1. The sum of all of the interior angles of seven polygons is $180 \cdot 17$. Find the total number of sides of the polygons.

   Answer: 31

   The angle sum of a polygon with n sides is $180(n-2)$. The total sum is then $180(n_1-2)+180(n_2-2)+\cdots+180(n_7-2) = 180(n_1+n_2+\cdots n_7) - 7\cdot 2\cdot 180 = 180\cdot 17$. Dividing through by 180 gives $n_1+n_2+\cdots+n_7-14 = 17$, so the total number of sides $14+17 = 31$.

2. The pattern in the figure below continues inward infinitely. The base of the biggest triangle is 1. All triangles are equilateral. Find the shaded area.

   Answer: $\frac{\sqrt{3}}{5}$

   Rank the shaded triangles by their area, largest to smallest. The largest shaded triangle has an area of $\frac{\sqrt{3}}{16}$. There are three of them, call them the first set. The i^{th} set of triangles has base $\frac{1}{4}$ that of the $(i-1)^{th}$ set, so the area of the i^{th} set is $\frac{1}{16}$ that of the of the $(i-1)^{th}$ set. So the total shaded area becomes an infinite geometric series:

$$\frac{\sqrt{3}}{16}\cdot 3\cdot\left(\frac{1}{1-\frac{1}{16}}\right) = \frac{\sqrt{3}}{5}.$$

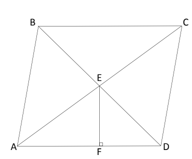
3. Given a regular pentagon, find the ratio of its diagonal, d, to its side, a.

   Answer: $2\cos(36)$

   Consider the triangle formed by the diagonal and two sides of the pentagon. The interior angle of the pentagon is $108°$, so the other two angles are both $36°$. By the law of sines, $\frac{\sin(36)}{a} = \frac{\sin(108)}{d}$. Thus, $\frac{d}{a} = \frac{\sin(108)}{\sin(36)} = \frac{\sin(72)}{\sin(36)} = \frac{\sin((2)(36))}{\sin(36)} = 2\cos(36)$.

4. $ABCD$ form a rhobus. E is the intersection of AC and BD. F lie on AD such that $EF \perp FD$. Given $EF = 2$ and $FD = 1$. Find the area of the rhobus $ABCD$.

   Answer: 20



   By the Pythogorean theorem, $ED = \sqrt{5}$. Since $ABCD$ is a rhombus, $AE \perp ED$. So triangle $\triangle FDE \sim \triangle EDA$. Thus we obtain the following ratio:

$$\frac{DF}{ED} = \frac{EF}{AE}$$
$$\frac{1}{\sqrt{5}} = \frac{2}{AE}.$$

   So $AE = 2\sqrt{5}$. Thus, the area is $\frac{1}{2}(2\times AE)(2\times DE) = \frac{1}{2}(4\sqrt{5})(2\sqrt{5}) = 20$.

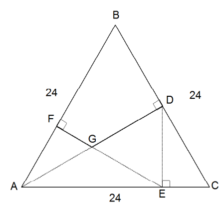
5. In the 2009 Rice Olympics, Willy and Sammy are two bikers. The circular race track has two lanes, the inner lane with radius 11, and the outer with radius 12. Willy will start on the inner lane, and Sammy on the outer. They will race for one complete lap, measured by the inner track. What is the square of the distance between Willy and Sammy's starting positions so that they will both race the same distance? Assume that they are of point size and ride perfectly along their respective lanes.

   Answer: $\mathbf{265 - 132\sqrt{3}}$

   Denote r_1 the inner radius and r_2 the outer radius. Then the inner lane has distance $2\pi r_1$ and outer lane $2\pi r_2$. But since Sammy will only be racing for $2\pi r_1$, there is $2\pi(r_2 - r_1)$ distance along the outer lane which he will skip. Let W denote Willy's starting position, S Sammy's starting position, and O the origin of the circular race track. Let θ be the angle between WO and SO. Then $2\pi(r_2-r_1) = \frac{\theta}{360}(2\pi r_2)$. Plugging in $r_1 = 11$ and $r_2 = 12$ and solving for θ, we get $\theta = 30°$. Using coordinate geometry, $W = (11, 0)$ and $S = (12\cos 30, 12\sin 30) = (6\sqrt{3}, 6)$. Thus, $(WS)^2 = (11-6\sqrt{3})^2 + 36 = 265 - 132\sqrt{3}$.

6. Equilateral triangle ABC has side length of 24. Points D, E, F lie on sides BC, CA, AB such that $AD \perp BC$, $DE \perp AC$, and $EF \perp AB$. G is the intersection of AD and EF. Find the area of the quadrilateral $BFGD$.

   Answer: $\mathbf{\frac{117\sqrt{3}}{2}}$



   AD bisects BC since $\triangle ABC$ is equilateral, so $CD = 12$. $\triangle ACD$ is a 30-60-90 degree triangle, so $AD = 12\sqrt{3}$. Likewise, $\triangle DCE$ is also 30-60-90, so $EC = 6$ and $ED = 6\sqrt{3}$. So $AE = AC - EC = 24 - 6 = 18$. $\triangle EAF$ is also 30-60-90, so $AF = 9$ and $EF = 9\sqrt{3}$. Since $\angle AEG = \angle AEF = 30°$, $\angle GED = 60°$. Likewise, $\angle CDE = 30°$ implies $EDG = 60°$. So $\angle DGE$ must also be $60°$ and $\triangle GED$ is equilateral, so $EG = GD = ED = 6\sqrt{3}$. $FG = EF - EG = 9\sqrt{3} - 6\sqrt{3} = 3\sqrt{3}$. $FB = AB - AF = 24 - 9 = 15$ and $BD = BC - CD = 12$. So area of quadrilateral $BFGD$ is $area\triangle BFG + area\triangle BDG = \frac{1}{2}(3\sqrt{3})(15) + \frac{1}{2}(6\sqrt{3})(12) = \frac{117\sqrt{3}}{2}$.

7. Four disks with disjoint interiors are mutually tangent. Three of them are equal in size and the fourth one is smaller. Find the ratio of the radius of the smaller disk to one of the larger disks.

   Answer: $\mathbf{\sqrt{3} - \frac{3}{2}}$

   Let r be the radius of the three largest circles and s be the radius of the smallest circles. Consider the equilateral triangle $\triangle ABC$ formed by the centers of the three largest circles. This triangle has side length $2r$ and altitude $r\sqrt{3}$. Let O be the center of the smallest circle, and consider altitude AM, passing through O. $AO = r + s = \frac{2}{3}AM$, so the altitude is also $\frac{3}{2}(r + s)$. Equating these and solving for the ratio gives $\frac{s}{r} = \frac{2\sqrt{3}}{3} - 1$.

8. Three points are randomly placed on a circle. What is the probability that they lie on the same semicircle?

   Answer: $\mathbf{\frac{3}{4}}$

   Suppose the first two points are separated by an angle α. Note that α is therefore randomly chosen between 0 and π. If we are to place the third point so that the three lie on the same semicircle, we have an arc of measure $2\pi - \alpha$ to choose from. The probability of this placement is therefore $1 - \frac{\alpha}{2\pi}$. This varies evenly from 1 at $\alpha = 0$ to $\frac{1}{2}$ at $\alpha = \pi$. The average is therefore $\frac{3}{4}$.

9. Two circles with centers A and B intersect at points X and Y. The minor arc $\angle XY = 120°$ with respect to circle A, and $\angle XY = 60°$ with respect to circle B. If $XY = 2$, find the area shared by the two circles.

   Answer: $\frac{10\pi - 12\sqrt{3}}{9}$**, or** $\frac{10\pi}{9} - \frac{4\sqrt{3}}{3}$

   $\angle XAY = 120°$, so the radius of circle A is $\frac{2\sqrt{3}}{3}$. $\angle XBY$ in $60°$, so the radius of circle B is 2. The area of the sector AXY is $\frac{1}{3}$ the area of circle A, so the area formed between segment XY and arc XY in circle A is the area of sector AXY inus the area of $\triangle AXY$.

$$\frac{1}{3}\pi\left(\frac{2\sqrt{3}}{3}\right)^2 - \frac{1}{2}\cdot 2\cdot\frac{\sqrt{3}}{3} = \frac{4\pi - 3\sqrt{3}}{9}.$$

   Similarly, sector BXY is $\frac{1}{6}$ of the area of circle B, so the area formed between segment XY and arc XY in circle B is

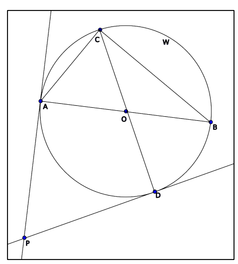
$$\frac{1}{6}\pi(2)^2 - (2)^2\frac{\sqrt{3}}{4} = \frac{2\pi - 3\sqrt{3}}{3}.$$

   The total area shared by the two circles is then:

$$\frac{4\pi - 3\sqrt{3}}{9} + \frac{2\pi - 3\sqrt{3}}{3} = \frac{10\pi - 12\sqrt{3}}{9}$$

10. Right triangle ABC is inscribed in circle W. $\angle CAB = 65°$, and $\angle CBA = 25°$. The median from C to AB intersects W at D. Line l_1 is drawn tangent to W at A. Line l_2 is drawn tangent to W at D. The lines l_1 and l_2 intersect at P. Compute $\angle APD$.

    Answer: 50°



    Note that $\angle ACB = 90°$, so AB must be the diameter of W. Then CO is the median from C to AB, where O is the origin of W, and CD passes through O. Then $CO = BO$ and $\angle BCD = \angle CBA = 25°$. We calculate $\angle COB = 180° - 2 \times 25° = 130°$. Then $\angle AOD = 130°$. Consider the quadrilateral $PDOA$. $\angle P = 360° - \angle BAD - \angle CDP - \angle AOD = 360° - 90° - 90° - 130° = 50°$.