1. Find the sum of the largest and smallest value of the following function: $f(x) = |x - 23| + |32 - x| - |12 + x|$ where the function has domain [-37,170].

   Answer: 69

   Solution: This function is the same as the piecewise function

   $f(x) = (23 - x) + (32 - x) - (-12 - x)$ when $x \leq -12$
   $f(x) = (23 - x) + (32 - x) - (12 + x)$ when $-12 \leq x \leq 23$
   $f(x) = (x - 23) + (32 - x) - (12 + x)$ when $23 \leq x \leq 32$
   $f(x) = (x - 23) + (x - 32) - (12 + x)$ when $32 \leq x$

   Simplifying these expressions, we find

   $f(x) = 67 - x$ when $x \leq -12$
   $f(x) = 43 - 3x$ when $-12 \leq x \leq 23$
   $f(x) = -3 - x$ when $23 \leq x \leq 32$
   $f(x) = x - 67$ when $32 \leq x$

   Thus, as x increases, $f(x)$ decreases when $x \leq -12$, decreases when $-12 \leq x \leq 23$, decreases when $23 \leq x \leq 32$, and increases when $32 \leq x$. Also, $f(-37) = 104$, $f(-12) = 79$, $f(23) = -26$, $f(32) = -35$, and $f(170) = 103$.

   Therefore, the smallest value is -35 and the largest is 104. The sum is $-35 + 104 = \boxed{69}$

2. Given a convex equiangular hexagon with consecutive side lengths of $9, a, 10, 5, 5, b$, where a and b are whole numbers, find the area of the hexagon.

   Answer: $\mathbf{\frac{97\sqrt{3}}{2}}$

   Solution: Since the hexagon is equiangular, and each angle must measure 120 degrees, we can extend its side lengths to form three smaller equilateral triangles and one large equilateral triangle composed of the hexagon and the three smaller triangles. Since the side lengths of this large triangle must all be equal, we can set up a system of equations where $b+9+a = a+10+5 = 5+5+b$. Solving this, we get that $a = 1$ and $b = 6$. Therefore, the large triangle has a side length of 16, giving it a total area of $64\sqrt{3}$. To find the area of the hexagon, we can subtract the area of the three smaller equilateral triangles to get an answer of $64\sqrt{3} - 9\sqrt{3} - \frac{\sqrt{3}}{4} - \frac{25\sqrt{3}}{4} = \frac{97\sqrt{3}}{4}$.

3. Let $S = \{1, 2, \ldots, 100\}$. Compute the minimum possible integer n such that, for any subset $T \subseteq S$ with size n, every integer a in S satisfies the relation $a \equiv bc \bmod 101$, for some choice of integers b, c in T.

   Answer: 51

   Solution: If $|T| = 51$ and $a \in S$, $aT^{-1} = \{at^{-1} \bmod 101 | t \in T\}$ also has size 51, so T and aT^{-1} must have some element in common. Then for some $b, c \in T$, $ab^{-1} \equiv c \Rightarrow a \equiv bc \bmod 101$, so any T of size 51 has the requested property.

   To show that 51 is the minimum such size, consider the 50-element subset
   $T = \{2^2, 2^4, \ldots 2^{100}\} \bmod 101$. We can see that T is closed under multiplication, so it does not have the desired property. Similarly, any size less than 50 also does not work, so the minimum size is $\boxed{51}$.

4. Let C be the circle of radius 2 centered at $(4, 4)$ and let L be the line $x = -2$. The set of points equidistant from C and from L can be written as $ax^2 + by^2 + cxy + dx + ey + f = 0$ where a, b, c, d, e, f are integers and have no factors in common. What is $|a + b + c + d + e + f|$?

Answer: 7

Solution 1: The distance between point (x, y) and the circle is $\sqrt{(x-4)^2 + (y-4)^2} - 2$. The distance between (x, y) and the line $x = -2$ is $|x + 2|$. Clearly, for any (x, y) equidistant from the line and circle, $x > -2$. Hence, the set of points for which these distances are equal is $\sqrt{(x-4)^2 + (y-4)^2} - 2 = x + 2$. Adding 2 to both sides, squaring, and rearranging gives us $y^2 - 16x - 8y + 16 = 0$. Thus, our answer is $|0 + 1 + 0 - 16 - 8 + 16| = \boxed{7}$

Solution 2: Notice that for any point P outside of the circle, the distance from P to the circle is precisely 2 less than the distance between P and the center of the circle. Thus, the locus of points is equidistant from the point $(4, 4)$ and the line $x = -4$. But this is simply a parabola with focus $(4, 4)$ and directrix $x = -4$. We can then easily see that the vertex is at $(0, 4)$ giving us the equation $4ax = (y-4)^2$, where $a = 4$ is the distance from the vertex to the focus. Expanding and rearranging, we get the equation $y^2 - 8y + 16 - 16x$, which gives us the answer $\boxed{7}$.

5. If a is picked randomly in the range $\left(\frac{1}{4}, \frac{3}{4}\right)$ and b is chosen such that

$$\int_a^b \frac{1}{x^2} dx = 1,$$

compute the expected value of $b - a$.

Answer: $\mathbf{\ln 9 - \frac{3}{2}}$

First, we find an equation to relate b to a:

$$\begin{aligned}
\int_a^b \frac{1}{x^2} dx &= 1 \\
-\frac{1}{x}\Big|_a^b &= 1 \\
-\frac{1}{b} - \left(-\frac{1}{a}\right) &= 1 \\
-a + b &= ab \\
b(1-a) &= a \\
b &= \frac{a}{1-a}
\end{aligned}$$

Hence, $b - a = \frac{a}{1-a} - a = \frac{a^2}{1-a}$. Let E be the expected value of this expression if a is chosen randomly in the range $\left(\frac{1}{4}, \frac{3}{4}\right)$. Then,

$$\begin{aligned}
E &= \frac{\int_{1/4}^{3/4} \frac{a^2}{1-a} da}{\frac{3}{4} - \frac{1}{4}} \\
&= 2\int_{1/4}^{3/4} \frac{a^2}{1-a} da \\
&= 2\int_{1/4}^{3/4} \Big(- a - 1 + \frac{1}{1-a} \Big) da \\
&= 2\Big(- \frac{a^2}{2} - a - \ln(1-a) \Big)\Big|_{1/4}^{3/4} \\
&= 2\Big(- \frac{9}{32} - \frac{3}{4} - \ln\left(\frac{1}{4}\right) + \frac{1}{32} + \frac{1}{4} + \ln\left(\frac{3}{4}\right) \Big) \\
&= 2\Big(- \frac{3}{4} + \ln(3) \Big) \\
&= \boxed{\ln(9) - \frac{3}{2}}.
\end{aligned}$$

6. Let $A_1, A_2, \ldots, A_{2020}$ be a regular 2020-gon with a circumcircle C of diameter 1. Now let P be the midpoint of the small-arc $A_1 - A_2$ on the circumcircle C. Then find:

$$\sum_{i=1}^{2020} |PA_i|^2$$

Answer: 1010

Solution: First observe that points A_i, A_{i+1010} for all $i \in [1, 1010]$ are diametrically opposite points. Thus, since point P is on the circumference, as P, A_i, A_{i+1010} form a right angled triangle, we can use the Pythagorean theorem

$$|PA_i|^2 + |PA_{i+1010}|^2 = 1$$

Summing up over $i \in [1, 1010]$, we get that

$$\sum_{i=1}^{2020} |PA_i|^2 = \boxed{1010}$$

7. A certain party of 2020 people has the property that, for any 4 people in the party, there is at least one person of those 4 that is friends with the other three (assume friendship is mutual). Call a person in the party a *politician* if they are friends with the other 2019 people in the party. If n is the number of politicians in the party, compute the sum of the possible values of n.

Answer: 6055

Solution: We'll use the language of graph theory to make terminology more concise. Thus, consider the 2020 people as vertices, and have two vertices be adjacent if they are friends. Then

n is the number of vertices that are adjacent to every other vertex.

Clearly, $n = 2020$ is a possibility. Otherwise, some vertices a and b are not adjacent, so for every pair of distinct vertices c and d, c must be adjacent to d (otherwise no vertex in the group a, b, c, d is adjacent to the other three). If a is not adjacent to c (without loss of generality), then d must be adjacent to each of a, b, and c. This is true for all distinct d, so $n = 2017$ is a possibility. Otherwise, both a and b are adjacent to every other vertex (besides each other), so $n = 2018$ in this case.

So the sum of the possible values of n is $2020 + 2018 + 2017 = \boxed{6055}$.

8. Let S be an n-dimensional hypercube of sidelength 1. At each vertex draw a hypersphere of radius $\frac{1}{2}$; let Ω be the set of these hyperspheres. Consider a hypersphere Γ centered at the center of the cube that is externally tangent to all the hyperspheres in Ω. For what value of n does the volume of Γ equal to the sum of the volumes of the hyperspheres in Ω.

   Answer: 9

   Solution: Let d_i denote the diameter of Γ, note that $d_i = \sqrt{n} - 1$. We also have that the diameter of the smaller hyperspheres are each 1. Note that Ω contains 2^n hyperspheres. As the ratio of the volumes similar odjects in n-dimensional space is the ratio of the lengths to the nth power, it follows that for the areas to be equal, we must have $(\frac{d_i}{1})^n = 2^n$ substituting and solving, we see that $d_i = 2$ and $n = \boxed{9}$.

9. Solve for C:

$$\frac{2\pi}{3} = \int_0^1 \frac{1}{\sqrt{Cx - x^2}} dx.$$

   Answer: $\frac{4}{3}$

   Solution:

$$\frac{2\pi}{3} = \int_0^1 \frac{1}{\sqrt{Cx - x^2}} dx = \int_0^1 \frac{1}{\sqrt{\frac{C^2}{4} - (x - \frac{C}{2})^2}} dx = \int_0^1 \frac{1}{\frac{C}{2}\sqrt{1 - \frac{2}{C}(x - \frac{C}{2})^2}} dx$$

   We make the substitution $\cos\theta = \frac{2}{C}(x - \frac{C}{2})$ and convert the limits of integration accordingly.

$$\frac{2}{C}dx = -\sin\theta d\theta$$
$$\frac{2\pi}{3} = \int_{\theta_1}^{\theta_2} \frac{-\sin\theta}{\sqrt{1-\cos^2\theta}} d\theta = \int_{\theta_1}^{\theta_2} -1 d\theta = \theta_1 - \theta_2$$
$$\theta_1 = \cos^{-1}\left[\frac{2}{C}\left(0 - \frac{C}{2}\right)\right] = \pi$$
$$\theta_2 = \cos^{-1}\left[\frac{2}{C}\left(1 - \frac{C}{2}\right)\right] = \cos^{-1}\left(\frac{2}{C} - 1\right)$$
$$\frac{2\pi}{3} = \pi - \cos^{-1}\left(\frac{2}{C} - 1\right)$$
$$\frac{2}{C} - 1 = \cos\frac{\pi}{3} = \frac{1}{2}$$
$$C = \frac{4}{3}$$

Thus C should be $\boxed{\frac{4}{3}}$.

10. Nathan and Konwoo are both standing in a plane. They each start at (0,0). They play many games of rock-paper-scissors. After each game, the winner will move one unit up, down, left, or right, chosen randomly. The outcomes of each game are independent, however, Konwoo is twice as likely to win a game as Nathan. After 6 games, what is the probability that Konwoo is located at the same point as Nathan? (For example, they could have each won 3 games and both be at (1,2).)

Answer: $\mathbf{\frac{25}{256}}$

Solution: Consider the vector that is the difference of Konwoo's and Nathan's positions. This vector is initially (0,0) and changes by 1 unit in a random direction after each game regardless of the winner. Thus the probability that Konwoo and Nathan are still on the same point is the same as the probability that a random lattice walk of length 6 returns to the origin. For such a lattice walk to return to origin, there needs to be the same number of up and down moves, and the same number of left and right moves. This condition is equivalent to having 3 moves that are left or up (LU), and 3 moves that are right or up (RU). Moreover, knowing whether a move is LU and whether it is RU uniquely determines what the move is, so it suffices to designate 3 LU moves and 3 RU moves, giving $(\binom{6}{3})^2 = 400$ random walks. Then, the answer is $\frac{400}{4^6} = \boxed{\frac{25}{256}}$

11. Suppose that x, y, z are real positive numbers such that $\left(1 + x^4y^4\right)e^z + \left(1 + 81e^{4z}\right)x^4e^{-3z} = 12x^3y$. Find all possible values of $x + y + z$.

Answer: $\mathbf{4/3 - \ln(3)}$

Solution: We can see that

$$\left(1 + x^4y^4\right)e^z + \left(1 + 81e^{4z}\right)x^4e^{-3z} = 12x^3y$$
$$\iff \frac{1}{x^4} + y^4 + e^{-4z} + 81 = 12\frac{ye^{-z}}{x}$$

We note that if $a^4+b^4+c^4+d^4=4abcd$ for $a,b,c,d\geq 0$ then $a=b=c=d$ for a,b,c,d positive real numbers. Since AM-GM tells us that

$$\frac{a^4+b^4+c^4+d^4}{4}\geq\sqrt[4]{a^4b^4c^4d^4}=abcd$$

with equality if and only if $a^4=b^4=c^4=d^4$. Since a,b,c,d are positive this implies that $a=b=c=d$. So, $1/x=y=e^{-z}=3$. So, $x=1/3, y=3, z=-\ln(3)$. So, their sum is $\boxed{10/3-\ln(3)}$.

12. Given a large circle with center (x_0,y_0), one can place three smaller congruent circles with centers $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ that are pairwise externally tangent to each other and all internally tangent to the outer circle. If this placement makes $x_0=x_1$ and $y_1>y_0$, we call this an "up-split". Otherwise, if the placement makes $x_0=x_1$ and $y_1<y_0$, we call it a "down-split."

Alice starts at the center of a circle C with radius 1. Alice first walks to the center of the upper small circle C_u of an up-slitting of C. Then, Alice turns right to walk to the center of the upper-right small circle of a down-splitting of C_d. Alice continues this process of turning right and walking to the center of a new circle created by alternatingly up- and down-splitting. Alice's path will form a spiral converging to (x_A,y_A). On the otherhand, Bob always up-splits the circle he is in the center of, turns right and finds the center of the next small circle. His path will converges to (x_B,y_B). Compute $|\frac{1}{x_A}-\frac{1}{x_B}|$.

Answer: $\mathbf{\frac{4\sqrt{3}+6}{3}}$.

Solution: Notice that Alice always turns $60°$ right so that the angle between incoming and outgoing paths at each circle center is $120°$. Moreover, her moves are all along three vectors: $v_1=(0,1), v_2=(\frac{\sqrt{3}}{2},\frac{1}{2}), v_3=(\frac{\sqrt{3}}{2},-\frac{1}{2})$. Her first move is along v_1, second is along v_2, third along v_3, fourth along $-v_1$, fifth along $-v_2$, sixth along $-v_3$, and repeat. The ratio between one move and the next is always constant. We call this constant α. Thus, her total movement from $\pm v_1$ movements is $M(1-\alpha^3+\alpha^6-+\cdots)=\frac{M}{1+\alpha^3}$, where M is the length of her first step and we also calculate this later. Similarly, she moves $\frac{M\alpha}{1+\alpha^3}$ units from $\pm v_2$ movements and $\frac{M\alpha^3}{1+\alpha^3}$ units from $\pm v_3$ movements. Therefore, $x_A=\frac{M\sqrt{3}}{2}\frac{\alpha+\alpha^2}{1+\alpha^3}=\frac{M\sqrt{3}}{2}\frac{\alpha}{1-\alpha+\alpha^2}$.

Bob always turns $120°$ right and moves along the same $u_1=v_1$, along $u_2=v_3$, and along $u_3=-v_2$. Then the calculations for these three movements is the same as Alice's, except that the denominators are $1-\alpha^3$. We find that $x_B=\frac{M\sqrt{3}}{2}\frac{\alpha-\alpha^2}{1-\alpha^3}=\frac{M\sqrt{3}}{2}\frac{\alpha}{1+\alpha+\alpha^2}$. Therefore, $|\frac{1}{x_A}-\frac{1}{x_B}|=\frac{2\cdot 2\alpha}{M\alpha\sqrt{3}}=\frac{4}{M\sqrt{3}}$. Notice that this expression is independent of α, so it suffices to calculate M.

The original circle has radius 1. When we up-split, say that the smaller circles have radii r. The distance from the center of the large circle to any of the small circles' centers is $\frac{2r}{\sqrt{3}}$. Therefore, $1=r(1+\frac{2}{\sqrt{3}})=r\frac{\sqrt{3}+2}{\sqrt{3}}$. Incidentally, we've calculated α, although what matters is that $M=\frac{2r}{\sqrt{3}}=\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}+2}=4-2\sqrt{3}$. Therefore, the answer is $\boxed{\frac{4\sqrt{3}+6}{3}}$.

13. Compute the sum of all natural numbers b less than 100 such that b is divisible by the number of factors of the base-10 representation of 2020_b.

Answer: 152

Solution: Suppose $b = 2^k \cdot n$, where n is an odd integer. Then $2020_b = 2b(b^2+1) = 2^{k+1}(n)(b^2+1)$. Due to the multiplicty of τ and b^2+1 being relatively prime to both 2 and n, it follows that the number of factors of 2020_b, $\tau(2020_b)$, is equal to $\tau(2^{k+1})\cdot\tau(n)\cdot\tau(b^2+1)$. Since $\tau(2^{k+1}) = k+2$, the condition is satisfied if and only if $\frac{b}{(k+2)\tau(n)\tau(b^2+1)}$ is an integer.

Because $b = 2^k \cdot n$, k must be between 0 and 6 inclusive. Doing casework on the value of k yields several cases. In the case where $k = 0$, b must be a multiple of $0+2 = 2$, which can never happen as b is an odd number. Similarly, in the case where $k = 2$, $b = 4n$ must be a multiple of $2+2 = 4$, meaning n is a multiple of $\tau(b^2+1)$. However, since b^2+1 is never a perfect square when b is a natural number, $\tau(b^2+1)$ must be even, which means n is not a multiple of $\tau(b^2+1)$ as n is odd. Therefore, there are no solutions when $k = 0$ or when $k = 2$.

In the case where $k = 1$, b must be a multiple of $1+2 = 3$. Since $b = 2n < 100$, n must be an odd multiple of 3 that is also less than 50. Furthermore, since $\tau(b^2+1)$ must be even, $\tau(n)$ must be odd, implying n is a perfect square. Taking both of these conditions into account, the only number that is a perfect square, a multiple of 3, and less than 50 is $n = 9$. However, this means that $b = 18$ and $b^2+1 = 325 = 5^2\cdot 13$, which has 6 factors. Because $\frac{b}{(k+2)\tau(n)\tau(b^2+1)} = \frac{18}{(3)(3)(6)} = \frac{1}{3}$ is not an integer, 18 is not a solution.

In the case where $k = 3$, b must be a multiple of $3+2 = 5$. Since b is less than 100, the only number that fits these conditions is $b = 40$, which occurs when $n = 5$. Because $b^2+1 = 1601$ is prime, $\frac{b}{(k+2)\tau(n)\tau(b^2+1)} = \frac{40}{(5)(2)(2)} = 2$, which means 40 is a solution.

In the case where $k = 4$, b must be a multiple of $4+2 = 6$, meaning n must be a multiple of 3. Since b is less than 100, the only number that fits these conditions is $b = 48$, which occurs when $n = 5$. Because $b^2+1 = 2305 = 5\cdot 461$, $\frac{b}{(k+2)\tau(n)\tau(b^2+1)} = \frac{48}{(6)(4)(2)} = 1$, which means 48 is a solution.

In the case where $k = 5$, b must be a multiple of $5+2 = 7$, which means b must be a multiple of $2^5\cdot 7 = 224 > 100$, meaning there are no solutions. Lastly, in the case where $k = 6$, the only possible number is $b = 64$, which occurs when $n = 1$. Because $b^2+1 = 4097 = 17\cdot 241$, $\frac{b}{(k+2)\tau(n)\tau(b^2+1)} = \frac{64}{(8)(1)(4)} = 2$, which means 64 is a solution.

Therefore, the only solutions are the numbers 40, 48, and 64, which sum to 152.

14. Iris is playing with her random number generator. The number generator outputs real numbers from 0 to 1. After each output, Iris computes the sum of her outputs, if that sum is larger than 2, she stops. What is the expected number of outputs Iris will receive before she stops?

Answer: $\boldsymbol{e^2 - e}$

Solution: Let $f(x)$ be the expected number of additional outputs needed given the sum of the current outputs is x. Then $f(x) = 0$ for $x \geq 2$ and we wish to find $f(0)$. In addition, $f(x) = 1 + \int_x^{x+1} f(x)dx$ as the expected number of outputs needed is equal 1 more than to the average expected number outputs for each possible new total.
For $1 \leq x < 2$, $f(x) = 1 + \int_x^2 f(x)dx + \int_2^{1+x} f(x)dx = 1 + \int_x^2 f(x)dx$. Differentiating both sides, we have $f'(x) = -f(x)$, thus $f(x) = Ae^{-x}$. Substituting back: $Ae^{-x} = 1 + \int_x^2 Ae^{-x}dx \Leftrightarrow Ae^{-x} = 1 - Ae^{-2} + Ae^{-x} \Leftrightarrow 0 = 1 - Ae^{-2} \Leftrightarrow A = e^2$.

Thus $f(x) = \begin{cases} 0 & x \geq 2 \\ e^{2-x} & 1 \leq x < 2 \\ TBD & 0 \leq x \leq 1 \end{cases}$

We now solve $f(x)$ for $0 \leq x \leq 1$. Then we have $f(x) = 1 + \int_x^{x+1} f(x)dx = 1 + \int_x^1 f(x)dx +$

$\int_1^{1+x} f(x)dx \Leftrightarrow$

$f(x) = 1+\int_x^1 f(x)dx+\int_1^{1+x} e^{2-x}dx$ Differentiating: $f'(x) = -f(x)+e^{1-x} \Leftrightarrow f(x)+f'(x) = e^{1-x}$. This equation has solutions of the form $f(x) = Be^{-x} + xe^{1-x}$. Note that $f(x)$ should be continuous, save the discontinuity at $x = 2$, thus $f(1) = e^{2-1} = Be^{-1} + 1e^{1-1} \Leftrightarrow e = \frac{B}{e} + 1 \Leftrightarrow$ $B = e^2 - e$. Now we know $f(x) = \begin{cases} 0 & x \geq 2 \\ e^{2-x} & 1 \leq x < 2 \\ (e^2 - e)e^{-x} + xe^{1-x} & 0 \leq x \leq 1 \end{cases}$

and the expected number of outputs needed, $f(0) = \boxed{e^2 - e}$

Solution: Let X be the number of rolls needed for the sum to be greater than 2. Than

$$E[X] = \sum_{n=0}^{\infty} P(X > n)$$

Now for $n = 0, 1, 2$, clearly $P(X > n) = 1$. Otherwise we have that $P(X > n)$ is the probability that the first n numbers sum less than 2, where the first n numbers are bounded by 1. We can see that this gives a n-dimensional area that we can calculate. The area of the n-dimensional triangle with side length 2 bounded by the equation $x_1 + ...x_n \leq 2$ is of area $2^n/n!$. Now since we are bounded our coordinates each by 1, this chops off n triangles in each dimension. Looking at say the triangle where $x_1 \geq 1$, we have that if $x_1 = 1 + y_1$ then we have each of the other coordinates bounded by 1 and $y_1 \leq 1$. So, this is the n dimensional triangle bounded by $y_1 + x_2 + ... + x_n \leq 1$. This has area $1/n!$. Thus, we have that $P(X > n) = 2^n/n! - n/n!$. So, we have

$$\sum_{n=0}^{\infty} P(X > n) = 3 + \sum_{n=3}^{\infty} \frac{2^n}{n!} - \frac{n}{n!} = 3 + e^2 - 5 - (e - 2) = \boxed{e^2 - e}$$

15. Evaluate

$$\int_0^{\frac{\pi}{2}} \ln\left(9\sin^2\theta + 121\cos^2\theta\right) d\theta$$

Answer: Answer: $\boldsymbol{\pi \ln (7)}$

Solution: Let

$$J(a, b) = \int_0^{\frac{\pi}{2}} \ln\left(a^2\sin^2\theta + b^2\cos^2\theta\right) d\theta$$

First, we evaluate the following integral for constant b:

$$I(a) = \int_0^{\infty} \frac{\ln\left(1 + a^2x^2\right)}{1 + b^2x^2} dx$$

Differentiating under the integral with respect to a gives:

$$I'(a) = \int_0^{\infty} \frac{2ax^2}{(1 + b^2x^2)(1 + a^2x^2)} dx = \frac{2a}{a^2 - b^2} \int_0^{\infty} \frac{1}{1 + b^2x^2} - \frac{1}{1 + a^2x^2} dx$$

$$= \frac{2a}{a^2 - b^2} \cdot \left(\frac{1}{b} - \frac{1}{a}\right) \arctan x\Big|_0^{\infty} = \frac{\pi}{b(a + b)}$$

Thus,

$$I(a) = \frac{\pi}{b} \ln(a+b) + C$$

Note that $I(0) = 0$, so

$$C = -\frac{\pi}{b} \ln(b)$$

Thus,

$$I(a) = \frac{\pi}{b} \ln\left(\frac{a}{b} + 1\right)$$

Now, consider substituting $\tan\theta = bx$ into the original definition of $I(a)$. We obtain

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{1}{b} \ln\left(1 + \frac{a^2 \tan^2\theta}{b^2}\right) d\theta = \frac{1}{b} J(a,b) - \frac{2}{b} \int_0^{\frac{\pi}{2}} \ln(b\cos\theta)\, d\theta$$

Thus,

$$I(a) = \frac{1}{b} J(a,b) - \frac{2}{b} \left(\frac{\pi \ln b}{2} + \int_0^{\frac{\pi}{2}} \ln(\cos\theta) d\theta\right)$$

The last integral is well known to equal $\frac{-\pi\ln(2)}{2}$. Finally,

$$I(a) = \frac{\pi}{b} \ln\left(\frac{a}{b} + 1\right) = \frac{1}{b}\left(J(a,b) - \pi \ln b + \pi \ln 2\right)$$

So,

$$J(a,b) = \pi \ln\left(\frac{a+b}{2}\right)$$

and our final answer is

$$J(3,11) = \boxed{\pi \ln 7}$$