1. MacGyver is on a train, being pursued by a pack of velociraptors, starting with a 400m head start. He frantically begins to build an anti-velociraptor weapon out of his paper clip, fork, spoon, energy bar, and spring-loaded click pen; he needs 35 seconds to finish. The antiquated train is running at 10m/s, and dramatically crashes to a stop after 20 seconds, but MacGyver finishes as the raptors close to only 5m away. How fast were the raptors running, in m/s?

   Answer: 17

   Let the answer be v. The raptors gain $20(v-10)$ while the train is moving, then $13v$ after it stops, for a total of $33v-200$. In this time they have gained 595, so we have $395 = 35v - 200$, so $v = 17$.

2. I attach my pet snake, Earl, to one corner of my barn with a leash. The barn is square, with sides of length 10, and the leash has a length of twenty, which wraps around the barn. I would like to make sure that I am being humane to Earl, and would therefore like to know that area of my lawn he can traverse while on the leash. What is this area?

   Answer: 350π

   We see that since this is at the corner of the barn, the snake is free to travel in a three quarter arc around this point such that it does not intersect the barn. On the corners, we notice that the leash will bend, and basically act as a shorter leash fixed at the other two corners. Thus, there is a leash of length 10 for each of two quarter circles, which sum to half of an arc:

$$\frac{3}{4}(20)^2\pi + \frac{1}{2}(10)^2\pi = 350\pi.$$

3. A parallelogram is given with a base of length $2x + 15$, and a height of $10 - x$. Find x such that the area is maximized.

   Answer: $\frac{5}{4}$

   This can be expanded, and we see that it is quadratic:

$$(2x + 15)(10 - x) = -2x^2 + 5x + 150 \Longrightarrow \frac{-b}{2a} = \frac{5}{4} = x$$

4.

5. Find the sum of the distinct real roots of $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$.

   Answer: 1

   It is apparent that 1 is a root. You can then factor the left side as $(x-1)(x^3 - 2x^2 + 2x - 1)$. The second factor also has 1 as a root, so you can further factor it as $(x-1)(x-1)(x^2 - x + 1)$. The last factor in this has no real roots, so 1 is the only real root.

6.

7. In a paralleogram, the measure of one angle is four times that of another angle. Find the measure of the larger angle.

   Answer: 144°

   The two different angles in a parallelogram sum to $180°$. If x is the smaller angle, we have $x + 4x = 5x = 180°$, so $x = 36°$ and the larger angle is $4x = 144°$.

8. A fireworks factory can currently produce Fizzbangs for two dollars each. Upgrades are available for the machinery, each of which will decrease cost by ten cents. The first upgrade costs \$100, the second costs \$200, and so on. What is the cheapest cost (in dollars) to produce 10001 fizzbangs?

   Answer: 15501

   Each upgrade will reduce the total costs by \$1000.1. Therefore the first ten upgrades, up to the one that costs \$1000, will save money. The cost of the fizzbangs will then be one dollar each, costing \$10001, and the upgrades will cost a total of \$5500.

9. The line $y = mx$ intersects the line $y = 10 - x$ at a distance of $\sqrt{82}$ from the origin. Find the product of all possible values of m.

Answer: 1

Suppose the point (x, y) is the intersection point. The distance from (y, x) to the origin is the same, and it will also be on the line $y = 10 - x$. The slope of the line from the origin to (y, x) is $1/m$, providing a second solution. The product of these two solutions is 1.

10. At the grocery store, Jeffrey notices berries on sale and decides to make an extremely large number of berry tarts. Each tart uses $\frac{1}{4}$ of a container of blueberries, $\frac{1}{7}$ a container of blackberries, and $\frac{1}{10}$ a container of strawberries. If he bakes an integer number of tarts, using all the berries, and resists eating berries before baking, what is the fewest number of containers he could have used?

Answer: 69

The number of tarts made is 140, the least common multiple of 4, 7, and 10. This uses $\frac{140}{4} + \frac{140}{7} + \frac{140}{10} = 35 + 20 + 14$

11. An ice cream cone has a radius of 5 and a height of 8. After five hours in the sun, the ice cream melts, filling the cone up to a height of 3. What is the volume of the melted ice cream?

Answer: $\boldsymbol{\frac{225\pi}{64}}$

The cone of melted ice cream is similar to the original cone, so it has a radius of $\frac{3}{8} \cdot 5 = \frac{15}{8}$. The volume of the cone is therefore $\frac{1}{3} \cdot 3 \cdot \left(\frac{15}{8}\right)^2$.

12. A brick wall containing several windows is built. In sections without windows, the wall uses 120 bricks per meter, while in sections with windows, it uses only 80 bricks per meter. If the wall is 20 meters long and contains 2120 bricks, how many meters of wall without windows are there?

Answer: 13

Let x be the length of wall without windows. The length of wall with windows is $20 - x$, so the total number of bricks used is $2120 = 120x + 80(20 - x)$. Solving for x gives $x = \frac{2120-1600}{40} = 13$.

13. A mouse factory makes 3- and 5-button mice. The factory normally uses 207 buttons a day, but one day accidentally switches the orders and makes 5-button mice instead of 3-button mice, and vice versa, and ends up using 281 buttons. How many 3-button mice does the factory normally make?

Answer: 49

Let x be the normal number of 3-button mice, and y the normal number of 5-button mice. We have $3x + 5y = 207$ and $5x + 3y = 281$. Subtracting 3 times the first equation from 5 times the second gives $25x - 9x = 5 \cdot 281 - 3 \cdot 207$, so $16x = 784$ and $x = 49$.

14. Find the largest prime factor of 3599.

Answer: 61

Notice that $3599 = 60^2 - 1^2$. Factoring the difference of two squares gives $60^2 - 1^2 = (60 - 1)(60 + 1)$. 59 and 61 are both prime, so the answer is 61.

15. How many factors does 12345 have?

Answer: 8

The number of factors of $p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$, where each of p_i is a prime, $1 \le i \le m$, is

$$(n_1 + 1)(n_2 + 1) \dots (n_m + 1).$$

Since $12345 = 3 \times 5 \times 823$, we have $(1 + 1)(1 + 1)(1 + 1) = 8$ factors.

16. Given a drawer with 8 white gloves, 12 black gloves, and 6 gray gloves, find the number of gloves you need to pull out to ensure you have a pair of matching gloves. Assume that each glove has a matching pair.

    Answer: 14

    We have three different colors, but each glove must have a matching pair of gloves. Thus, there are $\frac{8}{2} = 4$ white lefthanded gloves, $\frac{12}{2} = 6$ black lefthanded gloves, and $\frac{6}{2} = 3$ gray lefthanded gloves. We have $4 + 6 + 3 = 13$. But pulling out one more glove ensures a pair since all the remaining gloves are righthanded. Thus, we need 14 gloves to make sure we have a matching pair.

17. Bill has made a bet with Tom. Bill will flip a fair coin 20 times; if all 20 come up heads, Bill wins a million dollars. The first 19 coins come up heads. What is the probability that Bill will win?

    Answer: $\mathbf{\frac{1}{2}}$

    The first 19 flips are irrelevant. The probability that the last flip will be heads is still $\frac{1}{2}$.

18. As we know, a mathematician is a device for turning coffee into theorems. Nathan, a mathematician, can prove a theorem in six hours (given enough coffee, of which he has an infinite supply). Silas's invention, the Lemm-o-Matic 1729, can prove a theorem in five hours. Working together, how long will it take them to prove 100 theorems? Express your answer to the nearest hour.

    Answer: 273

    Nathan can proves theorems at a rate of $\frac{1}{6}$ theorems per hour, while the Lemm-o-Matic 1729 can prove theorems at a rate of $\frac{1}{5}$ theorems per hour. Working together, Nathan and the Lemm-o-Matic can prove $\frac{1}{5} + \frac{1}{6} = \frac{11}{30}$ theorems per hour. To prove 100 theorems, it will Nathan and the Lemm-o-Matic a total of $\frac{100}{11/30} \approx 273$ hours.

19. In the parliament of Pythonistan, the Silly Party controls N seats, and the Sensible Party controls 25 seats. The Silly members always vote yes on everything, but they need at least $\frac{2}{3}$ of the total members of parliament to vote yes in order to pass a bill. For some values of N, they will find that by kicking out one of their own members (and reducing the total membership by one), they will need fewer votes from the Sensible party to pass bills. How many such values of N are there?

    Answer: 0

    There are 0 such values of N. By kicking out one of their own members, the Silly party can at most reduce the number people needed to pass a bill by one, but at the same time, they reduce the number of voters in the Silly party by one. Thus, the number of votes needed from the Sensible party does not decrease.

20. Find the sum of the distinct real roots of $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$.

    Answer: 1

    You can see immediately that 1 is a root, so factor the equation as $(x - 1)(x^3 - 2x^2 + 2x - 1)$. The second factor still has 1 as a root, so factor it further as $(x - 1)^2(x^2 - x + 1)$. The second factor now has no real roots. The answer is 1, since 1 is the only real root.

21. Compute $1 + 2 + 3 + \ldots + 1000000$.

    Answer: 500000500000

    Pair the terms up: 1 pairs with 1000000, 2 pairs with 999999, etc. There are 500000 pairs, each of which sums to 1000001, so the sum is 500000500000.

22. I want to join several equilateral triangles along their edges (so that vertices coincide) to form a 7-sided polygon. What is the minimum number of triangles that I will need?

    Answer: 5

    Suppose we have just three triangles. The only way to join them (up to rotations and reflections) is to make an isosceles trapezoid. If we add a fourth triangle to this, we can get at most six sides. However, with a fifth triangle, we can get 7 sides. The arrangement that works comes from making a regular hexagon out of six triangles, then removing one of them.

23. An ellipse has semimajor axis 2 and semiminor axis 1. Find the distance between its foci.

    Answer: $2\sqrt{3}$

    The focus, at points $(c, 0)$ and $(-c, 0)$, satisfy the equation $a^2 - b^2 = c^2$ where a and b are the lengths of the semimajor and the semiminor axes. Thus, we have that $c^2 = 2^2 - 1^2 = 3$. The distance between the two foci are then $2\sqrt{3}$.

24. Find all positive integers n such that $n^4 + n^2 + 1$ is prime.

    Answer: 1

    The expression factors as $(n^2 - n + 1)(n^2 + n + 1)$. If it is to be prime, one of these must equal 1. But the only way that can happen, for positive integers n, is if $n = 1$.

25. I flip 4 fair coins and eat all of the coins that land on tails. I flip all the uneaten coins again and then eat all the coins that land on tails. What is the probability that I have eaten at least 3 coins?

    Answer: $\frac{189}{256}$

    Probability is independent, so the probability that any given coin is eaten is $1 - (1/2 * 1/2) = 3/4$. The probability of getting 3 or 4 coins is $\binom{4}{3}(3/4)^3(1/4) + \binom{4}{4}(3/4)^4 = 108/256 + 81/256 = 189/256$.

26. Cody can eat a 2 meter diameter pizza in 1 minute. Frank can eat a 2 meter diameter pizza in 2 minutes. Jeffrey can eat a 2 meter diameter pizza in 3 minutes. If they combine their powers together, how many minutes does it take them to eat one pizza of diameter 6?

    Answer: $\frac{54}{11}$ minutes

    The area of pizza Cody, Frank, and Jeffrey can eat per minute are π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, respectively. So together, they can eat $\pi + \frac{\pi}{2} + \frac{\pi}{3} = \frac{11\pi}{6}$ square meters of pizza per minute. Thus, it takes $\frac{6}{11\pi} \times 9\pi = \frac{54}{11}$ minutes to eat a pizza of area 9π.

27. How many consecutive zeros occur at the end of the decimal expansion of $(8!)!$?.

    Answer: 10076

    The answer is the number of times $8! = 40320$ is divisible by 5. This is equal to $\lfloor \frac{40320}{5} \rfloor + \lfloor \frac{40320}{25} \rfloor + \lfloor \frac{40320}{125} \rfloor + ...$, which is $8064 + 1612 + 322 + 64 + 12 + 2 = 10076$.