Time limit: 60 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions.
No calculators.

Short Answer Questions: For the short answer questions, all answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written inside the boxes on the answer sheet will be considered for grading.

1. Given $x + y = 7$, find the value of x that minimizes $4x^2 + 12xy + 9y^2$.

2. There are real numbers b and c such that the only x-intercept of $8y = x^2 + bx + c$ equals its y-intercept. Compute $b + c$.

3. Consider the set of 5 digit numbers $ABCDE$ (with $A \neq 0$) such that $A+B = C$, $B+C = D$, and $C + D = E$. What's the size of this set?

4. Let D be the midpoint of BC in $\triangle ABC$. A line perpendicular to D intersects AB at E. If the area of $\triangle ABC$ is four times that of the area of $\triangle BDE$, what is $\angle ACB$ in degrees?

5. Define the sequence $c_0, c_1, \ldots$ with $c_0 = 2$ and $c_k = 8c_{k-1} + 5$ for $k > 0$. Find $\lim_{k\to\infty} \frac{c_k}{8^k}$.

6. Find the maximum possible value of $|\sqrt{n^2 + 4n + 5} - \sqrt{n^2 + 2n + 5}|$.

7. Let $f(x) = \sin^8(x) + \cos^8(x) + \frac{3}{8}\sin^4(2x)$. Let $f^{(n)}(x)$ be the nth derivative of f. What is the largest integer a such that 2^a divides $f^{(2020)}(15^\circ)$?

8. Let $\mathbb{R}^n$ be the set of vectors $(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are all real numbers. Let $||(x_1, \ldots, x_n)||$ denote $\sqrt{x_1^2 + \ldots + x_n^2}$. Let S be the set in $\mathbb{R}^9$ given by

$$S = \{(x, y, z) : x, y, z \in \mathbb{R}^3, 1 = ||x|| = ||y - x|| = ||z - y||\}$$

If a point (x, y, z) is uniformly at random from S, what is $E[||z||^2]$?

9. Let $f(x)$ be the unique integer between 0 and $x - 1$, inclusive, that is equivalent modulo x to $\left(\sum_{i=0}^{2} \binom{x-1}{i}((x - 1 - i)! + i!)\right)$. Let S be the set of primes between 3 and 30, inclusive. Find $\sum_{x\in S} f(x)$.

10. In the Cartesian plane, consider a box with vertices $(0, 0), (\frac{22}{7}, 0), (0, 24), (\frac{22}{7}, 24)$. We pick an integer a between 1 and 24, inclusive, uniformly at random. We shoot a puck from $(0, 0)$ in the direction of $(\frac{22}{7}, a)$ and the puck bounces perfectly around the box (angle in equals angle out, no friction) until it hits one of the four vertices of the box. What is the expected number of times it will hit an edge or vertex of the box, including both when it starts at $(0, 0)$ and when it ends at some vertex of the box?

11. Sarah is buying school supplies and she has \$2019. She can only buy full packs of each of the following items. A pack of pens is \$4, a pack of pencils is \$3, and any type of notebook or stapler is \$1. Sarah buys at least 1 pack of pencils. She will either buy 1 stapler or no stapler. She will buy at most 3 college-ruled notebooks and at most 2 graph paper notebooks. How many ways can she buy school supplies?

12. Let O be the center of the circumcircle of right triangle ABC with $\angle ACB = 90^\circ$. Let M be the midpoint of minor arc $\widehat{AC}$ and let N be a point on line BC such that $MN \perp BC$. Let P be the intersection of line AN and the Circle O and let Q be the intersection of line BP and MN. If $QN = 2$ and $BN = 8$, compute the radius of the Circle O.

13. Reduce the following expression to a simplified rational:

$$\frac{1}{1-\cos\frac{\pi}{9}}+\frac{1}{1-\cos\frac{5\pi}{9}}+\frac{1}{1-\cos\frac{7\pi}{9}}$$

14. Compute the following integral:

$$\int_0^\infty \log\left(1+e^{-t}\right)dt.$$

15. Define $f(n)$ to be the maximum possible least-common-multiple of any sequence of positive integers which sum to n. Find the sum of all possible odd $f(n)$.