Comment: Version 1.1

RH12 1. Evaluate $\sqrt{2223^2 - 8888}$.

Answer: 2221

Solution: Note that $\sqrt{2223^2 - 8888} = \sqrt{(2222+1)^2 - 8888} = \sqrt{2222^2 + 4444 + 1 - 8888} = \sqrt{2222^2 - 4444 + 1} = \sqrt{(2222-1)^2} = \boxed{2221}$.

RH11 2. If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a + b$?

Answer: 202020

Solution: We have

$$2020^{202020-a} = a$$

$$2020^b = 202020 - b.$$

If $x = 202020 - a$, then $2020^x = 202020 - x$. Since b is the only real number that satisfies $2020^b = 202020 - b$, x must be equal to b. Therefore, $a + b = \boxed{202020}$.

SS04 3. Two cars driving from city A to city B leave at the same time. The first car drives at some constant speed during the whole trip. The second car travels at a speed 12 km/hr slower than the first car until the halfway point between city A and city B. After the halfway point, the second car travels at a constant speed of 72 km/hr. Both cars end up reaching city B at the same time. Calculate the speed of the first car in km/hr, given that it was faster than 40 km/hr.

Answer: 48

Solution: Let d be the distance between the cities and s be the speed of the first car in km/hrs. Then $\frac{d}{s}$ is the total time taken by the first car to get from city A to B. For the second car, $\frac{d/2}{s-12}$ is the time taken to get from city A to the halfway point and $\frac{d/2}{72}$ is the time taken to get from the halfway point to city B. Therefore,

$$\frac{d}{s} = \frac{d/2}{s-12} + \frac{d/2}{72}.$$

Dividing by $d/2$ and multiplying by the denominators,

$$144(s-12) = 72s + s(s-12).$$

Rearranging,

$$s^2 - 84s + 1728 = (s-36)(s-48) = 0,$$

so s must be either 36 or 48. However, given that the speed was greater than 40 km/hr, s, the speed of the first car in km/hrs, must be 48.

KW47 4. Find the value of bc such that $x^2 - x + 1$ divides $20x^{11} + bx^{10} + cx^9 + 4$.

Answer: −480

Solution: First we consider a polynomial $p(x) = p_9x^9 + p_8x^8 + ... + p_0$ such that $p(x)(x^2 - x + 1) = ax^{11} + bx^{10} + cx^9 + 4$. Clearly $p_0 = 4$. Then we can deduce since $-4x + p_1x = 0$ that $p_1 = 4$. Then we continue to see that $p_2x^2 - 4x^2 + 4x^2 = 0$, so $p_2 = 0$. Continuing this pattern we have that $p_3 = -4$, $p_4 = -4$, $p_5 = 0$, $p_6 = 4$, $p_7 = 4$, and $p_8 = 0$. Then $p_9x^9 + 4p_7x^9 = cx^9$, so $p_9 = c - 4$. However, we can see that since $x^2 - x + 1$ is monic, $a = p_9$. So, $a = c - 4$. In addition, $b = -a$. So, $bc = \boxed{-480}$.

WW19 5. Suppose $f(x)$ is a monic quadratic polynomial such that there exists an increasing arithmetic sequence $x_1 < x_2 < x_3 < x_4$ where $|f(x_1)| = |f(x_2)| = |f(x_3)| = |f(x_4)| = 2020$. Compute the absolute difference of the two roots of $f(x)$.

Answer: $10\sqrt{101}$

Solution: Suppose $f(x) = (x-h)^2 - k$ with vertex (h, k) where k is non-negative. Since the roots of $f(x)$ are $h \pm \sqrt{k}$, the absolute difference of the two roots of $f(x)$ is $2\sqrt{k}$. By symmetry, if $x_2 = h - d$, then $x_3 = h + d$, which means $x_1 = h - 3d$ and $x_4 = h + 3d$. Since $x_1 < x_2 < h$ and $f(x)$ is monic, $f(x_1) = f(x_4)$ must be positive and $f(x_2) = f(x_3)$ must be negative. Therefore, $f(x_1) = -f(x_2) = 2020$. Substituting, $9d^2 - k = -(d^2 - k)$, which implies $k = 5d^2$. Since $f(x_1) = 9d^2 - k = 4d^2 = 2020$, it follows that $k = 5d^2 = \frac{5}{4}(4d^2) = \frac{5}{4}(2020)$. Therefore, the absolute difference of the roots of $f(x)$ is $2\sqrt{k} = 5\sqrt{2020} = 10\sqrt{101}$.

HA01 6. Let $f : A \to B$ be a function from $A = \{0, 1, \ldots, 8\}$ to $B = \{0, 1, \ldots, 11\}$ such that the following properties hold:

$$f(x + y \mod 9) \equiv f(x) + f(y) \mod 12$$

$$f(xy \mod 9) \equiv f(x)f(y) \mod 12$$

for all $x, y \in A$. Compute the number of functions f that satisfy these conditions.

Answer: 2

Solution: Note $f(n) \equiv f(\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}}) = n * f(1) \mod 12$, so f can be completely determined if we know $f(1)$. In addition, $f(0) = f(0 + 0) \equiv f(0) + f(0) \mod 12 \Rightarrow f(0) = 0$. Now consider $f(0) = 0 = f(\underbrace{1 + 1 + \ldots + 1}_{9 \text{ times}}) \equiv 9f(1) \mod 12$. Thus $f(1)$ equals 0, 4, or 8.

If $f(1) = 0$, then $f(n) = 0$ for all n, which is well-defined. If $f(1) = 4$, then $f(n) \equiv 4n \mod 12$, which is well-defined as $f(n) \equiv f(n+9) = 4n+36 \mod 12 \equiv 4n$, and $f(m) * f(n) = 4m * 4n \equiv 4mn \mod 12 = f(mn)$. However, if $f(1) = 8$, $f(1) = f(1*1) = f(1)*f(1) = 8*8 \not\equiv 8 \mod 12$, so this f doesn't satisfy the second condition. So there are only $\boxed{2}$ functions that satisfy the given properties, which are the f defined by $f(1) = 0$ and $f(1) = 4$.

WW05 7. Let a_n be a sequence where $a_0 = \sqrt{3}, a_1 = \sqrt{2}, a_3 = -1$ (not a_2) and $a_n = a_{n-1}a_{n-2} - a_{n-3}$ for $n \geq 3$. Compute a_{2020}.

Answer: $-\frac{\sqrt{6}+\sqrt{2}}{2}$

Solution: First, observe that $a_0 = 2\cos(30°), a_1 = 2\cos(45°), a_2 = \frac{\sqrt{6}-\sqrt{2}}{2} = 2\cos(75°), a_3 = 2\cos(120°)$. Note that if $a_k = 2\cos(a - b), a_{k+1} = 2\cos(b), a_{k+2} = 2\cos(a)$, $a_{k+3} = 4\cos(a)\cos(b) - 2\cos(b - a) = 2\cos(b + a)$. Therefore, by a simple inductive argument, $a_n = 2\cos(15F_{n+3})$ where F_n is the n^{th} Fibonacci number. Since cos has a period of 360 degrees and the Fibonacci numbers mod 24 have a period of 24, it follows that the sequence has a period of 24. Therefore, it follows that $a_{2020} = a_4 = 2\cos(195°) = -\dfrac{\sqrt{6} + \sqrt{2}}{2}$.

JZ03 8. For how many integers n with $3 \leq n \leq 2020$ does the inequality

$$\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4k+1} 9^k > 3 \sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{n}{4k+3} 9^k$$

hold?

Answer: 672

Solution: By the binomial theorem, we have $(1+\sqrt{3}i)^n = \sum_{k=0}^{n} \binom{n}{k}(\sqrt{3}i)^k$, and hence the imaginary part of $(1+\sqrt{3}i)^n$ is given by

$$\begin{aligned}&\binom{n}{1}\sqrt{3} - \binom{n}{3}(\sqrt{3})^3 + \binom{n}{5}(\sqrt{3})^5 - \binom{n}{7}(\sqrt{3})^7 + \dots \\ &= \sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4k+1}(\sqrt{3})^{4k+1} - \sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{n}{4k+3}(\sqrt{3})^{4k+3} \\ &= \sqrt{3}\left(\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4k+1}9^k - 3\sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{n}{4k+3}9^k \right).\end{aligned}$$

It follows that $\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4k+1}9^k > 3\sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{n}{4k+3}9^k$ if and only if the imaginary part of $(1+\sqrt{3}i)^n$ is positive. Since $1+\sqrt{3}i = 2(\cos(2\pi/3)+i\sin(2\pi/3))$, by de Moivre's formula we have $(1+\sqrt{3}i)^n = 2^n(\cos(2\pi n/3)+i\sin(2\pi n/3))$, and hence the imaginary part of $(1+\sqrt{3}i)^n$ is positive if and only if $\sin(2\pi n/3) > 0$, which is true if and only if $n \equiv 1, 2 \mod 6$. So now we ask how many integers n between 3 and 2020, inclusive, satisfy $n \equiv 1, 2 \mod 6$. Since $2016/6 = 336$, we know there are $2 \times 336 = 672$ integers between 1 and 2016, inclusive, with remainder 1 or 2. Thus there are 670 such integers between 3 and 2016, inclusive. Since 2017 and 2018 also have remainders 1 and 2, the total number of such integers between 3 and 2020 inclusive is $\boxed{672}$.

WX07 9. A sequence of numbers is defined by $a_0 = 2$ and for $i > 0$, a_i is the smallest positive integer such that $\sum_{j=0}^{i} \frac{1}{a_j} < 1$. Find the smallest integer N such that $\sum_{i=N}^{\infty} \frac{1}{\log_2(a_i)} < \frac{1}{2^{2020}}$.

Answer: 2022

Solution: We claim that our sequence follows the recurrence $a_{n+1} = a_n^2 - a_n + 1$.

First, we will show that $\sum_{j=0}^{k} \frac{1}{a_j} = \frac{a_k^2 - a_k - 1}{a_k^2 - a_k}$ for all k. This is obvious for $k = 0$. In fact, we see that if this is true for all k, then it implies our recurrence. Therefore, to finish, we just need to show that $a_{k+1} = a_k^2 - a_k + 1$ implies that $\sum_{j=0}^{k+1} \frac{1}{a_j} = \frac{a_{k+1}^2 - a_{k+1} - 1}{a_{k+1}^2 - a_{k+1}}$.

This is just a matter of brute force algebra.

$$\begin{aligned}\sum_{j=0}^{k+1} \frac{1}{a_j} &= \frac{a_k^2 - a_k - 1}{a_k^2 - a_k} + \frac{1}{a_{k+1}} = \frac{a_k^2 - a_k - 1}{a_k^2 - a_k} + \frac{1}{a_k^2 - a_k + 1} \\ &= \frac{a_k^4 - 2a_k^3 + 2a_k^2 - a_k - 1}{a_k^4 - 2a_k^3 + 2a_k^2 - a_k} = \frac{a_{k+1}^2 - a_{k+1} - 1}{a_{k+1}^2 - a_{k+1}}\end{aligned}$$

(To better understand what's going on, let $P(n)$ be the assertion that $a_{n+1} = a_n^2 - a_n + 1$ and $S(n)$ be the assertion that $\sum_{j=0}^{n} \frac{1}{a_j} = \frac{a_n^2 - a_n - 1}{a_n^2 - a_n}$. It is easy to see that $S(n) \implies P(n)$ and we have shown that $P(n) \implies S(n+1)$. Since $S(0)$ is easily seen to be true, we have a domino effect which shows that $P(n)$ is true for all n.)

Now, it is easy to show that by induction, $2^{2^{i-1}} \le a_i \le 2^{2^i}$ for all $i \ge 0$. The motivation for this is noticing that the recurrence approximately grows as $a_{n+1} = a_n^2$. Therefore, for any nonnegative integer k,

$$\frac{1}{2^{k-1}} = \sum_{i=k}^{\infty} \frac{1}{2^i} \le \sum_{i=k}^{\infty} \frac{1}{\log_2(a_i)} \le \sum_{i=k}^{\infty} \frac{1}{2^{i-1}} \le \frac{1}{2^{k-2}}.$$

Therefore, our answer is 2022.

WX02 10. Let $f(a,b)$ be a third degree two-variable polynomial with integer coefficients such that $f(a,a)=0$ for all integers a and the sum

$$\sum_{\substack{a,b\in\mathbf{Z}^+\\ a\neq b}} \frac{1}{2^{f(a,b)}}$$

converges. Let $g(a,b)$ be the polynomial such that $f(a,b)=(a-b)g(a,b)$. If $g(1,1)=5$ and $g(2,2)=7$, find the maximum value of $g(20,20)$.

Answer: 43

Solution: Since we have that $f(a,a)=0$ for all integers a, we can write $f(a,b)=(a-b)\cdot(m_0a^2+m_1ab+m_2b^2+m_3a+m_4b+m_5)$ for some integers m_0,m_1,m_2 since every term of f has degree 3.

Thus, $g(a,b)=m_0a^2+m_1ab+m_2b^2m_3a+m_4b+m_5$.

A necessary condition for the sum to converge is that $g(n,n-1)$ must be positive for all natural numbers $n>N_1$ for some N_1. Otherwise, $f(n,n-1)$ is negative for infinitely many pairs $(n,n-1)$ and the sum wouldn't converge.

Similarly, $g(n-1,n)$ must be negative for all natural numbers $n>N_2$ for some N_2. This can be rewritten as

$$m_0(n^2)+m_1(n)(n-1)+m_2(n-1)^2+m_3(n)+m_4(n-1)+m_5>0$$

$$m_0(n-1)^2+m_1(n-1)(n)+m_2(n)^2+m_3(n-1)+m_4(n)+m_5<0.$$

For all $n>\max(N_1,N_2)$. In other words, we have:

$$(m_0+m_1+m_2)n^2-(m_1+2m_2+m_3+m_4)n+m_2-m_4+m_5>0$$

$$(m_0+m_1+m_2)n^2-(m_1+2m_0+m_3+m_4)n+m_0-m_3+m_5<0.$$

In order for this to be possible, we must have $m_0+m_1+m_2=0$. Therefore, $g(a,a)=(m_3+m_4)a+m_5$. From the values of $g(1,1)$ and $g(2,2)$ given, we know that $m_5=3$ and $m_3+m_4=2$. Thus, $g(20,20)=20*2+3=43$.

An example of a function that works is $f(a,b)=(a-b)(3a^2-3ab+2a+3)$. Our sum becomes:

$$\sum_{\substack{a,b\in\mathbf{Z}^+\\ a\neq b}} \frac{1}{2^{f(a,b)}}=\sum_{\substack{a,b\in\mathbf{Z}^+\\ a\neq b}} \frac{1}{2^{(a-b)(3a^2-3ab+2a+3)}}=\sum_{\substack{a,b\in\mathbf{Z}^+\\ a\neq b}} \frac{1}{2^{(a-b)^2(3a+\frac{2a+3}{a-b})}}.$$

Notice that the summation can be rewritten in terms of a and $a+k, k\neq 0$ instead of a and b. This gives us:

$$\sum_{k=1}^{\infty}\sum_{a=1}^{\infty}\frac{1}{2^{k^2(3a+\frac{2a+3}{k})}}+\sum_{k=1}^{\infty}\sum_{a=1}^{\infty}\frac{1}{2^{k^2(3a-\frac{2a+3}{k})}}$$

However, since $3a+\frac{2a+3}{k}\geq 3a$ and $3a-\frac{2a+3}{k}\geq a-3$, our summation is less than or equal to

$$\sum_{k=1}^{\infty}\frac{1}{2^{k^2}}\sum_{a=1}^{\infty}\frac{1}{2^{3a}}+\sum_{k=1}^{\infty}\frac{1}{2^{k^2}}\sum_{a=1}^{\infty}\frac{1}{2^{a-3}}<1\cdot 1+1\cdot 8=9.$$