1. In the future, each country in the world produces its Olympic athletes via cloning and strict training programs. Therefore, in the finals of the 200 m free, there are two indistinguishable athletes from each of the four countries. How many ways are there to arrange them into eight lanes?

   **Answer: 2520**

$$\frac{8!}{(2!)^4} = \frac{7!}{2} = 2520$$

2. Factor completely the expression $(a-b)^3 + (b-c)^3 + (c-a)^3$.

   **Answer: $3(a-b)(b-c)(c-a)$**

   The expression is zero when any two of $a$, $b$, and $c$ are equal. So it must have $(a-b)(b-c)(c-a)$ as a factor. But the original polynomial is degree 3, and so is this one, so the remaining factor must be a constant. The original polynomial contains a term $3ab^2$, but $(a-b)(b-c)(c-a)$ only contains a term $ab^2$, so the constant must be 3.

3. If $x$ and $y$ are positive integers, and $x^4 + y^4 = 4721$, find all possible values of $x+y$.

   **Answer: 13**

   Consider the equation modulo 5. All fourth powers are either 0 or 1 mod 5. So one of $x$ and $y$ must be divisible by 5; suppose it's $x$. Then we must in fact have $x = 5$, since $x = 10$ is too large. This gives $y = 8$, and this is the only possible solution. So the answer is 13.

4. How many ways are there to write 657 as a sum of powers of two where each power of two is used at most twice in the sum? For example, 256+256+128+16+1 is a valid sum.

   **Answer: 41**

   A recursion relationship describing this problem is
   $a_1 = 1, a_2 = 2, a_{2n} = a_n + a_{n-1}, a_{2n+1} = a_n$
   where $a_n$ is the number of valid sums for n. Thus,
   $a_{657} = a_{328} = a_{164} + a_{163} = a_{82} + 2a_{81} = a_{41} + 3a_{40} = 3a_{19} + 4a_{20}$
   $= 4a_{10} + 7a_9 = 4a_5 + 11a_4 = 4a_2 + 11a_4 = 4 \cdot 2 + 11 \cdot 3 = 41.$

5. Compute

$$\int_0^\infty t^5 e^{-t} dt$$

   **Answer: 120**

   Define $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$. Using integration by parts,

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty t^n e^{-t} dt \\ &= -t^n e^{-t}|_0^\infty + \int_0^\infty n t^{n-1} e^{-t} dt \\ &= 0 + n \int_0^\infty t^{n-1} e^{-t} dt \\ &= n\Gamma(n). \end{aligned}$$

   Next we evaluate $\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}|_0^\infty = 0 - -1 = 1$. Thus, $\Gamma(n+1) = n\Gamma(n) = \ldots = n!\Gamma(1) = n!$. So for the problem, $\Gamma(6) = 5! = 120$.

6. Rhombus ABCD has side length 1. The size of $\angle A$ (in degrees) is randomly selected from all real numbers between 0 and 90. Find the expected value of the area of ABCD.

   **Answer: $\frac{2}{\pi}$**

$$\begin{aligned}\text{Area of Rhombus } ABCD &= 4 * \frac{1}{2} * \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ &= 2 * \cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta\end{aligned}$$

$$\begin{aligned}\mathbf{E}[\text{Rhombus } ABCD] &= \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \sin\theta\, dx \\ &= \frac{2}{\pi} * 1 \\ &= \frac{2}{\pi}.\end{aligned}$$

7. An isosceles trapezoid has legs and shorter base of length 1. Find the maximum possible value of its area.

   **Answer:** $\mathbf{\frac{3\sqrt{3}}{4}}$

   Let the angle between the longer base and the leg be $\theta$.
   The Area of the Trapezoid $\Delta(\theta) = \sin\theta + \sin\theta * \cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$
   The area reaches extrema when its derivative is zero:
   $\Delta' = \cos\theta + \cos 2\theta = 0$
   We use the formula $\cos 2\theta = 2 * \cos^2\theta - 1$
   $2 * \cos^2\theta + \cos\theta - 1 = 0$
   $\cos\theta = \dfrac{-1 \pm \sqrt{9}}{4} = \dfrac{1}{2}$ or $-1$ (omitted)
   $\sin\theta = \dfrac{\sqrt{3}}{2}$
   $\Delta_{Max} = \boxed{\frac{3\sqrt{3}}{4}}$

8. Simplify

$$\sum_{k=1}^{n} \frac{k^2(k-n)}{n^4}.$$

   **Answer:** $\mathbf{\frac{-n^2+1}{12n^2}}$

$$\begin{aligned}\sum_{k=1}^{n} \frac{k^2(k-n)}{n^4} &= \sum_{k=1}^{n} \frac{k^3 - k^2 n}{n^4} \\ &= \sum_{k=1}^{n} \frac{k^3}{n^4} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\ &= \frac{1}{n^4}\sum_{k=1}^{n} k^3 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\ &= \left(\frac{1}{n^4}\right)\left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{1}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{n^4 + 2n^3 + n^2}{4n^4} - \frac{2n^3 + 3n^2 + n}{6n^3} \\ &= \frac{-n^4 + n^2}{12n^4} \\ &= \frac{-n^2 + 1}{12n^2}.\end{aligned}$$

9. Find the shortest distance between the point (6,12) and the parabola given by the equation $x = \frac{y^2}{2}$.

**Answer:** $\mathbf{2\sqrt{17}}$

Find the point on the parabola closest to the point (6,12). Call it $(x, y)$ This point is where the normal line at $x$ crosses the parabola. We find the derivative by:

$$\begin{aligned} x &= y^2 \\ dx &= y dy \\ \frac{dy}{dx} &= \frac{1}{y} \end{aligned}$$

The normal line will have slope of $-y$. It will contain $(6, 12)$. Its equation is:

$$\begin{aligned} y - 12 &= -y(x - 6) \\ y &= -xy + 6y + 12 \\ y &= -\frac{y^3}{2} + 6y + 12 \\ 2y &= -y^3 + 12y + 24 \\ 0 &= y^3 - 10y - 24 \end{aligned}$$

The roots are 4 and two other imaginary answers, so 4 is the only one that works.

$$\begin{aligned} y - 12 &= -y(x - 6) \\ -8 &= -4(x - 6) \\ x &= 8 \end{aligned}$$

Find the distance between $(8, 4)$ and $(6, 12)$. The answer is $2\sqrt{17}$.

10. Evaluate $\displaystyle\sum_{n=2009}^{\infty} \frac{\binom{n}{2009}}{2^n}$.

**Answer: 2**

More generally, define a function $G$ by

$$G(m) = \sum_{n=m}^{\infty} \frac{\binom{n}{m}}{2^n}.$$

Thus we wish to evaluate $G(2009)$. Observe that for all $m \geq 1$:

$$\begin{aligned} G(m) &= \sum_{n=m}^{\infty} \frac{\binom{n}{m}}{2^n} \\ &= \sum_{n=m}^{\infty} \frac{\binom{n-1}{m-1} + \binom{n-1}{m}}{2^n} \\ &= \frac{1}{2} \sum_{n=m-1}^{\infty} \frac{\binom{n}{m-1}}{2^n} + \frac{1}{2} \sum_{n=m-1}^{\infty} \frac{\binom{n}{m}}{2^n} \\ &= \frac{1}{2}(G(m-1) + G(m)) \end{aligned}$$

And thus $G(m) = G(m-1)$. Thus it suffices to evaluate $G(0)$. However, this is simply a geometric series:

$$\begin{aligned} G(0) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= 2. \end{aligned}$$

NOTE: By noticing that $\binom{n}{2009}$ is $\frac{1}{2009!}n^{2009}$ asymptotically, one can see this summation as a discrete analogue of the Euler $\Gamma$ function, which is defined by $\Gamma(x) = \int_0^\infty \frac{t^{x-1}}{e^t}dt$. The solution above is similar to the proof that $\Gamma(n+1) = n\Gamma(n)$.

11. Let $z_1$ and $z_2$ be the zeros of the polynomial $f(x) = x^2 + 6x + 11$. Compute $(1 + z_1^2 z_2)(1 + z_1 z_2^2)$.

**Answer: 1266**

$$\begin{aligned} (1 + z_1^2 z_2)(1 + z_1 z_2^2) &= 1 + z_1^2 z_2 + z_1 z_2^2 + z_1^3 z_2^3 \\ &= 1 + z_1 z_2(z_1 + z_2) + (z_1 z_2)^3. \end{aligned}$$

Since $z_1 + z_2 = -6$ and $z_1 z_2 = 11$,

$$\begin{aligned} (1 + z_1^2 z_2)(1 + z_1 z_2^2) &= 1 + 11(-6) + 11^3 \\ &= 1266. \end{aligned}$$

12. A number N has 2009 positive factors. What is the maximum number of positive factors that $N^2$ could have?

**Answer: 13689**

$2009 = 7^2 \times 41$

We know for a number $n = a_1^{\alpha_1} \times a_2^{\alpha_2} \times \ldots \times a_n^{\alpha_n}$, it has $(\alpha_1 + 1) \times (\alpha_2 + 1) \times \ldots (\alpha_n + 1)$ factors.
Hence, for number N, we have the following options:
$\alpha_1 = 7 - 1 = 6$, $\alpha_2 = 7 \times 41 - 1 = 289 - 1 = 288$
$\alpha_1 = 7 - 1 = 6$, $\alpha_2 = 7 - 1 = 6$, $\alpha_3 = 41 - 1 - 40$
By the same fact mentioned above, $N^2$ has: $(2 * \alpha_1 + 1) \times (2 * \alpha_2 + 1) \times \ldots (2 * \alpha_n + 1)$ factors.
Calculating this number for both, we get the 2nd option gets us a bigger number: $13 \times 13 \times 81 = \boxed{13689}$

13. Find the remainder obtained when $17^{289}$ is divided by 7?

**Answer: 3**

$$\begin{aligned} &17^{289} \equiv (14+3)^{289} \equiv \binom{289}{1} 14^{288} 3 + \ldots + \binom{289}{n} 13^{289-n} 3^n + \ldots \\ &3^{289} \equiv 3^{289} (\mod 7) \end{aligned}$$

Note that $3^3 \equiv 27 \equiv -1(\mod 7)$. Then $3^{289} \equiv 3^{3\dot{9}6}\dot{3}^1 \equiv (-1)^{96}\dot{3}^1 \equiv 3 \mod 7$. Thus, the remainder is 3.

14. Let $a$ and $b$ be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

**Answer: 17**

equation modulo 23, we get $-6(a-b) \equiv -10 \; (\mod 23)$. Since -4 is an inverse of -6 modulo 23, then we multiply to get $(a-b) \equiv 17 \; (\mod 23)$. Therefore, the smallest possible positive value for (a-b) is 17. This can be satisfied by $a = 5$, $b = -12$.

15. What is the largest integer n for which $\frac{2008!}{31^n}$ is an integer?

**Answer: 66**

$\lfloor \frac{2008}{31} \rfloor + \lfloor \frac{2008}{31^2} \rfloor + \lfloor \frac{2008}{31^3} \rfloor + \lfloor \frac{2008}{31^4} \rfloor + \cdots = 64 + 2 + 0 + 0 + \cdots = 66.$