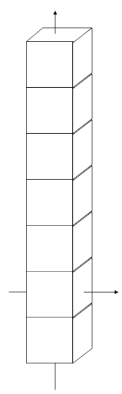
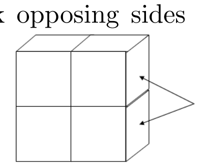
Power Question Solutions

1. **Answer: 105**. Since the tower is of odd height, it is now the case that the top and bottom faces are opposite faces, so every opposite pair of faces sums to 7. Each of the 7 dice has 2 pairs of faces showing horizontally, plus the one vertical pair up and down the tower. This gives (15*7) =

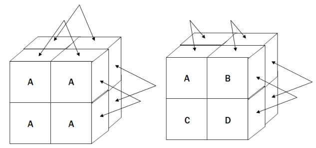


105.

2. **Answer: 56**. Having all 4 panels of the 2x2 face match implies that adjacent die are rotated by 180 degrees. The blocks along the outside (not the 2x2 face) therefore have the property that any neighboring blocks on a 1x2 face contain opposite die sides. In other words, the arrows in the accompanying figure point to blocks that sum to 7. There are 4 such pairs of adjacent blocks and 4 pairs of front/back opposing sides on the 2x2 face, leading to a total sum of (4+4)*7 = 56.



3. **Answer: {64, 72, 80, 88, 96, 104} and {68, 76, 92, 100}** . There are two families of solutions. The first is to take two blocks from Problem 2, flip one over, and stack them on top of each other. This gives 8 pairs of matching sides but leaves the two 2x2 faces are all equal and allows their die value to be set arbitrarily. The sum of all faces can take on the values of $56+8i, i = 1, \ldots, 6$. This gives the first family in the answer. The other valid arrangement, shown on the right, attempts to form a 2x2 face with every die different. The other faces have matching pairs (the arrows in the diagram on the right show equal numbered faces rather than opposite

valued faces). The end result will be 8 opposite pairs that sum to 7 plus four pairs that are 90 degrees away from each other on the die rather than opposites. This gives the second family of values in the solution, plus some redundant values with the first family. 

4a. For A to be better than B, we must have $x > 3$. For D to be better than A, we must have $y > x$. For C to be better than D, we must have $y < 6$. From these we get that the only option is $x = 4$ and $y = 5$.

4b. Since B is always going to roll 3, the winning number die must roll at least 3. So if C rolls 6, C wins, and otherwise must lose. The probability of C winning is $\frac{1}{3}$. For D to win, C must not roll 6, and D must roll 5. The probability of this is $\frac{2}{3} * \frac{1}{2} = \frac{1}{3}$ also. For A to win, neither of these must happen, and A must roll 4. This happens with probability $\frac{1}{3} * \frac{2}{3} = \frac{2}{9}$. The remainder, $\frac{1}{9}$, is the probability that B will win.

5a. One way to do it (there are many) is:

A: 1, 6, 8

B: 3, 5, 7

C: 2, 4, 9

5b. Suppose WLOG that the 9 goes on die A. This already gives A a $\frac{1}{3}$ chance of winning, so we must make sure A can't win unless it rolls a 9. However, when A doesn't roll a 9, B and C must have an equal chance of winning overall. This means that when only B and C are rolled, each has an equal chance of winning. Thus B is not better than C, a contradiction.

5c. Here's one way to do it:

A: 3, 3, 3

B: 5, 2, 2

C: 4, 4, 1

6a. You can do it like this:

D_1: 1, $4n+2$, $6n+2$, $8n+2$, ..., $2n(2n+1)+2$

D_2: 2, $2n+2$, $6n+3$, $8n+3$, ..., $2n(2n+1)+3$

D_3: 3, $2n+3$, $4n+3$, $8n+4$, ..., $2n(2n+1)+4$

D_4: 4, $2n+4$, $4n+4$, $6n+4$, ..., $2n(2n+1)+5$

...

D_{2n}: $2n$, $4n$, $6n$, $8n$, ..., $2n(2n+1)+(2n+1)$

D_{2n+1}: $2n+1$, $4n+1$, $6n+1$, $8n+1$, ..., $2n(2n+1)+1$

Notice how in each column, we write down the next $2n+1$ integers, but each time we start one row lower. Each die in this set is better than the previous n in the list (wrapping around if necessary) and better than the following n.

6b. Here is a way to do it:

A: 1, 9, 14

B: 2, 10, 12

C: 3, 6, 15

D: 4, 7, 13

E: 5, 8, 11

6c. Construct the dice like this:

D_1: 1, $3n+3$, $6n+2$

D_2: 2, $3n+4$, $6n$

D_2: 3, $3n+5$, $6n-2$

...

D_n: n, $4n+2$, $4n+4$

D_{n+1}: $n+1$, $2n+2$, $6n+3$

D_{n+2}: $n+2$, $2n+3$, $6n+1$

D_{n+3}: $n+3$, $2n+4$, $6n-1$

...

D_{2n}: $2n$, $3n+1$, $4n+5$

D_{2n+1}: $2n+1$, $3n+1$, $4n+3$

The procedure we've used here is as follows: First, write down the numbers from 1 to $2n+1$, going straight down the list of dice. Then do the same with the numbers from $2n+2$ to $4n+2$, but start at die D_{n+1}. Finally, fill in the numbers from $4n+3$ to $6n+3$, but as you go down the list, write the even numbers first in descending order, followed by the odd numbers in descending order. In this way, each die is better than the n that precede it, and is worse than the n that follow it.

7. Let the dice be:

A: a_1, a_2, ..., a_6

B: b_1, b_2, ..., b_6

C: c_1, c_2, ..., c_6

D: d_1, d_2, ..., d_6

Assume each of the dice has the numbers on its faces written in descending order (so $a_1 > a_2 > ... > a_6$, etc). Consider just A and B. Suppose we want to make A beat B with a probability of greater than $\frac{2}{3}$. There are 36 possible outcomes of rolling both dice, so A must win at least 25 of these. Suppose $a_4 < b_4$. Then A can roll a_4 and win in at most two of the 36 cases. Since a_5 and a_6 are even worse, they can also account for at most two winning cases for A. And obviously, a_1, a_2, and a_3 can only account for six winning cases each. This is a total of 24, which is not enough. This shows that we must have $a_4 > b_4$.

But the same argument applies to B and C, C and D, and D and A. So we find that we need $a_4 > b_4 > c_4 > d_4 > a_4$, which is impossible.

8. Suppose n is even. Bob's score will always be nx, which is less than n. If Alice scores n or better, she will win. But this requires Alice to roll 2 on at least half of her rolls, which she happens with probability greater than $\frac{1}{2}$. So if n is even, Alice always has the better chance of winning.

   Now suppose n is odd. To score n or better (which would guarantee her a win), Alice must roll 2 on more than half of her rolls. Since n is odd, the probability of this occurring is exactly $\frac{1}{2}$. If Alice also wins when she rolls 2 on $\frac{n-1}{2}$ of her n rolls (i.e. just under half), then she has the better chance of winning. Otherwise, they have an equal chance of winning. The condition for them to have equal chances is that $nx > 2 * \frac{n-1}{2}$, or $x > \frac{n-1}{n}$.

   To summarize, Bob never has a greater chance of winning. They have equal chances if n is odd and $x > \frac{n-1}{n}$, and otherwise Alice has the better chance.