1. No math tournament exam is complete without a self referencing question. What is the product of the smallest prime factor of the number of words in this problem times the largest prime factor of the number of words in this problem?

   Answer: 1681

   There are 41 words in the problem statement. Since 41 is itself a prime, the answer is $41^2 = 1681$.

2. King Midas spent $\frac{100}{x}\%$ of his gold deposit yesterday. He is set to earn gold today. What percentage of the amount of gold King Midas currently has would he need to earn today to end up with as much gold as he started?

   Answer: $\boldsymbol{\frac{100}{x-1}\%}$

   After yesterday, the fraction of the initial gold remaining is $1-\frac{1}{x}=\frac{x-1}{x}$. Therefore, in order to to reach the original amount of gold, we must multiply by $\frac{x}{x-1}=1+\frac{1}{x-1}$. Thus, the gold must be increased by $\frac{100}{x-1}$ percent.

3. Find all integer pairs (a, b) such that $ab + a - 3b = 5$.

   Answer: $\mathbf{(5, 0), (4, 1), (1, -2),}$ **and** $\mathbf{(2, -3)}$

   We factor the expression as follows:

$$\begin{aligned} ab + a - 3b - 3 &= 5 - 3 \\ (a-3)(b+1) &= 2 \end{aligned}$$

   We can use a table to find appropriate values for a and b.
   Thus, $(5, 0), (4, 1), (1, -2)$, and $(2, -3)$ are the desired solutions.

$a-3$	$b+1$	a	b
2	1	5	0
-2	-1	1	-2
1	2	4	1
-1	-2	2	3

4. Find all the solutions for which $f(x) + xf\left(\frac{1}{x}\right) = x$.

   Answer: $\boldsymbol{x = 1}$

$$\begin{aligned} f(x) + xf\left(\frac{1}{x}\right) &= x \\ f\left(\frac{1}{x}\right) + \frac{1}{x}f(x) &= \frac{1}{x} \\ f\left(\frac{1}{x}\right) &= \frac{1}{x} - \frac{1}{x}f(x) \\ f(x) + x\left(\frac{1}{x} - \frac{1}{x}f(x)\right) &= x \\ x &= 1 \end{aligned}$$

5. Find the minimum possible value of $2x^2 + 2xy + 4y + 5y^2 - x$ for real numbers x and y.

   Answer: $\boldsymbol{-\frac{5}{4}}$

We complete the square:

$$\begin{aligned} 2x^2+2xy+4y+5y^2-x &= (x^2+2xy+y^2)+(x^2-x+\frac{1}{4})+(4y^2+4y+1)-(\frac{1}{4}+1) \\ &= (x+y)^2+(x-\frac{1}{2})^2+(2y+1)^2-\frac{5}{4} \end{aligned}$$

Notice that $x=\frac{1}{2}$ and $y=-\frac{1}{2}$ would yield the minimum, which is $-\frac{5}{4}$.

6. The dollar is now worth $\frac{1}{980}$ ounce of gold. After the n^{th} \$7001 billion "No Bank Left Behind" bailout package passed by congress, the dollar gains $\frac{1}{2^{2^{n-1}}}$ of its $(n-1)^{th}$ value in gold. After four bank bailouts, the dollar is worth $\frac{1}{b}\left(1-\frac{1}{2^c}\right)$ in gold, where b,c are positive integers. Find $b+c$.

Answer: 506

Let P_n be the value of the dollar in gold after the n^{th} bailout. Let $s=\frac{1}{2}$. Then after the n^{th} bailout, the dollar is a factor of $(1+s^{2^{n-1}})$ of its $(n-1)^{th}$ value. Thus,

$$\begin{aligned} P_4 &= \frac{1}{980}(1+s)(1+s^2)(1+s^4)(1+s^8) \\ &= \frac{1}{980}(1+s+s^2+s^3)(1+s^4)(1+s^8) \\ &= \frac{1}{980}(1+s+s^2+s^3+s^4+s^5+s^6+s^7)(1+s^8) \\ &= \frac{1}{980}(1+s+s^2+s^3+s^4+s^5+s^6+s^7+s^8+s^9+s^{10}+s^{11}+s^{12}+s^{13}+s^{14}+s^{15}) \\ &= \frac{1}{980}\left(\frac{1-s^{16}}{1-s}\right). \end{aligned}$$

Plug in $s=\frac{1}{2}$, and we find that $P_4=\frac{1}{490}\left(1-\frac{1}{2^{16}}\right)$. So $b+c=490+16=506$.

7. Evaluate $\sum\limits_{k=1}^{2009}\lfloor\frac{k}{60}\rfloor$

Answer: 32670

Largest multiple of 60 below 2009 is 1980, so find the sum for $k=1$ to 1979, so that we have each value of $\lfloor k/60\rfloor$ exactly 60 times. This sum is therefore $60(1+2+\ldots+32)=60(1+32)\frac{32}{22}=31680$. The remaining terms are all 33, and there are $2009-1980+1=30$ of them, giving an answer of $31680+30\times 33=32670$.

8. "Balanced tertiary" is a positional notation system in which numbers are written in terms of the digits $\bar{1}$ (negative one), 0, and 1 with the base 3. For instance, $10\bar{1}1=(1)3^0+(-1)3^1+(0)3^2+1(3)^3=25_{10}$. Calculate $(1\bar{1}00)(\bar{1}1)+(1\bar{1}1)$ and express your answer in balanced tertiary.

Answer: $\bar{1}0\bar{1}1$

$(1\bar{1}00)(\bar{1}1)=\bar{1}1000+1\bar{1}00=\bar{1}\bar{1}00$
$\bar{1}\bar{1}00+1\bar{1}1=\bar{1}0\bar{1}1$

9. All the roots of x^3+ax^2+bx+c are positive integers greater than 2, and the coefficients satisfy $a+b+c+1=-2009$. Find a.

Answer: -58

Let the roots be r, s, and t. Then they satisfy $r+s+t=-a$, $rs+st+rt=b$, and $rst=-c$. So we have $-(a+b+c+1)=r+s+t-rs-rt-st+rst-1=(r-1)(s-1)(t-1)=2009=7*7*41$.

Thus the roots are 8, 8, and 42, and $a=-(r+s+t)=-58$.

10. Let $\delta(n)$ be the number of 1s in the binary expansion of n (e.g. $\delta(1) = 1, \delta(2) = 1, \delta(3) = 2, \delta(4) = 1$). Evaluate:

$$10\left(\frac{\sum_{n=1}^{\infty}\frac{\delta(n)}{n^2}}{\sum_{n=0}^{\infty}\frac{(-1)^{n-1}\delta(n)}{n^2}}\right).$$

Answer: 20

For convenience, set $x = \sum_{n=1}^{\infty}\frac{\delta(n)}{n^2}$ and $y = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}\delta(n)}{n^2}$.

The crucial observation is that $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$ give the same summation as x, restricted to the terms with odd n and even n respectively. The latter summation is easily related to x using the fact that $\delta(2n) = \delta(n)$ (since multiplying by 2 is simply appending a 0 in the binary expansion), as follows.

$$\begin{aligned}
\frac{1}{2}(x-y) &= \sum_{\text{even } n\geq 2}\frac{\delta(n)}{n^2} \\
&= \sum_{n=1}^{\infty}\frac{\delta(2n)}{(2n)^2} \\
&= \sum_{n=1}^{\infty}\frac{\delta(n)}{4n^2} \\
&= \frac{1}{4}x.
\end{aligned}$$

Thus we have $\frac{1}{2}(x-y) = \frac{1}{4}x$. It follows that $x = 2y$, so $x/y = 2$. Thus, the desired answer is 20.