt{3}i}{2}}$. Then $x^2 = \frac{1+\sqrt{3}i}{2}$. Converting to polar form, $\frac{1+\sqrt{3}i}{2} = (e^{\frac{\pi}{3}i})^{\frac{1}{2}} = e^{\frac{\pi}{6}i} = \frac{\sqrt{3}+i}{2}$

8. Frank alternates between flipping a weighted coin that has a $\frac{2}{3}$ chance of landing heads and a $\frac{1}{3}$ chance of landing tails and another weighted coin that has a $\frac{1}{4}$ chance of landing heads and a $\frac{3}{4}$ chance of landing tails. The first coin tossed is the "$2/3 - 1/3$" weighted coin. What is the probability that he sees two heads in a row before he sees two tails in a row?

   Answer: $\boldsymbol{\frac{13}{33}}$

   If the first toss comes up heads (2/3 probability), Frank has a 1/4 chance of getting another heads, a $(3/4) * (1/3) = 1/4$ chance of getting two successive tails, and a $(3/4) * (2/3) = 1/2$ chance of getting tails-heads and winding up back at his current position of tossing the "$1/4-3/4$" coin with the previous toss being a heads. Expressing the probabilities as geometric series (or just the weighted probability of the two nonrepeating options), he has a 1/2 chance of getting HH first and a 1/2 chance of getting TT first. If instead, the first toss comes up tails (1/3 probability), he has a 3/4 chance of getting another tails, a $(1/4) * (2/3) = 2/12$ chance of getting two successive heads, and a $(1/4) * (1/3) = 1/12$ chance of getting heads-tails and winding up back at my current state. Expressing the probabilities as a geometric series, he has a 2/11 chance of getting HH first and a 9/11 chance of getting TT first. The probability of getting HH before TT is $(2/3) * (1/2) + (1/3) * (2/11) = 13/33$.

9. The triangular numbers $T_n = 1, 3, 6, 10, \ldots$ are defined by $T_1 = 1$ and $T_{n+1} = T_n + (n+1)$. The square numbers $S_n = 1, 4, 9, 16, \ldots$ are defined by $S_1 = 1$ and $S_{n+1} = T_{n+1} + T_n$. The pentagonal numbers $P_n = 1, 5, 12, 22, \ldots$ are defined by $P_1 = 1$ and $P_{n+1} = S_{n+1} + T_n$. What is the 20th pentagonal number P_{20}?

   Answer: 590

   Expanding out the recurrence relations, we confirm that the triangular numbers are $T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and the square numbers are $S_n = n^2$. A general formula for the pentagonal numbers is therefore $P_n = n^2 + n(n-1)/2 = n(3n-1)/2$. Substituting $n = 20$ gives $P_{20} = 20(60-1)/2 = 590$.

10. Evaluate $e^{i\pi/3} + 2e^{2i\pi/3} + 2e^{3i\pi/3} + 2e^{4i\pi/3} + e^{5i\pi/3} + 9e^{6i\pi/3}$.

    Answer: 6

    $e^{i\pi/3} + e^{2i\pi/3} + e^{3i\pi/3} + e^{4i\pi/3} + e^{5i\pi/3} + e^{6i\pi/3}$ sum to 0 because the terms are sixth roots of unity (i.e. they satisfy $z^6 - 1 = 0$, which is a 6th degree polynomial whose 5th degree coefficient is 0). Likewise, $e^{2i\pi/3} + e^{4i\pi/3} + e^{6i\pi/3}$ sum to zero because the terms are cubic roots of unity. $e^{3i\pi/3} + e^{6i\pi/3}$ sum to 0 because they are square roots of unity. Subtracting these sums from the original expression, we are left with only $6e^{6i\pi/3}$, which is $6(\cos(2\pi) + i\sin(2\pi)) = 6$.