Comment: Version 1.0

EW05

1. In your drawer you have two red socks and a blue sock. You randomly select socks, without replacement, from the drawer. However, every time you take a blue sock, another one magically appears in the drawer. What is the probability that you get a red pair before a blue pair?

   Answer: $\mathbf{\frac{11}{18}}$

   Solution: Note that at most 3 socks will be drawn. Based on this, there are three cases where a red pair is drawn before a blue pair.

   (a) Red Red: The probability of the first sock being red is $\frac{2}{3}$. After removing the red sock, the probability of the second sock being red is $\frac{1}{2}$.

   (b) Red Blue Red: The probability of the first sock being red is $\frac{2}{3}$. After removing the red sock, the probability of the second sock being blue is $\frac{1}{2}$. After replacing the blue sock, the probability of the third sock being red is $\frac{1}{2}$.

   (c) Blue Red Red: The probability of the first sock being blue is $\frac{1}{3}$. After replacing the blue sock, the probability of the second sock being red is $\frac{2}{3}$. After removing the red sock, the probability of the third sock being red is $\frac{1}{2}$.

   Adding the probabilities over these three cases, the probability that a red pair is drawn before a blue pair is equal to $(\frac{2}{3})(\frac{1}{2}) + (\frac{2}{3})(\frac{1}{2})(\frac{1}{2}) + (\frac{1}{3})(\frac{2}{3})(\frac{1}{2}) = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$.

WW21

2. Suppose a, b, c are positive integers such that $lcm(a, b) = 400$, $lcm(b, c) = 2000$, $lcm(c, a) = 1000$, and $gcf(a, b, c) = 10$. Given that a is a three-digit number, what is the value of $a + b + c$?

   Answer: 530

   Solution: Looking at the prime factorizations $400 = 2^4 \cdot 5^2, 2000 = 2^4 \cdot 5^3, 1000 = 2^3 \cdot 5^3$, consider the exponents of 2 in the prime factorizations of a, b, c. It follows that the exponent of b must be 4 and the exponents of a and c must be 1 and 3 in some order. Similarly for the exponents of 5, the exponent of c must be 3 and the exponents of a and b must be 1 and 2 in some order. In order for a to be a three-digit number, a must equal $2^3 \cdot 5^2 = 200$. Therefore, b must equal $2^4 \cdot 5^1 = 80$ and c must equal $2^1 \cdot 5^3 = 250$. Since $a = 200$, $b = 80$, and $c = 250$, $a + b + c = 530$.

KW02

3. Consider a 3×3 grid with the first 9 positive integers placed in the grid. Take the greatest integer in each row and let r be the smallest of those numbers. Take the smallest integer in each column and c be the greatest of those numbers. How many arrangements are there such that $r \leq c \leq 4$?

   Answer: 38880

   Solution: We first note that r and c are have minimum value 3. We proceed using casework. If $r = c = 3$, then we need to split 1,2,3 into different columns for $c = 3$. Then they necessarily need to be in the same row for $r = 3$. There are 9 places to choose where the 3 is, 2 more ways to arrange 1,2 in the row and 6! ways to arrange the other numbers. Thus, there are $9 \cdot 2 \cdot 6! = 720 \cdot 18 = 12960$ arrangements.
   If $r = 3, c = 4$. Then again 1,2,3 need to be in the same row, forcing them to be in different columns. But we need 4 to be the smallest integer in some column, hence causing a problem. So, there are 0 arrangements.
   Finally if $r = 4, c = 4$, we can choose 9 places for the 4 to be. Then we are forced to place two of the numbers 1,2,3 in the row, giving us 6 ordered ways. Then the final number can not be placed in the same column as the 4, hence there are 4 places to put it. Finally there are 5! ways to place the final numbers. So there are $9 \cdot 6 \cdot 4 \cdot 5! = 25920$.
   Thus there are $12960 + 25920 = \boxed{38880}$ arrangements.