EQUIVALENT NOTIONS FOR THE DIMENSION OF NOETHERIAN LOCAL RINGS

AARON LANDESMAN

1. THE MAIN RESULTS

The goal of this document is to give a proof of Theorem 1.4. The material of Sections 2 and 4 are well known, but, as far as I’m aware, Section 3 is original. While the standard proof from Atiyah MacDonald bounds each notion of dimension successively by the next, we give a very different proof, first showing the first two notions are equivalent, and then showing the first two notions are equivalent to the third. One reason this proof seems nice to me, is that after black-boxing several standard results from dimension theory, the proof is almost automatic. We first give some definitions, so that we can state the main Theorem 1.4.

Definition 1.1. For $A$ a ring, define the Krull Dimension of $A$, to be the maximum $n$ such that there exists a sequence of primes $p_0 \subset p_1 \subset \cdots \subset p_n$. We notate the Krull dimension of $A$ by $\dim A$.

Theorem 1.2. Let $(A, m)$ be a Noetherian local ring. Then, the function $n \mapsto \dim_{A/m}(m^n/m^{n+1})$ is eventually polynomial, and is an isomorphism invariant of the ring $A$.

The proof of this theorem is the only proof we shall omit from this post. See Vakil, Theorem 18.6.1 for a proof.

Definition 1.3. Let $(A, m)$ be a local ring with maximal ideal $m$ and $H_A : \mathbb{Z} \to \mathbb{Z}$ denote the Hilbert polynomial of $A$, defined by $H_A(n)$ to be the polynomial from Theorem 1.2.

Theorem 1.4. For $A$ a ring, the following are equivalent

1. $\dim A$
2. The minimal $n$ such that there exists $f_1, \ldots, f_n \subset m, m = (f_1, \ldots, f_n)$
3. $\deg H_A$ where $H_A$ is the Hilbert polynomial of $A$.

We will first show that (1) is equivalent to (2), and then show that (1), (2) together are equivalent to (3).

We will assume Krull’s Principal Ideal Theorem, Noether Normalization, and that the dimension of a finitely generated algebras over a field is preserved by base change to a field, which will be proven at the end.

2. PROVING (1) $\iff$ (2) IN THEOREM 1.4

2.1. Reducing to Krull’s Principal Theorem. We wish to prove the equivalence of (1) $\iff$ (2) in Theorem 1.4. This shall follow without much difficulty from the following, which we prove later in Subsection 4.1.

Theorem 2.1. (Krull’s Principal Ideal Theorem) Suppose $f \in A$, with $A$ Noetherian. Then $\text{codim}_A(f) \leq 1$ and if $f$ is a non-zerodivisor then $\text{codim}_A(f) = 1$.

First, let us deduce 1 $\iff$ 2 from 2.1
Lemma 2.2. If \( p_1 \subseteq p_2 \subseteq p_3 \) is a sequence of ideals, with \( f \in p_3 \), then there is an \( q, p_1 \subseteq q \subseteq p_3 \) with \( f \in q \).

Proof. Replace \( A \) by \( A/p_1 \). Further, localize at \( p_3 \). We then obtain a Noetherian ring, and by Krull’s principle ideal theorem 2.1, we know \( f \) is contained in some prime of codimension 1. In other words, some prime containing \( p_1 \), but not equal to the maximal ideal \( p_3 \). Let that prime be \( q_1 \). Taking its preimage in \( A \) yields the desired prime \( q \). □

Lemma 2.3. Given any chain of primes \( p_0 \subseteq \cdots \subseteq p_n \), with \( f \in p_n \), there is a chain of primes \( p_0 \subseteq q_1 \subseteq \cdots \subseteq p_n \) with \( f \in q_1 \).

Proof. Immediate from Lemma 2.2. □

Here is an important lemma, which provides a crucial link between zero divisors and minimal primes. While in Noetherian rings, the zero divisors are exactly the elements in some associated prime, it is in general true that all elements in minimal primes are zero divisors.

Lemma 2.4. Let \( A \) be a ring (not necessarily Noetherian). If \( a \in p \) with \( p \) a minimal prime, then \( a \) is a zero divisor.

Proof. Observe that \( A_p \) is a ring with a single prime ideal \( p \). Therefore, \( p \) is the nilradical of \( A \), implying it is nilpotent. Hence, \( a \in p \) implies \( a^n = 0 \). Then, lifting this back to \( A \), we see there is some \( b \in A \setminus p \) with \( ba^n = 0 \) implying \( a \) is a zero-divisor. □

Corollary 2.5. For \( (A, m) \) a local ring with \( f \in m \), and \( f \) a non-zerodivisor, we have \( \dim A/(f) = \dim A - 1 \).

Proof. We know \( \dim A/(f) \geq \dim A - 1 \) 2.3, as we have produced a chain of primes of length \( \dim A \) with each prime containing \( f \). However, we also know \( (f) \) is not contained in any minimal prime of \( A \) if \( f \) is a non-zerodivisor, by Lemma 2.4 and so \( \dim A/(f) \leq \dim A - 1 \). □

Corollary 2.6. For \( (A, m) \) a local ring. Suppose \( f_1, \ldots, f_n \) are such that \( f_i \) is a non-zerodivisor in \( A/(f_1, \ldots, f_{i-1}) \). Then \( \dim A/(f_1, \ldots, f_n) = \dim A - n \).

Proof. This follows by applying Corollary 2.5 \( n \) times. □

We can now show \((1) \iff (2)\) in Theorem 1.4.

Proof. (Proof of \((1) \iff (2)\) in Theorem 1.4)

First, let us show the Krull dimension is at least the number of generators of a radical of the maximal ideal. Suppose \( \dim A = n \). Then, choosing \( f_i \) to be a non-zerodivisor in \( A/(f_1, \ldots, f_{i-1}) \) we obtain \( \dim A/(f_1, \ldots, f_n) = \dim A - n = 0 \) by Corollary 2.6, implying that the dimension is at least the minimal number of generators. Conversely, the minimal number of generators is at least the Krull dimension, because by Krull’s Principal Ideal Theorem 2.1, we have \( \dim A/(f) \geq \dim A - 1 \), and so the number of generators cannot be less than the dimension, as after quotienting by all generators we get a zero dimensional ring. □

3. Proving \((1), (2) \iff (3)\) in Theorem 1.4

Assuming the above Theorem 1.2, which is clearly necessary to make sense of Theorem 1.4 let us proceed to prove Theorem 1.4.

Proposition 3.1. Let \( B = k[x_0, x_1, \ldots, x_n]/I \), with \( I \) homogeneous, and let \( m = (x_0, x_1, \ldots, x_n) \). Define the local ring \( A = B_m \). For such a local ring, we have conditions \((2), (3)\) from Theorem 1.4 are equivalent.
Proof. (Proof of (1), (2) ⇐⇒ (3) from Theorem 1.4, assuming Proposition 3.1)

We can now complete the main theorem, assuming we know it holds in the case that \( A = k[x_0, \ldots, x_n]/I \), as stated in Proposition 3.1.

Let \( k = A/m \) and let \( f_1, \ldots, f_s \) be generators of \( m \). Then, define the associated graded ring \( R = \oplus_{i=0}^{\infty} m^i/m^{i+1} \). Observe that \( R \) is a finitely generated graded ring over \( k \), hence isomorphic to \( k[x_1, \ldots, x_s]/I \) for some \( I \). Note that the Hilbert polynomial of \( A \) is equal to the Hilbert polynomial of \( S_n \), where \( n \) is the maximal ideal generated by the images of the \( f_i \) in \( S \). The two are equal because \( n^i/n^{i+1} \cong m^i/m^{i+1} \) as \( k \) vector spaces, by construction of \( R \). Additionally, \( \dim A = \dim R \) using the equivalence of (1) ⇐⇒ (2), proven above, because the image of a minimal system of generators \((f_1, \ldots, f_n)\) for \( m \) are also a minimal system of generators for \( n \), and visa versa.

To complete our proof, it suffices to prove Proposition 3.1.

Proof of Proposition 3.1. First, let us reformulate the definition of Hilbert polynomial in terms of the \( i \)th graded component of \( B \). Observe that \( m^i/m^{i+1} \) is the \( i \)th homogeneous graded part of \( B \), where \( B \) is graded by degree of the corresponding polynomials. So, denoting the \( i \)th homogeneous graded part of \( B \) by \( B_i \), we have \( \dim B_i = H_B(i) \), since \( \dim B_i = \dim m^i/m^{i+1} \). To prove the proposition, it suffices to show that the polynomial eventually equal to \( i \mapsto \dim A_i \) has degree equal to \( \dim B \).

Next, we may note that the proposition holds when \( I = 0 \), i.e., when \( B \) is affine. Indeed, in this case \( \dim A_i = \binom{n+i}{n} \), which is certainly polynomial in \( i \), of degree \( n \). Additionally, we know \( \dim k[x_0, \ldots, x_n] = n + 1 \) by Theorem 4.16, and so the Proposition holds in this case.

Next, we reduce to the case that the field \( k \) is infinite, using the fact that dimension is preserved under base change by a field, see Proposition 4.12. Indeed, we only need to worry about the case that \( k \) is finite. Letting \( L \) be an infinite extension of \( k \), for example the algebraic closure, we see by Proposition 4.12 that \( \dim B = \dim B \otimes_k L \). Additionally, it is clear that \( \dim B_i = \dim (B \otimes_k L)_i \), as vector spaces over \( k, L \) respectively, since a basis of the first is mapped to a basis of the second, under tensoring up.

The key point of reducing to the infinite case, was that we can now apply Noether Normalization in the infinite case, see Theorem 4.14, which says we can construct a finite injective map \( \phi : k[x_1, \ldots, x_n] \to B \), to actually be a linear map. In particular, we can take \( x_1, \ldots, x_n \) to map to elements of the maximal ideal of \( B \), in which case the corresponding map is actually a graded map.

Since \( \phi \) is a finite extension, we know \( \dim k[x_1, \ldots, x_i] \leq \dim B_i \leq C \cdot \dim k[x_1, \ldots, x_i] \), where \( C \) is the number of finite number generators of \( B \) over \( k[x_1, \ldots, x_i] \) as a module. The first inequality comes from the fact that the map \( \phi \) is injective, while the second inequality comes from the fact that the map is a map of graded rings, and \( B \) is finitely generated. However, it follows from Lemma 4.11 that both \( k[x_1, \ldots, x_i], B \) have the same dimension.

Hence, it suffices to check the theorem holds for affine space, which we have already done above. \( \square \)

4. Additional Proofs

4.1. The Proof of Theorem 2.1. We now proceed to prove Theorem 2.1.

Lemma 4.1. Any Noetherian ring with one prime ideal is Artinian.

Proof. Observe that \( m^n = 0 \) for some \( n \). Additionally, \( m^i/m^{i+1} \) are finite dimensional vector spaces. We then use the fact that if we have an exact sequence of modules \( 0 \to M \to N \to K \to 0 \), we have \( N \) Artinian if and only if \( M, K \) are both Artinian, applied where \( N = m^i, M = m^{i+1}, K = m^i/m^{i+1} \). \( \square \)
We prove the theorem in series of (easy) exercises. For the solutions to these exercises in the form of a complete proof see Vakil, 11.5.2.

**Proof.**

**Exercise 4.2.** Show that if \( f \) is a non-zerodivisor then it must lie in a prime of codimension 1, assuming we know that \((f)\) is either of codimension 0 or 1. Hint: Use Lemma 2.4.

**Exercise 4.3.** Let \( p \) be a minimal prime. Reduce to showing that if \( q \subset p \) then \( A_q \) is Artinian.

**Exercise 4.4.** Consider the “symbolic power” \( q^{(n)} := \{ a \in A | \exists s \in A \setminus q, sa \in q^n \} \). Show that there is some \( k \) for which \( q^{(k)} \subset q^{(k+1)} + (f) \).

**Exercise 4.5.** Show \( q^{(k)} = q^{(k+1)} + (f)q^{(k)} \).

**Exercise 4.6.** Show \( q^{(k)} = q^{(k+1)} \). Hint: Use Nakayama’s lemma, applied to the module \( q^{(k)}/q^{(k+1)} \) and the previous exercise.

**Exercise 4.7.** Show \( q^kA_q = q^{k+1}A_q \).

**Exercise 4.8.** Show \( q^kA_q = 0 \), again using Nakayama’s lemma.

**Exercise 4.9.** Prove the theorem. Hint: By the previous exercise, \( A_q \) is Artinian.

\[ \square \]

4.2. Dimension is Preserved Under Base Change by a Field.

**Lemma 4.10.** Let \( A \to B \) be an integral extension. Then \( B \otimes_A A_p/p_p \) is zero dimensional.

**Proof.** First, integral morphisms are preserved under base change, so to show the fiber above every point has dimension 0, it suffices to show that any integral morphism \( Y \to \text{Spec} \ k \) must have dimension 0. To see this, it suffices to show that no prime in \( Y \) contains another prime of \( Y \). Therefore, we can assume \( Y \) is affine, and further that it is an integral domain. We wish to show \( Y \) is actually a field. But this follows because any integral domain which is an integral extension of a field, is actually a field. To see this, suppose \( k \to A \) is an integral extension. Let \( a \in A \) satisfy a minimal polynomial \( \sum_{i=0}^{n} k_i a^i = 0 \). Then, \( k_0 = a(\sum_{i=1}^{n} k_i a^{i-1}) \) and so \( f(a) = a^{-1} \).

**Lemma 4.11.** Let \( A \to B \) be a integral extension of rings. Then \( \dim A = \dim B \).

**Proof.** First, by the going up theorem, we know every chain in \( X = \text{Spec} A \) pulls back to some chain in \( Y = \text{Spec} B \). So, \( \dim Y \geq \dim X \). To show the converse, it suffices to show that the chain in \( Y \) is saturated. Equivalently, there are no two primes in \( Y \), one contained in the other, which map to the same prime in \( X \). However, this follows from the previous lemma.

**Proposition 4.12.** Suppose \( A \) is a, finitely generated as a \( k \) algebra, and of dimension \( n \), and \( L/k \) is an algebraic extension. Then, \( A \otimes_k L \) is also \( n \) dimensional

**Proof.** Let \( p \in A \otimes_k K \) be a minimal prime. Let \( \phi : A \to A \otimes_k K \). We will show that the map \( \phi : A \to A \otimes_k K/p \) is injective. To see this, if \( \phi(a) = 0 \) then \( a \) maps to a zero divisor in \( A \otimes_k K \), since \( \phi(a) \) lies in a minimal prime, and all elements of minimal primes are zero divisors. However, \( A \otimes_k K \cong \otimes_k A^{K/k} \) is free as an \( A \) module. Therefore, multiplication by any \( a \in A \) is injective. Hence, no \( a \) can be sent to a zero divisor, except 0 and so \( a = 0 \). Thus, \( \phi \) is injective, implying \( A \otimes_k K/p \) have the same dimension, by the previous lemma. Then, it follows that \( A \otimes_k K \) have the same dimension, because \( A \otimes_k K/p \) corresponds to each irreducible component of \( A \otimes_k K \) as \( p \) ranges over the minimal primes of \( A \otimes_k K \).

**Remark 4.13.** Without too much additional work, this theorem even holds for non-algebraic extensions.
4.3. Noether Normalization. Let us now prove Noether Normalization:

**Theorem 4.14. (Noether Normalization For Infinite Fields)** For $A$ a finitely generated integral domain over an infinite field $k$, there are $x_1, \ldots, x_n \in A$, algebraically independent over $k$, so that $A$ is finite over $k[x_1, \ldots, x_n]$ for $n = \text{tr. deg} (A)/k$.

**Proof.** Say $A$ is generated by $y_1, \ldots, y_m$. If $m = n$, we are done, as $y_1, \ldots, y_m$ are necessarily independent, as otherwise, the transcendence degree would be lower. So, we may assume $m > n$. We will show that there are some $z_1, \ldots, z_{m-1} \in A$ so that $A$ is finite over $k[z_1, \ldots, z_{m-1}]$. Applying this transformation $m - n$ times, we can conclude the theorem, because the composition of finite maps is a finite map.

Now, our aim is to produce $z_i = y_i - k_i y_m$, for $k_i \in k$, so that $f(z_1, \ldots, y_m)$ is a monic polynomial in $y_m$. This would complete the proof, as it would exhibit $A$ as finite over $k[z_1, \ldots, z_{m-1}]$, thus reducing $m$ by 1.

Indeed, if $n = \deg f$, we can write $f(y_1 - k_1 y_m, \ldots, y_n - k_n y_m)$ as a polynomial in the $k_i$ of the form $g(k_1, \ldots, k_n)$. Letting $g_n(k_1, \ldots, k_n)$ be the degree $n$ part of $g(k_1, \ldots, k_n)$, it suffices to show there is some value for which $g_n \neq 0$, as if $g_n = k$, we can then replace the polynomials $g$ and $f$ by $\frac{1}{g} f, \frac{1}{g_n} f$ without changing their solutions. Then, choosing $k_1, \ldots, k_n$ so that $\frac{1}{g_n}(k_1, \ldots, k_n) = 1$, we have $h(y_n) = \frac{1}{g_n}(f(y_1 - k_1 y_m, \ldots, y_n - k_n y_m)$ is a monic polynomial in $y_m$, completing the proof. □

**Remark 4.15.** Note that Noether Normalization still holds for finite fields, but we may not be able to take the projection to be linear.

4.4. Transcendence Degree. Finally, we shall prove that the dimension of finite type schemes over a field is equal to their transcendence degree. Once again, we prove the theorem in series of (easy) exercises. For the solutions to these exercises in the form of a complete proof see Vakil, 11.2.7.

**Theorem 4.16.** If $A$ is an integral domain finitely generated over $k$ then $\dim A = \text{tr. deg} K(A)/k$.

**Proof.**

**Exercise 4.17.** Reduce to the case of showing $\dim k^n_A = n$. Hint: Use Noether Normalization and Lemma 4.11.

**Exercise 4.18.** Observe that $\dim k^n_A \geq n$. Hint: Produce a chain of $n$ increasing ideals.

**Exercise 4.19.** Reduce, by induction, to showing that $k^n_A$ is $n$ dimensional, assuming $k^{n-1}_A$ is $n - 1$ dimensional.

**Exercise 4.20.** Suppose we have a chain of prime ideals $0 \subset p_1 \subset \cdots p_m$, with $m > n$ in $k^n_A$. Then, choose $f \in p_1$, irreducible, and reduce to showing that the dimension of $k[x_1, \ldots, x_n]/f$ is $n - 1$, by noting its transcendence degree is $n - 1$.

**Exercise 4.21.** Show that $\dim k[x_1, \ldots, x_n]/f \leq n - 1$, using Noether normalization, and the inductive hypothesis.

**Exercise 4.22.** Conclude $\dim k[x_1, \ldots, x_n]/f = n - 1$, and hence conclude the theorem. □

5. Acknowledgements

Thanks to Dennis Gaitsgory for discussing this proof with me, and for pointing out the idea of using 4.12 to deal with the case that the variety is over a finite field.