1. How many positive three-digit integers $\underline{a}\,\underline{b}\,\underline{c}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{b}\,\underline{c}$ corresponds to the day in that month, or $\underline{a}\,\underline{b}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

   Answer: 273

   Solution: The integers which are valid have a 1-1 correspondence to days in the first 9 months – this is straightforward to see for all positive integers that do not have a 1 in the hundreds place and just requires careful inspection of the case where 1 is in the hundreds place. There are $365 - 31 - 30 - 31 = \boxed{273}$ such days.

2. Consider the numbers $\{24, 27, 55, 64, x\}$. Given that the mean of these five numbers is prime and the median is a multiple of 3, compute the sum of all possible positive integral values of x.

   Answer: 60

   Solution: The restriction on the median means either $x \leq 27$ or $3 \mid x$ and $x < 55$. Hence, the sum of all five numbers is $24 + 27 + 55 + 64 + x = 170 + x < 225$, so the average is $> \frac{170}{5} = 34$ and $< \frac{225}{5} = 45$. The only prime numbers in this range are 37, 41, and 43, which yield $x = 15$, $x = 35$, or $x = 45$. 35 is greater than 27 but not a multiple of 3, so it doesn't work. Hence, the answer is $15 + 45 = \boxed{60}$.

3. Nick has a terrible sleep schedule. He randomly picks a time between 4 AM and 6 AM to fall asleep, and wakes up at a random time between 11 AM and 1 PM of the same day. What is the probability that Nick gets between 6 and 7 hours of sleep?

   Answer: $\mathbf{\frac{3}{8}}$

   Solution: Consider the rectangle with lower-left corner at $(4, 11)$ and upper-right corner at $(6, 13)$. We want to compute the probability that a randomly generated point inside the rectangle falls above the line $y = x + 6$ and below the line $y = x + 7$. These lines cut out a trapezoid of area $\frac{3}{2}$, so therefore the probability that a randomly generated point in the rectangle falls in this trapezoid is $\frac{\frac{3}{2}}{4} = \boxed{\frac{3}{8}}$.

4. Given the digits 1 through 7, one can form $7! = 5040$ numbers by forming different permutations of the 7 digits (for example, 1234567 and 6321475 are two such permutations). If the 5040 numbers obtained are then placed in ascending order, what is the 2013^{th} number?

   Answer: 3657214

   Solution: When the numbers are ordered, the first $6! = 720$ numbers all have 1 in the millions place value. The next 720 numbers all have 2 in the millions place value. The 2013^{th} number must then lie in the next batch, with 3 as the millions place value digit. Within the third batch of 720 numbers, the first $5! = 120$ have a 1 in the hundred thousands place value, the next 120 have a 2 in the hundred thousands place, the next 120 have a 4, and so on. Continuing in the same manner, we can deduce that the 2013^{th} number is $\boxed{3657214}$.

5. An unfair coin lands heads with probability $\frac{1}{17}$ and tails with probability $\frac{16}{17}$. Matt flips the coin repeatedly until he flips at least one head and at least one tail. What is the expected number of times that Matt flips the coin?

   Answer: $\mathbf{\frac{273}{16}}$

Solution: Let E be the desired expected value, E_h the expected number of additional flips given that the previous flip was a head, and E_t the expected number of additional flips given that the previous flip was a tail. Then we can write

$$E = \frac{1}{17}(E_t + 1) + \frac{16}{17}(E_h + 1)$$

$$E_t = \frac{16}{17} + \frac{1}{17}(E_t + 1)$$

$$E_h = \frac{1}{17} + \frac{16}{17}(E_h + 1)$$

where the second equation comes from the fact that if the previous flip was a tail and the next flip is a head, then the sequence is over, whereas if the next flip is also a tail then the situation is unchanged; similarly for the third equation. Solving the second and third equations gives $E_t = \frac{17}{16}$ and $E_h = 17$ (note as a potential shortcut that these are the reciprocals of the probabilities given in the problem), and plugging into the first equation gives $E = \boxed{\frac{273}{16}}$ as desired.

6. A positive integer $b \geq 2$ is *neat* if and only if there exist positive base-b digits x and y (that is, x and y are integers, $0 < x < b$ and $0 < y < b$) such that the number $x.y$ base b (that is, $x + \frac{y}{b}$) is an integer multiple of x/y. Find the number of *neat* integers less than or equal to 100.

Answer: 39

Solution: The constraint that $x.y$ is an integer multiple of x/y is equivalent to the claim that there exists an integer n such that

$$\frac{nx}{y} = x + \frac{y}{b} \implies nx = xy + \frac{y^2}{b} \implies x(n-y) = \frac{y^2}{b} \implies x = \frac{y^2}{b(n-y)}.$$

We see that b cannot be prime, since then $b \mid y^2$ would imply that $b \mid y \implies y \geq b$. In fact, for exactly the same reason, b cannot be the product of distinct primes.

We now claim that any b that is not the product of distinct primes is *neat*. Say b has one prime factor p that occurs $m > 1$ times in its prime factorization. Then, set $y = \frac{b}{p}$ and $n = y + 1$. $b \mid y^2$ because y^2 has a factor of p^{2m-2} and $m > 1 \implies 2m - 2 \geq m$, and all other prime factors of b are also clearly contained in y^2 in sufficient numbers. Finally, $x = \frac{y^2}{b} < \frac{b^2}{b} = b$ because $y < b$, so it is also a base-b digit.

Hence, we just need to count the number of integers less than or equal to 100 that have at least one prime factor repeated more than once. This prime factor can be either 2, 3, 5, or 7 (since $11^2 > 100$). We can count using the Principle of Inclusion-Exclusion: considering only positive integers greater than 1 and less than or equal to 100, there are 25 multiples of 2^2, 11 multiples of 3^2, 4 multiples of 5^2, and 2 multiples of 7^2. We've double-counted two multiples of 36 (36 and 72), as well as 100, but any other number that might be multiple of more than one of these squares would have to be too big. Hence, report $25 + 11 + 4 + 2 - 3 = \boxed{39}$.

7. Robin is playing notes on an 88-key piano. He starts by playing middle C, which is actually the 40th lowest note on the piano (i.e. there are 39 notes lower than middle C). After playing a note, Robin plays with probability $\frac{1}{2}$ the lowest note that is higher than the note he just played,

and with probability $\frac{1}{2}$ the highest note that is lower than the note he just played. What is the probability that he plays the highest note on the piano before playing the lowest note?

Answer: $\mathbf{\frac{13}{29}}$

Solution: Let a_i be the probability that Robin plays the highest note before the lowest note given a starting position of the ith lowest note. Clearly, $a_1 = 0$ and $a_{88} = 1$. Furthermore, for all intermediate i, we have that $a_i = \dfrac{a_{i-1} + a_{i+1}}{2}$. From here, it is apparent that $a_i = \dfrac{i-1}{87}$, so therefore $a_{40} = \boxed{\dfrac{13}{29}}$.

8. Farmer John owns 2013 cows. Some cows are enemies of each other, and Farmer John wishes to divide them into as few groups as possible such that each cow has at most 3 enemies in her group. Each cow has at most 61 enemies. Compute the smallest integer G such that, no matter which enemies they have, the cows can always be divided into at most G such groups?

   Answer: 16

   Solution: Let $N = 2013, E = 61, L = 3$. Suppose we have G groups. Consider a set of $E+1$ cows such that each cow is enemies with all E of the others. Each group can have at most $L+1$ of these cows, so a valid partition is not always possible if $(L+1)G < E+1$. Therefore, we must have $G \geq \frac{E+1}{L+1}$ for a valid partition to always exist.

   We will now prove that this is also a sufficient condition. Take any partition of the cows into G groups. Choose any cow with more than L enemies in her group. If all groups have more than L of her enemies, then $E \geq (L+1)G$. So if $E < (L+1)G \implies E+1 \leq (L+1)G \implies G \geq \frac{E+1}{L+1}$, then there exists a group with at most L of her enemies, and we can move her to this group. Making this move strictly decreases the total number of pairs of enemies within the groups, since the only affected pairs are those involving the moved cow, and we removed more than L pairs of enemies from the old group but created at most L in the new group. Therefore, we can repeatedly move a cow in a group with more than L of her enemies to a group with at most L of her enemies. This process cannot continue indefinitely since the number of pairs cannot decrease below 0, so it must yield a partition in which no cow has more than L enemies. Therefore, if $G \geq \frac{E+1}{L+1}$, then a valid partition is always possible.

   Therefore, the minimal number of groups such that a valid partition is always possible is $G = \left\lceil \frac{E+1}{L+1} \right\rceil = \left\lceil \frac{61+1}{3+1} \right\rceil = \boxed{16}$.

9. Big candles cost 16 cents and burn for exactly 16 minutes. Small candles cost 7 cents and burn for exactly 7 minutes. The candles burn at possibly varying and unknown rates, so it is impossible to predictably modify the amount of time for which a candle will burn except by burning it down for a known amount of time. Candles may be arbitrarily and instantly put out and relit. Compute the cost in cents of the cheapest set of big and small candles you need to measure exactly 1 minute.

   Answer: 58

   Solution: The way to achieve 58 is as follows: burn a big candle together with two small candles, one after the other, leaving one 2-minute candle. Burn the 2-minute candle together with two small candles, in parallel, leaving two 5-minute candles. Burn one of the 5-minute candles together with two small candles, leaving two 2-minute candles. Burn the other 5-minute candles together with two 2-minute candles, one after the other, leaving a 1-minute candle. That's 1 big candle and 6 small candles for $16 + 7 \cdot 6 = 58$ cents.

To motivate that we can see this quickly, note that $5 \cdot 7 - 2 \equiv 1 \pmod{16}$. Note that if we buy 5 small candles, 1 big candle, and then buy one extra small candle, we can make that small candle a 2-minute candle as outlined above and then be bought those additional 2 minutes so we can get a 1-minute candle.

To show that we can't do better, we just check a lot of possibilities. If we use 3 big candles, we can use 1 small candle. If we use 2 big candles, we can use up to 3 small candles. If we use 1 big candle, we can use up to 5 small candles. If we can show that it is impossible in all of these cases, then we are done.

Case 1: 3 big candles, 1 small candle. In this case, we can extract a 9-minute candle at best by burning a big candle and a small candle in parallel.

Case 2: 2 big candles, 3 small candles. In this case, we can extract a 9-minute candle at the cost of one small candle. This can get us a 2-minute candle, but we clearly can't extract a 1-minute candle as a consequence.

Case 3: 1 big candle, 5 small candles. We can burn one big candle and one small candle in parallel to get one 9-minute candle and four 7-minute candles. We could do this with one 7-minute candle and three 2-minute candles, but then we would need five 7-minute candles to begin with. Having more than one 9-minute candle is similarly ineffective.

Thus, the cheapest possible cost is $\boxed{58}$ cents.

10. Compute the number of positive integers b where $b \leq 2013$, $b \neq 17$, and $b \neq 18$ such that there exists some positive integer N such that $\frac{N}{17}$ is a perfect 17th power, $\frac{N}{18}$ is a perfect 18th power, and $\frac{N}{b}$ is a perfect bth power.

Answer: 652

Solution: We claim that b is a valid positive integer if it satisfies either of the following conditions:

(a) b is relatively prime to both 17 and 18

(b) $\dfrac{b}{2}$ is a perfect square relatively prime to 17 and 3.

There are 632 numbers that fit the first condition, and 20 additional numbers which don't satisfy the first condition that fit the second condition. This gives us an answer of $\boxed{652}$.

It remains to prove that these conditions are necessary and sufficient.

We first prove that, for any set of pairwise relatively prime integers $x_1, \ldots, x_n$, there exists some integer N such that $\dfrac{N}{x_i}$ is a perfect x_ith power for all x_i. This follows from the Chinese Remainder Theorem. Let n have prime factorization $p_1^{a_1} \ldots p_k^{a_k}$. We have n modular recurrences for each prime, each modulo being relatively prime, so by CRT, there exists some solution for the a_i and therefore some N exists.

This covers the first case. It remains to prove that the second case is a sufficient condition and that the two given conditions are necessary conditions over the given range of numbers.

To prove that the second case holds, note that the power of two in N must be $1 \pmod{18}$ and also $1 \pmod{2r^2}$, which is acceptable if r is relatively prime to 3 and 17 because then the exponents of 3 and 17 remain unaffected and there is no conflict on the parity of the exponent of 2.

It remains to show that no other integer is valid. Any other integer which is a scalar multiple of 17 will be multipled by some prime power p^k. It must be the case that the prime p must be 0 (mod 17) and also $k \pmod{17p^k}$, which is a contradiction unless p^k is a 17th power, but that is impossible in our desired range. The same logic holds for the scalars of 2 and 3. This completes the proof that no other integer is valid.