1. Find the exact value of $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \ldots$.

   Answer: $\sin 1$

   By Taylor Expansion, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$. Let x=1, and the desired value equals $\sin \mathbf{1}$.

2. At RMT 2009 is a man named Bill who has an infinite amount of time. This year, he is walking continuously at a speed of $\frac{1}{1+t^2}$, starting at time $t = 0$. If he continues to walk for an infinite amount of time, how far will he walk?

   Answer: $\frac{\pi}{2}$

   $d = \int_0^\infty \frac{1}{1+t^2} dt = \tan^{-1}(t)|_0^\infty = \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(0) = \frac{\pi}{2}$.

3. Evaluate $\lim\limits_{x\to 0} \dfrac{10x^2}{\sin^2(3x)}$.

   Answer: $\frac{10}{9}$

   By l'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{10x^2}{\sin^2(3x)} &= \lim_{x\to 0} \frac{20x}{6\sin(3x)\cos(3x)} \\
&= \lim_{x\to 0} \frac{20x}{3\sin(6x)} \\
&= \lim_{x\to 0} \frac{20}{18\cos(6x)} \\
&= \frac{10}{9}.
\end{aligned}$$

4. Compute $\displaystyle\int_0^1 \tan^{-1}(x)dx$

   Answer: $\frac{\pi - 2\ln(2)}{4}$ or equivalent expression

   We integrate by parts:

$$\begin{aligned}
\int_0^1 1 \cdot \tan^{-1}(x)dx &= \left[x\tan^{-1}(x)\right]_0^1 - \int_0^1 \frac{x}{x^2+1}dx \\
&= \frac{\pi}{4} - 0 - \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 \\
&= \frac{\pi}{4} - \frac{\ln(2)}{2}.
\end{aligned}$$

5. Let $a(t) = \cos^2(2t)$ be the acceleration at time t of a point particle traveling on a straight line. Suppose at time $t = 0$, the particle is at position $x = 1$ with velocity $v = -2$. Find its position at time $t = 2$.

   Answer: $-\frac{\cos(8)}{32} + \frac{33}{32}$

$$v(t) = \int a(t)dt = \int \cos^2(2t)dt = \int \frac{1+\cos(4t)}{2}dt = \frac{\sin(4t)}{8} + \frac{t}{2} + c_1,$$

   where c_1 is a constant. Plug in $t = 0$ to find $v(0) = c_1 = -2$. So $v(t) = \frac{\sin(4t)}{8} + \frac{t}{2} - 2$.

$$x(t) = \int v(t)dt = \int \frac{\sin(4t)}{8} + \frac{t}{2} - 2dt = -\frac{\cos(4t)}{32} + \frac{t^2}{4} - 2t + c_2.$$

   Plug in $t = 0$ to get $x(0) = -\frac{1}{16} + c_2 = 1$, so $c_2 = \frac{33}{32}$. Thus,

$$x(2) = -\frac{\cos(8)}{16} + \frac{33}{32}.$$

6. Find

$$\sum_{n=2}^{\infty} \frac{d^n}{dx^n}(e^{-ax})$$

for $|a| < 1$.

Answer: $\boldsymbol{\frac{a^2}{1+a}e^{-ax}}$

Since $\frac{d^n}{dx^n}(e^{-ax}) = (-a)^n e^{-ax}$,

$$\sum_{n=2}^{\infty} \frac{d^n}{dx^n}(e^{-ax}) = e^{-ax}\sum_{n=2}^{\infty}(-a)^n.$$

This forms a geometric series with common ratio $-a$ and first element a^2, which converges since $|a| < 1$. Thus the answer is $\frac{a^2}{1+a}e^{-ax}$.

7. Compute

$$\lim_{n\to\infty}\sum_{k=1}^{n} \frac{n-k}{n^2}\cos\left(\frac{4k}{n}\right).$$

Answer: $\boldsymbol{\frac{1-\cos(4)}{16}}$

Define a partition on $[0,1]$ with n elements by setting $x_i = \frac{i}{n}$ for $0 \le i \le n$. Then $x_i - x_{i-1} = \frac{1}{n}$ for all i. If we let $f(y) = (1-y)\cos(4y)$ and put $y_k = \frac{k}{n}$ for $1 \le k \le n$, then we have

$$\sum_{k=1}^{n} \frac{n-k}{n^2}\cos\left(\frac{4k}{n}\right) = \sum_{k=1}^{n} f(y_k)(x_i - x_{i-1}).$$

Thus, we may conclude that

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n} f(y_k)(x_i - x_{i-1}) &= \int_0^1 f(y)dy \\
&= \int_0^1 (1-y)\cos(4y) \\
&= \left[\left(\frac{1-y}{4}\right)\sin(4y) - \frac{\cos(4y)}{16}\right]_0^1 \\
&= \frac{-\cos(4)}{16} + \frac{1}{16}.
\end{aligned}$$

8. Evaluate $\int_0^\infty 4\lfloor x+7\rfloor e^{-2x}dx$. Remember to express your answer as a single fraction.

Answer: $\boldsymbol{\frac{14e^2-12}{e^2-1}}$**, or** $\boldsymbol{\frac{14-12e^{-2}}{1-e^{-2}}}$

To evaluate the floor function, split the integral into unit intervals:

$$\begin{aligned}
\int_0^\infty 4\lfloor x+7\rfloor e^{-2x}dx &= \sum_{k=0}^{\infty}\int_k^{k+1} 4(k+7)e^{-2x}dx \\
&= (14e^{-0} - 14e^{-2}) + (16e^{-2} - 16e^{-4}) + (18e^{-4} - 18e^{-6}) + \ldots \\
&= 12 + 2(e^{-0} + e^{-2} + e^{-4} + \ldots) \\
&= 12 + \frac{2}{1-e^{-2}} = \frac{14-12e^{-2}}{1-e^{-2}} = \frac{14e^2-12}{e^2-1}.
\end{aligned}$$

9. Compute $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{1}{5}\right)^n$.

Answer: $\mathbf{\frac{5}{16}}$

Let $S = \sum_{n=0}^{\infty} \frac{n}{5^n}$. Then

$$\frac{1}{5}S = \sum_{n=0}^{\infty} \frac{n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{n+1-1}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{n+1}{5^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{n}{5^n} - \frac{\frac{1}{5}}{1-\frac{1}{5}} = S - \frac{1}{4}$$

$$\Rightarrow \frac{S}{5} = S - \frac{1}{4} \Rightarrow \frac{4S}{5} = \frac{1}{4} \Rightarrow S = \frac{5}{16}.$$

10. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{50 + n^2/80000}$, as a decimal to the nearest tenth.

Answer: 62.8

This sum is difficult to evaluate exactly. However, it can be closely approximated by the improper integral of the same function, which is easily evaluated using u-substitution.

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{50 + x^2/80000} &= \frac{1}{50}\int_0^{\infty} \frac{dx}{1 + (x/20000)^2} \\
&= \frac{1}{50}\int_0^{\infty} \frac{2000du}{1+u^2} \\
&= \frac{2000}{50}\left[\tan^{-1} u\right]_0^{\infty} \\
&= 40 \lim_{b\to\infty} (\tan^{-1}(b) - \tan^{-1}(0)) \\
&= 40 \lim_{b\to\infty} \tan^{-1}(b) \\
&= 40\frac{\pi}{2} \\
&= 20\pi \\
&\approx 62.83.
\end{aligned}$$

To see that this integral is correct to the nearest tenth, we observe that since the integrand is a monotonic function, we can bound it above and below by Riemann sums. More precisely:

$$\sum_{n=1}^{\infty} \frac{1}{50 + n^2/80,000} \le 20\pi \le \sum_{n=0}^{\infty} \frac{1}{50 + n^2/80,000}.$$

By rearranging terms, this implies that:

$$20\pi - \frac{1}{50} \le \sum_{n=1}^{\infty} \frac{1}{50 + n^2/80,000} \le 20\pi.$$

From this it follows that 62.8 is indeed correct to the nearest tenth.