1. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

   Answer: $\mathbf{\frac{1}{\sqrt{2}}}$

   From $a + b = 2\frac{\sqrt{3}}{3}i$ we can let $a = \frac{\sqrt{3}}{3}i + x$ and $b = \frac{\sqrt{3}}{3}i - x$. Then $a^2 + b^2 = 2((\frac{\sqrt{3}}{3}i)^2 + x^2) = 2(x^2 - \frac{1}{3}) = \frac{2\sqrt{3}}{3}i$. So $x^2 = \frac{1+\sqrt{3}i}{3} = \frac{2}{3}e^{i\pi/3}$, $x = \pm\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}+i}{2}$. Since $|\operatorname{Re}(a)| = |\operatorname{Re}(x)|$, the answer is $\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$.

2. Consider the curves $x^2 + y^2 = 1$ and $2x^2 + 2xy + y^2 - 2x - 2y = 0$. These curves intersect at two points, one of which is $(1, 0)$. Find the other one.

   Answer: $\left(-\frac{3}{5}, \frac{4}{5}\right)$

   From the first equation, we get that $y^2 = 1 - x^2$. Plugging this into the second one, we are left with

$$
\begin{aligned}
2x^2 \pm 2x\sqrt{1-x^2} + 1 - x^2 - 2x \mp 2\sqrt{1-x^2} = 0 &\Rightarrow (x-1)^2 = \mp 2\sqrt{1-x^2}(x-1) \\
&\Rightarrow x - 1 = \mp 2\sqrt{1-x^2} \text{ assuming } x \neq 1 \\
&\Rightarrow x^2 - 2x + 1 = 4 - 4x^2 \Rightarrow 5x^2 - 2x - 3 = 0.
\end{aligned}
$$

   The quadratic formula yields that $x = \frac{2\pm 8}{10} = 1, -\frac{3}{5}$ (we said that $x \neq 1$ above but we see that it is still valid). If $x = 1$, the first equation forces $y = 0$ and we easily see that this solves the second equation. If $x = -\frac{3}{5}$, then clearly y must be positive or else the second equation will sum five positive terms. Therefore $y = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Hence the other point is $\left(-\frac{3}{5}, \frac{4}{5}\right)$.

3. If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

   Answer: $\mathbf{\frac{9}{4}}$

   First notice that the polynomial

$$g(x) = x^4\left(\frac{1}{x^4} + \frac{3}{x^3} + \frac{3}{x} + 2\right) = 2x^4 + 3x^3 + 3x + 1$$

   is a polynomial with roots $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{t}$, $\frac{1}{u}$. Therefore, it is sufficient to find the sum of the squares of the roots of $g(x)$, which we will denote as r_1 through r_4. Now, note that

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = (r_1 + r_2 + r_3 + r_4)^2 - (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) = (-\frac{a_3}{a_4})^2 - \frac{a_2}{a_4}$$

   by Vieta's Theorem, where a_n denotes the coefficient of x^n in $g(x)$. Plugging in values, we get that our answer is $(-\frac{3}{2})^2 - 0 = \frac{9}{4}$.

4. Find the 2011th-smallest x, with $x > 1$, that satisfies the following relation:

$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

   Answer: $\boldsymbol{x = e^{2011\pi/5}}$

   Set $y = \ln x$, and observe that

$$2\cos(3y)\sin(2y) = \sin(3y + 2y) - \sin(3y - 2y) = \sin(5y) - \sin(y),$$

   so that the equation in question is simply

$$\sin(5y) = 0.$$

   The solutions are therefore

$$\ln x = y = \frac{n\pi}{5} \implies x = e^{n\pi/5} \quad \text{for all } n \in \mathbb{N}.$$

5. Find the remainder when $(x+2)^{2011} - (x+1)^{2011}$ is divided by x^2+x+1.

   Answer: $\mathbf{(-3^{1005}-1)x + (-2\cdot 3^{1005}-1)}$

   The standard method is to use the third root of unity ω, $\omega^2+\omega+1=0$. Let $(x+2)^{2011}-(x+1)^{2011} = (x^2+x+1)Q(x)+ax+b$ and substitute $x=\omega$. Then $a\omega+b=(\omega+2)^{2011}-(\omega+1)^{2011}$. Note that $\omega+2$ has size $\sqrt{3}$ and argument $\pi/6$, so $(\omega+2)^6=-3^3$. Also $\omega+1$ has magnitude 1 and argument $\pi/3$, so $(\omega+1)^6=1$. Using this and $2011 = 6\cdot 335+1$, we get that $a\omega+b=(-3^{1005}-1)\omega+(-2\cdot 3^{1005}-1)$.

   Another solution is to note that $(x+2)^2 \equiv x^2+4x+4 \equiv -3x^2 \pmod{x^2+x+1}$ and $(x+1)^2 \equiv x^2+2x+1 \equiv x \pmod{x^2+x+1}$. Then we have $x^3\equiv 1 \pmod{x^2+x+1}$ and we can proceed by using periodicity.

6. There are 2011 positive numbers with both their sum and the sum of their reciprocals equal to 2012. Let x be one of these numbers. Find the maximum of $x+x^{-1}$.

   Answer: $\mathbf{\frac{8045}{2012}}$

   Let $y_1, y_2, \cdots, y_{2010}$ be the 2010 numbers distinct from x. Then $y_1+y_2+\cdots+y_{2010} = 2012-x$ and $\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_{2010}} = 2012-\frac{1}{x}$. Applying the Cauchy-Schwarz inequality gives

$$\left(\sum_{i=1}^{2010} y_i\right)\left(\sum_{i=1}^{2010}\frac{1}{y_i}\right) = (2012-x)(2012-\frac{1}{x}) \geq 2010^2$$

   so $2012^2 - 2012(x+x^{-1}) + 1 - 2010^2 \geq 0$, $x+x^{-1} \leq 8045/2012$.

7. Let $P(x)$ be a polynomial of degree 2011 such that $P(1)=0$, $P(2)=1$, $P(4)=2$, ... , and $P(2^{2011}) = 2011$. Compute the coefficient of the x^1 term in $P(x)$.

   Answer: $\mathbf{2-\frac{1}{2^{2010}}}$

   We analyze $Q(x)=P(2x)-P(x)$. One can observe that $Q(x)-1$ has the powers of 2 starting from $1, 2, 4, \cdots$, up to 2^{2010} as roots. Since Q has degree 2011, $Q(x)-1 = A(x-1)(x-2)\cdots(x-2^{2010})$ for some A. Meanwhile $Q(0)=P(0)-P(0)=0$, so

$$Q(0)-1=-1=A(-1)(-2)\cdots(-2^{2010}) = -2^{(2010\cdot 2011)/2}A.$$

   Therefore $A = 2^{-(1005\cdot 2011)}$. Finally, note that the coefficient of x is same for P and $Q-1$, so it equals $A(-2^0)(-2^1)\cdots(-2^{2010})((-2^0)+(-2^{-1})+\cdots+(-2^{-2010})) = \frac{A\cdot 2^{1005\cdot 2011}(2^{2011}-1)}{2^{2010}} = \boxed{2-\frac{1}{2^{2010}}}$.

8. Find the maximum of

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2}$$

   for reals a, b, c, and d not all zero.

   Answer: $\mathbf{\frac{\sqrt{5}+1}{4}}$

   One has $ab \leq \frac{t}{2}a^2 + \frac{1}{2t}b^2$, $bc \leq \frac{1}{2}b^2+\frac{1}{2}c^2$, and $cd \leq \frac{1}{2t}c^2+\frac{t}{2}d^2$ by AM-GM. If we can set t such that $\frac{t}{2} = \frac{1}{2t}+\frac{1}{2}$, it can be proved that $\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \leq \frac{\frac{t}{2}(a^2+b^2+c^2+d^2)}{a^2+b^2+c^2+d^2} = \frac{t}{2}$, and this is maximal because we can set a, b, c, d so that the equality holds in every inequality we used. Solving this equation, we get $t=\frac{1+\sqrt{5}}{2}$, so the maximum is $\frac{t}{2}=\frac{\sqrt{5}+1}{4}$.

9. It is a well-known fact that the sum of the first n k-th powers can be represented as a polynomial in n. Let $P_k(n)$ be such a polynomial for integers k and n. For example,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

   so one has

$$P_2(x) = \frac{x(x+1)(2x+1)}{6} = \frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{6}x.$$

Evaluate $P_7(-3) + P_6(-4)$.

Answer: −665

Since the equation

$$P_k(x) = P_k(x-1) + x^k$$

has all integers ≥ 2 as roots, the equation is an identity, so it holds for all x. Now we can substitute $x = -1, -2, -3, -4, \cdots$ to prove

$$P_k(-n) = -\sum_{i=1}^{n-1} (-i)^k$$

so $P_7(-3) + P_6(-4) = -(-1)^6 - (-2)^6 - (-3)^6 - (-1)^7 - (-2)^7 = -665$.

10. How many polynomials P of degree 4 satisfy $P(x^2) = P(x)P(-x)$?

Answer: 10

Note that if r is a root of P then r^2 is also a root. Therefore $r, r^2, r^{2^2}, r^{2^3}, \cdots$, are all roots of P. Since P has a finite number of roots, two of these roots should be equal. Therefore, either $r = 0$ or $r^N = 1$ for some $N > 0$.

If all roots are equal to 0 or 1, then P is of the form $ax^b(x-1)^{(4-b)}$ for $b = 0, ..., 4$.

Now suppose this is not the case. For such a polynomial, let q denote the largest integer such that $r = e^{2\pi i \cdot p/q}$ is a root for some integer p coprime to q. We claim that the only suitable $q > 1$ are $q = 3$ and $q = 5$.

First note that if r is a root then one of $\sqrt{r}$ or $-\sqrt{r}$ is also a root. So if q is even, then one of $e^{2\pi i \cdot p/2q}$ or $e^{2\pi i \cdot p+q/2q}$ should also be root of p, and both p/q and $(p+q)/2q$ are irreducible fractions. This contradicts the assumption that q is maximal. Therefore q must be odd. Now, if $q > 6$, then $r^{-2}, r^{-1}, r, r^2, r^4$ should be all distinct, so $q \leq 6$. Therefore $q = 5$ or 3.

If $q = 5$, then the value of p is not important as P has the complex fifth roots of unity as its roots, so $P = a(x^4 + x^3 + x^2 + x + 1)$. If $q = 3$, then P is divisible by $x^2 + x + 1$. In this case we let $P(x) = a(x^2 + x + 1)Q(x)$ and repeating the same reasoning we can show that $Q(x) = x^2 + x + 1$ or $Q(x)$ is of form $x^b(x-1)^{2-b}$.

Finally, we can show that exactly one member of all 10 resulting families of polynomials fits the desired criteria. Let $P(x) = a(x-r)(x-s)(x-t)(x-u)$. Then, $P(x)P(-x) = a^2(x^2 - r^2)(x^2 - s^2)(x^2 - t^2)(x^2 - u^2)$. We now claim that r^2, s^2, t^2, and u^2 equal r, s, t, and u in some order. We can prove this noting that the mapping $f(x) = x^2$ maps 0 and 1 to themselves and maps the third and fifth roots of unity to another distinct third or fifth root of unity, respectively. Hence, for these polynomials, $P(x)P(-x) = a^2(x^2 - r)(x^2 - s)(x^2 - t)(x^2 - u) = aP(x^2)$, so there exist exactly 10 polynomials that fit the desired criteria, namely the ones from the above 10 families with $a = 1$.