Comment: Version 1.3

DT02 1. If $f(x) = nx$, $g(x) = e^{2x}$, and $h(x) = g(f(x))$, find n such that $h'(0) = 100$.

Answer: 50

Solution: If $f(x) = nx$, $g(x) = e^{2x}$, and $h(x) = g(f(x))$, then $h(x) = e^{2(nx)} = e^{2nx}$. The derivative of $h(x)$ is $2n * e^{2nx}$, by the Chain Rule. Then, plugging in 0 for x gets us $2n = 100$, so, $n = 50$.

KW36 2. Farmer Joe will plant carrots to cover a rectangle in the first quadrant with a vertex at the origin and sides parallel to the x and y axes. However, he can not grow carrots on his neighbor's land. If the border between his and his neighbor's land is along the curve $y = -\ln(2x)$, what is the maximum area of carrotland Farmer Joe can create?

Answer: $\mathbf{\frac{1}{2e}}$

Solution: We note that the area of carrotland is $xy = -x\ln(2x)$. The maximum occurs when $(xy)' = 0$, or $-\ln(2x) + -1 = 0$. Hence $x = e^{-1}/2$ and $y = 1$. So, the maximum area is $\boxed{\frac{1}{2e}}$.

KW40 3. For all θ from 0 to 2π, Annie draws a line segment of length θ from the origin in the direction of θ radians. What is the area of the spiral swept out by the union of these line segments?

Answer: $\mathbf{\frac{4\pi^3}{3}}$

Solution: After drawing the spiral, it should become clear that we have the following calculation since our radius is θ

$$\frac{1}{2}\int_0^{2\pi} \theta^2 d\theta = \boxed{\frac{4\pi^3}{3}}$$

DT01 4. The Chebyshev Polynomials are defined as

$$T_n(x) = \cos(n\cos^{-1}(x)),$$

for $n = 0, 1, 2, ...$ Compute the following infinite series:

$$\sum_{n=1}^{\infty} \int_{-1}^{1} T_{2n+1}(x)dx.$$

If the series diverges, your answer should be "D."

Answer: 0

Solution: We can show that the Chebyshev Polynomials are odd for odd n. Recall that for an odd function, $f(-x) = -f(x)$. So, the integral of said function over $[-1, 1]$ should be 0. Thus, the sum of those integrals should also be 0.

KW49 5. What is

$$(2020)^2 + \frac{(2021)^2}{1!} + \frac{(2022)^2}{2!} + \frac{(2023)^2}{3!} + \frac{(2024)^2}{4!} + \dots$$

Answer: $\mathbf{4084442}e$

Solution: We start with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Then we can do a pattern of differentiating and multiplying by x.

$$x^{2020}e^x = \sum_{n=0}^{\infty} \frac{x^{n+2020}}{n!}$$

$$(2020x^{2019} + x^{2020})e^x = \sum_{n=0}^{\infty} \frac{(n+2020)x^{n+2019}}{n!}$$

$$(2020x^{2020} + x^{2021})e^x = \sum_{n=0}^{\infty} \frac{(n+2020)x^{n+2020}}{n!}$$

$$(2020^2x^{2019} + 4041x^{2020} + x^{2021})e^x = \sum_{n=0}^{\infty} \frac{(n+2020)^2x^{n+2019}}{n!}$$

So, our desired sum occurs when $x = 1$, and we obtain $\boxed{4084442e}$.

KW48 6. Let us define the sequence $a_n = (-1)^n/(n)$. Now, we define the partial sums

$$A_N = \sum_{n=1}^{N} a_n.$$

What is the difference

$$\sum_{N=1}^{\infty}\left(A_N - \lim_{M\to\infty} A_M\right)?$$

Answer: $-\log(2) + 1/2$

Solution: First we note that we are calculating the series

$$\sum_{N=1}^{\infty}\sum_{m=N+1}^{\infty} \frac{(-1)^{m+1}}{m} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{n+m}$$

Instead, we consider

$$F(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-x)^{n+m}}{n+m}.$$

Then we note that the answer should be $\lim_{x\to 1} -F(x)$. Now we can see that

$$F'(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)(-x)^{n+m-1} = \frac{x}{(1+x)^2}.$$

Then we can determine that

$$F(x) = \int \frac{x}{(1+x)^2}dx = \log(1+x) + \frac{1}{1+x} - 1$$

(Note that $F(0) = 0$ from its definition). Thus,

$$\lim_{x\to 1} -F(x) = -\log(2) - \frac{1}{2} + 1 = \boxed{-\log(2) + 1/2.}$$

KA23 7. Define $f_1(x) = x$ and for every integer $n \geq 2$, we define $f_n(x) = x^{f_{n-1}(x)}$. Compute

$$\lim_{n\to\infty} \int_e^{2020} \frac{f_n'(x)}{f_n(x)f_{n-1}(x)\ln x} - \frac{f_{n-1}'(x)}{f_{n-1}(x)} dx.$$

Answer: $\ln(\ln 2020)$

Solution: It turns out that the limit is unnecessary, as we can see by induction that

$$f_n'(x) = f_n(x)\left(f_{n-1}'(x)\ln x + \frac{1}{x}f_{n-1}(x)\right).$$

This means that the desired integral is $\int_e^{2020} \frac{1}{x\ln x}dx$. The anti-derivative is just $\ln(\ln x)$, so evaluating at endpoints gives $\boxed{\ln(\ln 2020)}$.

ZY02 8. Compute

$$\int_0^\infty \frac{dx}{x^4 - 6x^2 + 25}.$$

Answer: $\frac{\pi}{20}$

Solution: We first factor the denominator to get

$$\int_0^\infty \frac{dx}{x^4 - 6x^2 + 25} = \int_0^\infty \frac{dx}{(x^2 - 4x + 5)(x^2 + 4x + 5)}.$$

We can then decompose the integral into the partial fractions

$$\int_0^\infty \left[\frac{-x+4}{40(x^2 - 4x + 5)} + \frac{x+4}{40(x^2 - 4x + 5)}\right] dx.$$

Focusing on the first term, we notice that $\frac{d}{dx}(x^2 - 4x + 5) = 2x - 4$. This suggests that we further decompose the first term into

$$\int_0^\infty \left[\frac{-(x-2)}{40(x^2 - 4x + 5)} + \frac{2}{40(x^2 - 4x + 5)}\right] dx.$$

The first integral evaluates to $-\frac{1}{2}\ln(x^2 - 4x + 5)$. To evaluate the second integral, we complete the square in the denominator to get

$$\int_0^\infty \frac{2dx}{40(x-2)^2 + 40}.$$

We can then make the substitution $u = x - 2$ and use the fact that $\int \frac{dx}{x^2+1} = \tan^{-1}(x)$ to see that the second integral evaluates to $\frac{1}{20}\tan^{-1}(x-2)$. Decomposing the second integral in a similar manner, we find

$$\begin{aligned}\int_0^\infty \frac{dx}{x^4 - 6x^2 + 25} &= \Bigg[-\frac{1}{80}\ln(x^2 - 4x + 5) + \frac{1}{80}\ln(x^2 + 4x + 5) \\ &\quad + \frac{1}{20}\tan^{-1}(x-2) + \frac{1}{20}\tan^{-1}(x+2)\Bigg]_0^\infty \\ &= \left[\frac{1}{80}\ln\left(\frac{x^2 - 4x + 5}{x^2 + 4x + 5}\right) + \frac{1}{20}\left(\tan^{-1}(x-2) + \tan^{-1}(x+2)\right)\right]_0^\infty\end{aligned}$$

When $x = 0$, the resulting terms cancel to 0. When $x \to \infty$, the fraction in the ln term approaches 1, and $\ln 1 = 0$. On the other hand, $\tan^{-1}(x) \to \frac{\pi}{2}$, so our answer is $\frac{1}{20}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \boxed{\frac{\pi}{20}}$.

WX09 9. Define $a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}_{\text{n square roots}}$. For example $a_1 = \sqrt{2}$ and $a_2 = \sqrt{2+\sqrt{2}}$. Find the value of

$$\lim_{n\to\infty} 4^n(2 - a_n).$$

Answer: $\boldsymbol{\frac{\pi^2}{4}}$

Solution: It is not hard to show by induction that $a_n = 2\cos(\pi/2^{n+1})$. Therefore,

$$4^n(2-a_n) = 4^n\left(2 - \left(2 - 2\frac{\left(\frac{\pi}{2^{n+1}}\right)^2}{2!} + 2\frac{\left(\frac{\pi}{2^{n+1}}\right)^4}{4!} - \dots\right)\right) = \frac{\pi^2}{4} + O(1/4^n).$$

Thus, as $n \to \infty$, the limit approaches $\frac{\pi^2}{4}$.

AJ06 10. Let

$$I_m = \int_0^{2\pi} \sin(x)\sin(2x)\cdots\sin(mx)dx.$$

Find the sum of all integers $1 \le m \le 100$ such that $I_m \neq 0$.

Answer: 1300

Solution: No solution yet.