THE TORELLI THEOREM FOR CURVES

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Abstract. After a brief discussion of Jacobians, we prove the Torelli theorem for curves, which states that a curve can be recovered from its principally polarized Jacobian. We also discuss an infinitesimal version of the Torelli theorem.

1. Introduction

The goal is to prove the Torelli theorem for curves: That a curve C can be reconstructed from a principally polarized abelian variety. We will start with a brief discussion of Jacobians, then state and prove equivalent versions of the Torelli theorem, and finally discuss an infinitesimal version of the Torelli theorem. We primarily follow [Pet14], with slight modifications and extrapolations. We also learned much of the material from [McM].

2. Jacobians

Let C be a curve over the complex numbers, which for the purposes of these notes means a smooth, projective, 1 dimensional connected scheme over C. There are two constructions for the Jacobian of C: one algebraic and one analytic.

Definition 2.1 (Algebraic Definition). For C a curve, we define the Jacobian of C, notated Jac C to be Pic^0(C), the group scheme of degree 0 line bundles.

Definition 2.2 (Analytic Definition). Let C a curve over C. Let \Omega^1_C denote the sheaf of differentials of C and define

\[ \tau : H_1(C, \mathbb{Z}) \to H_0(C, \Omega^1_C) \]

\[ [\gamma] \mapsto \left( \omega \mapsto \int_{\gamma} \omega \right). \]

Then, the Jacobian of C is defined to be

\[ \text{Jac } C := \frac{H^0(C, \Omega^1_C)^\vee}{\tau(H_1(C, \mathbb{Z}))}. \]
In order to avoid a horrendous conflict of notation, we had better prove the following!

**Proposition 2.3.** Over $\mathbb{C}$, the analytic Jacobian and algebraic Jacobian are isomorphic.

**Proof.** Consider the exponential exact sequence

\[(2.1) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C^\times \longrightarrow 0.\]

Taking cohomology, we obtain a long exact sequence with terms

\[(2.2) \quad H^1(C, \mathbb{Z}) \longrightarrow H^1(C, \mathcal{O}_C) \overset{f}{\longrightarrow} H^1(C, \mathcal{O}_C^\times) \overset{g}{\longrightarrow} H^2(C, \mathbb{Z}).\]

Now, we obtain the identifications

\[(2.3) \quad H^1(C, \mathbb{Z}) \longrightarrow H^1(C, \mathcal{O}_C) \overset{f}{\longrightarrow} H^1(C, \mathcal{O}_C^\times) \overset{g}{\longrightarrow} H^2(C, \mathbb{Z})
\]

\[\downarrow \sim \downarrow \sim \downarrow \sim \]

\[H_1(C, \mathbb{Z}) \longrightarrow H^0(C, \Omega^1_C) \vee \quad \text{Pic}(C) \overset{\text{deg}}{\longrightarrow} \mathbb{Z}.\]

**Exercise 2.4** (Difficult exercise). Verify that the two squares above commute.

The first and last vertical arrows are gotten by Poincare duality. The second vertical arrow is gotten by Serre duality, and the third vertical map is a standard description of the Picard group via cocycles. Now, we claim $\text{Pic}^0(C) \cong H^0(C, \Omega^1_C) / \iota(H_1(C, \mathbb{Z}))$. To see this, observe that $\text{Pic}^0(C) = \ker g$ while $H^0(C, \Omega^1_C) / \iota(H_1(C, \mathbb{Z})) = \text{im } f$. Hence, the two are equal because $\ker g = \text{im } f$, by the long exact sequence on cohomology. □

3. **Statement and Equivalent Formulations of the Torelli Theorem**

In this section, we give various equivalent statements of the Torelli theorem. In order to state the main version of the Torelli theorem we will prove, we first define a particular map $u_q$.

**Definition 3.1.** Let $C$ be a curve over $\mathbb{C}$, let $\text{Jac}(C)$ be its Jacobian. Let $q$ be a point of $C$, and let $W_{g-1}(C)$ denote the image of $C^{g-1} \to J(C)$
under the map
\[ u_q : C^{g-1} \rightarrow \text{Jac}(C) \]
\[ (x_1, \ldots, x_{g-1}) \mapsto \left( \omega \mapsto \sum_i \int_{x_i}^{x_i + q} \omega \right) \]

**Remark 3.2.** Technically speaking, we should notate \( W_{g-1}(C) \) as \( W^q_{g-1}(C) \) to reflect dependence on \( q \). However, \( W^q_{g-1}(C) \) and \( W^p_{g-1}(C) \) are related by a translation (namely by \( u_q(p, \ldots, p) \)) and so this justifies the omission of dependence on \( q \).

We now state the main version of the Torelli theorem, whose proof we defer until section 4.

**Theorem 3.3 (Torelli for curves).** Any curve \( C \) can be recovered from the datum of \( \text{Jac}(C) \) and \( W_{g-1}(C) \).

Before moving on to further formulations, let’s check Torelli in the simple cases that \( g = 0, 1, 2 \).

**Example 3.4.** Proof of Torelli in the cases \( g = 0, 1, 2 \).

1. In the case \( g = 0 \), the only abelian variety is a point, and the only curve is \( \mathbb{P}^1 \), so we know the curve before even starting!
2. In the case of \( g = 1 \), we have \( C \cong \text{Jac}(C) \) given by choosing a point \( p \) on \( C \) and sending \( q \) to \( p - q \), using the elliptic curve group structure. This is injective because, using the algebraic construction we have \( \mathcal{O}_C(p - q) \neq \mathcal{O}_C \) for \( p \neq q \). (Recall that if \( \mathcal{O}(p - q) = \emptyset \), then there would be a function \( f \) with a pole at \( q \) and a zero at \( p \), which would give a degree 1 map \( C \rightarrow \mathbb{P}^1 \), which would then imply \( C \) has genus 0.) Then, the analytic construction yields that \( \text{Jac}(C) \) is a smooth genus 1 curve, and hence the map \( C \rightarrow \text{Jac}(C) \) is an isomorphism.
3. In the case of \( g = 2 \), the theorem says we get to keep track of \( \text{Jac}(C) \) together with the image of \( C^1 = C \). Note that this map is injective because any two points \( p \) and \( q \) correspond to different divisors, as \( \mathcal{O}(p - q) \) is only trivial if \( p = q \), as in the genus 1 case. It follows that \( C \) maps injectively into its Jacobian, and so we can recover \( C \) as the normalization of \( W_1(C) \). (In fact, \( W_1(C) \cong C \), but we won’t need this, so we won’t prove it!)

We now turn to some equivalent formulations of **Theorem 3.3**.
Theorem 3.5. Let \((A, \theta)\) be a principally polarized abelian variety together with its principal polarization, which is the Jacobian of a curve \(C\). Then, \(C\) is uniquely determined.

**Proof.** This would follow from Theorem 3.3 if we knew that \(\theta\) were equal to \(W_{g-1}(C)\), up to translation. Indeed this is the case, as follows from the theory of \(\theta\) functions, see [GH94, Riemann’s Theorem, p. 338]. □

We now give a formulation in terms of moduli spaces.

Theorem 3.6. Let \(A_g\) denote the moduli stack of principally polarized abelian varieties of dimension \(g\) and let \(\mathcal{M}_g\) denote the moduli stack of curves of genus \(g\). Then, the torelli map \(C \mapsto \text{Pic}^0(C)\) is an injection.

**Proof.** This is just a reformulation of Theorem 3.5 which implicitly uses the equivalence from Proposition 2.3. □

Finally, we have a hodge theoretic reformulation.

Theorem 3.7. A genus \(g\) curve is determined by its polarized hodge structure.

**Proof.** By Theorem 3.5, we only need know that one can obtain the Jacobian of a curve from its polarized hodge structure. Recall that a polarized hodge structure is just the datum of the lattice \(H^0(C, \Omega^1_C) \otimes \mathbb{Z}\) inside \(H^0(C, \Omega^1_C)\) together with the cup product form \(Q\) on this vector space, satisfying the Riemann bilinear relations

\[
Q(\omega, \omega') = \int_X \omega \wedge \omega' = 0, \\
iQ(\omega, \overline{\omega}) > 0 \text{ for } \omega \neq 0,
\]

Hence, the result follows from the well known fact that that any complex torus \(V/\Lambda\) (where here \(V = H^0(C, \Omega^1_C)\) and \(\Lambda = H^1(C, \mathbb{Z})\)) with corresponding Riemann bilinear form satisfying these two relations is an abelian variety. □

4. Proof of the Torelli Theorem

4.1. Description of the tangent space. As set up, let \(K_C\) denote the canonical divisor associated to \(\Omega^1_C\), and let \(\phi : C \to \text{Pic}^0(C, K_C)\) denote the canonical map.

**Lemma 4.1.** The map \(u_q\) defined in Definition 3.1 satisfies that the projectivized tangent space to \(u_q(x_1, \ldots, x_{g-1})\) contains the span of \(\phi(x)\) where there is some effective divisor \(E\) so that \(E + x = x_1 + \cdots + x_{g-1}\).
Proof. Indeed, we may view \( u_q(x_1, \ldots, x_{g-1}) \) as the point in the analytic jacobian defined by

\[
\omega \mapsto \sum_i \int_q^{x_i} \omega.
\]

Since the analytic Jacobian is isomorphic to the algebraic Jacobian, if two points correspond to isomorphic line bundles, they will agree. Then, the derivative with respect to the \( x_i \) coordinate is simply the function \( \omega \mapsto \omega(x_i) \), which is the point \( \phi(x_i) \in \mathbb{P}H^0(K_C)\. Therefore, the tangent space to \( W_{g-1}(C) \) contains the span of \( \phi(x_i) \). \qed

**Lemma 4.2.** We have \( \dim W_{g-1}(C) = g - 1 \). Furthermore, if \( D = x_1 + \cdots + x_{g-1} \), with \( x_1, \ldots, x_{g-1} \) distinct points, then \( h^0(C, D) = 1 \) if and only if \( \phi(x_1), \ldots, \phi(x_{g-1}) \) are linearly independent.

**Proof.** Since \( W_{g-1}(C) \) is the image of \( u_q : C^{g-1} \to \text{Jac}(C) \), \( \dim W_{g-1}(C) \leq g - 1 \). To show \( \dim W_{g-1}(C) = g - 1 \), by upper semicontinuity of fiber dimension, we only need show there is some point of \( W_{g-1}(C) \) whose fiber is a point. Using the identification between points of \( \text{Jac}(C) \) and line bundles, we only need verify there is some degree \( g - 1 \) effective divisor \( D \) with \( h^0(C, D) = 1 \). This will follow if we prove the second statement of the lemma, as there are certainly \( g - 1 \) independent points on \( \phi(C) \), since the canonical map is nondegenerate. The second statement of the lemma follows immediately from geometric Riemann-Roch. In more detail, writing \( D = x_1 + \cdots + x_{g-1} \), we have that \( h^0(C, D) - h^0(C, K_C - D) = g - 1 - g + 1 = 0 \). Therefore, \( h^0(C, D) = 1 \) if and only if \( h^0(C, K_C - D) = 1 \). The latter can be interpreted as saying there is only a 1-dimensional vector space space of divisors in \( K_C \) passing through all \( g - 1 \) points. Since \( \mathbb{P}H^0(C, K_C)^\vee \) is the projective space in which the curve is canonically embedded, this is the same as saying that there is only a 1-dimensional vector space of hyperplanes (i.e., a single hyperplane) containing all \( \phi(x_i) \), meaning that the \( x_i \) are independent. \qed

**Lemma 4.3.** Suppose \( x_1, \ldots, x_{g-1} \) are distinct points. The point \( u_q(x_1, \ldots, x_{g-1}) \) is smooth in \( W_{g-1}(C) \) if and only if \( \phi(x_1), \ldots, \phi(x_{g-1}) \) are linearly independent, in which case the projectivized tangent space to \( \mathbb{P}T_{x_1 + \cdots + x_{g-1}} W_{g-1}(C) \) is \( \text{Span} \langle \phi(x_1), \ldots, \phi(x_{g-1}) \rangle \).

**Proof.** Since \( W_{g-1}(C) \) is \( g - 1 \) dimensional by [Lemma 4.2], smoothness is equivalent to the projectivized tangent space being \( g - 1 \) dimensional. If \( \phi(x_1), \ldots, \phi(x_{g-1}) \) are linearly independent, By [Lemma 4.2]
\( h^0(C, D) = 1 \), and so by [Lemma 4.1], we obtain that the projectivized tangent space is spanned by \( \phi(x_1), \ldots, \phi(x_{g-1}) \), which is then \( g-2 \)-dimensional.

Conversely, we only need show that if \( \phi(x_1, \ldots, x_{g-1}) \) are dependent, (but we still have \( x_1, \ldots, x_{g-1} \) distinct,) then the projectivized tangent space is \( g-1 \)-dimensional. Indeed, by [Lemma 4.2], \( h^0(C, D) > 1 \). This implies that for every \( x \in C \) we can find an effective \( E \) with \( x + E = C \). Hence, the projectivized tangent space is that spanned by \( \phi(x) \) for every \( x \in C \), which is all of \( \mathbb{P}H^0(C, K_C)^\vee \), which is \( g-1 \) dimensional. □

Above we showed that a divisor \( D = x_1 + \cdots + x_{g-1} \) is a smooth point of \( W_{g-1}(C) \) when \( h^0(C, D) = 1 \), provided the \( x_i \) were distinct. For the proof of the Torelli theorem for hyperelliptic curves, we will also need to know this is the case when the \( x_i \) are not distinct. This is shown in ACGH, following a deformation theoretic argument:

**Proposition 4.4** ([ACGH85, Corollary (4.5), Chapter IV]). A point \( x_1 + \cdots + x_{g-1} \in W_{g-1}(C) \) is smooth if and only if \( h^0(C, D) = 1 \).

4.2. Completing the Proof of the Torelli theorem.

**Proof of Theorem 3.3 assuming Lemma 4.6 and Lemma 4.7.** As set up, recall we are given \( W_{g-1}(C) \subset \text{Jac } C \) and we wish to recover \( C \). To start consider the smooth locus of \( W_{g-1}(C) \), call it \( W^{sm} \). From this, we can construct the incidence correspondence

\[
\Phi = \{([D, [H]] : H = \text{PT}_{W^{sm}}) \subset \text{Jac } C \times \mathbb{P}H^0(C, K_C) \}.
\]

**Remark 4.5.** Note that although we have written \( H^0(C, K_C) \), so it appears that this incidence correspondence depends on \( C \), we have isomorphisms \( H^0(C, K_C) \cong H^1(C, \mathcal{O}_C) \cong H^1(\text{Jac } C, \mathcal{O}_{\text{Jac } C}) \), and so this is really independent of \( C \). Nevertheless, we introduce this notation so that we may more easily relate this incidence correspondence to the image of \( C \) under the canonical map.

We have maps

\[
(4.1) \quad \Phi \xleftarrow{\pi_1} \text{Jac } C \quad \xrightarrow{\pi_2} \mathbb{P}H^0(C, K_C).
\]
Let $B$ be the closure of the branch locus of $\pi_2$. In the case $C$ is not hyperelliptic, we find from Lemma 4.6 that $B = C^\vee$. Therefore, since $(X^\vee)^\vee = X$ we find $B^\vee = C$, so we can recover $C$.

In the case $C$ is hyperelliptic with Weierstrass points $p_i$, we find $B = \phi(C)^\vee \cup (\cup_i \phi(p_i)^\vee)$ is the closure of the branch locus by Lemma 4.7. Therefore, when $g > 1$, we can recover $\phi(C)$ and $\phi(p_i)$ from the duals of the components of $B$ using that curves of genus at least 2 are hyperelliptic in a unique way. The cases $g = 0, 1$ are covered in Example 3.4. Then, since $C \to \phi(C)$ is 2:1 and branched precisely at the $\phi(p_i)$, we can recover $C$.

Note finally that the Jacobian of a hyperelliptic curve cannot be isomorphic to the Jacobian of a non-hyperelliptic curve, because the branch locus of $\pi_2$ for a hyperelliptic curve has more components than that of a non-hyperelliptic curve. This holds because the dual of a smooth scheme is irreducible. (To see the dual to a smooth irreducible scheme $X$ is irreducible, one can construct the incidence correspondence of pairs of points on $X$ and hyperplanes tangent to $X$. This incidence correspondence is irreducible because it maps to $X$ with irreducible fibers of the constant dimension, and the dual variety is the image of this incidence correspondence in the space of hyperplanes.)

It only remains to prove Lemma 4.6 and Lemma 4.7.

**Lemma 4.6.** If $C$ is not hyperelliptic, then $B = C^\vee \subset \text{IPH}^0(C, K_C)$.

**Proof.** First, we will show $\pi_2$ is a quasi-finite map of degree $\binom{2g-2}{g-1}$. To see this, for any hyperplane plane $H \subset \text{IPH}^0(C, K_C)$, $H \cap C$ consists of $2g - 2$ points, counted with multiplicity. First, observe that for a general plane $H$, there will be $2g - 2$ points at which $H$ intersects $C$, and any subset of size $g - 1$ among these $2g - 2$ points will be independent. (To see this, note that the locus of $(g-1)$-tuples of points on $C$ which are dependent is a strict subset of $C^{g-1}$, so it has dimension at most $g - 2$, but $\Phi$ is $g - 1$ dimensional.) In this case, there are generically $\binom{2g-2}{g-1}$ such $g - 1$ tuples, which implies the degree of $\pi_2$ is $\binom{2g-2}{g-1}$.

We claim that whenever $H$ is not tangent to $C$, the map $\pi_2$ is unramified. In the case that $H$ is not tangent to $C$, the map will certainly be unramified if it has $\binom{2g-2}{g-1}$ preimages. This will only fail to be the case if some collection of $g - 1$ points in $C \cap H$ are dependent. But, in this case, this collection of points does not determine a divisor in the smooth locus of $W$, by Lemma 4.3. So, although the number of
preimages of such an \([H]\) under \(\pi_2\) drops, such an \([H]\) will not lie in the ramification locus. Therefore, the fiber of the map \(\pi_2\) will contain all \(g - 1\) tuples of independent points which are subsets of \(H \cap C\).

To conclude, we only need verify that a general \(H\) tangent to \(\phi(C)\) will lie in the branch locus. To see this, note that a general \(H\) tangent to \(\phi\) can be written as \(H \cap C = x_1 + x_2 + x_3 + \cdots + x_{2g-2}\), where \(x_2 = x_1\), and further so that \(x_1, x_3, \ldots, x_g\) form an independent set. Choose a divisor \(D\) of the form \(D = x_1 + x_3 + \cdots + x_g = x_2 + x_3 + \cdots + x_g\). Then, choosing an analytic path passing through \(H\), call it \(\{H_t\}_{t \in (-\epsilon, \epsilon)}\), so that \(H_0 = H\), define \(H_t \cap C := x_1^t + \cdots + x_{2g-2}^t\). Further, we can choose our path so that \(H_0\) is the only hyperplane tangent to \(C\) in the path. Then, choosing the families of divisors \(D_t\) defined to be \(x_1^t + x_3^t + \cdots + x_{2g-2}^t\), we see that these two families coincide over \(H_0\), but not over any other element \(H_t\). Hence \(\pi_2\) is branched over \([H_0]\). So, the closure of the ramification locus is precisely the hyperplanes tangent to \(C\). □

**Lemma 4.7.** If \(C\) is hyperelliptic with Weierstrass points \(p_i\), then 

\[
B = \phi(C)^\vee \cup \left( \bigcup_{i=1}^{2g+2} \phi(p_i)^\vee \right) \subset \text{Pic}^0(C, K_C).
\]

**Proof.** In this case, we again to show that the branch locus of \(\pi_2\) consists of \(\phi(C)^\vee \cup \left( \bigcup_{i=1}^{2g+2} \phi(p_i)^\vee \right)\), where \(\phi(p_i)^\vee\) denotes the locus of hyperplanes containing \(\phi(p_i)\).

By Proposition 4.4, the number of points in the preimage of \([H]\) is precisely the number of divisors \(D\) on \(C\) mapping to \(\phi(C) \cap H\) with \(h^0(C, D) = 1\). Say \(H \cap C = y_1 + \cdots + y_{g-1}\). Since the rational functions on a hyperelliptic curve \(y^2 = f(x)\) are generated by \(x\) and \(y\), we see that \(h^0(C, D) > 1\) if and only if \(D\) contains two distinct points \(x_i\) and \(x_j\) in its support, mapping to the same point under \(\phi\). Now, any hyperplane not tangent to \(C\) and not containing the image of a Weierstrass point has the same number of such preimages under \(\pi_2\), and hence does not lie in the branch locus. To conclude the proof, it suffices to verify that a general hyperplane meeting a Weierstrass point and a general hyperplane tangent to \(C\) will indeed be in the branch locus of \(\pi_2\).

First, let us deal with the case that \(H\) is tangent to \(C\), showing such \(H\) lie in the ramification locus. Indeed, here we can find two families of divisors limiting to the same divisor over a tangent hyperplane in a manner completely analogous to that done in the proof of Lemma 4.6.
Finally, we deal with the case that $H \cap C$ contains a Weierstrass point in its support, and is otherwise general. In this case, we proceed by constructing two families of divisor limiting to a given divisor in $\pi^{-1}_2(H)$, similarly to the proof of Lemma 4.6. More precisely, write $H$ so that $\phi(p_1) \in H \cap C$ with $p_1$ a Weierstrass point, and choose $D = p_1 + x_2 + \cdots + x_{g-1}$ so that $\phi(D) \subset H \cap C$. Then, we choose a path of hyperplanes $\{H_t\}_{t \in (-\epsilon, \epsilon)}$ with $H_0 := H$. Say $H \cap C = \phi(x^1_t) + \phi(x^2_t) + \cdots + \phi(x^t_{g-1})$. Then, we can choose two families of divisors $D^1_t, D^2_t$ so that $D^1_t = x^1_t + \cdots + x^t_{g-1}, D^2_t = \iota(x^1_t) + \cdots + x^t_{g-1},$ where $\iota$ denotes the hyperelliptic involution. Note that $D^1_0 = D^2_0 = D$, since $\iota(p_1) = p_1$. But, away from 0, these two divisors will be distinct. This shows two branches come together over $[H]$, and hence $[H]$ is a ramification point.

\[\square\]

5. INFINITESIMAL TORELLI

There is also an infinitesimal version of the Torelli theorem. While the Torelli theorem answered the question of whether one can recover a curve from its Jacobian, the infinitesimal Torelli theorem asks whether one can recover a curve from how its Jacobian varies in a small neighborhood. Two versions of this are given in Theorem 5.1 and Theorem 5.5.

**Theorem 5.1.** The torelli map $\tau : M_g \to A_g$ is an injection on tangent spaces at $[C]$ if and only if $C$ is not hyperelliptic or has genus $\leq 2$. Furthermore, the map on tangent spaces at $C$ can be identified with the dual of the multiplication map

$$\text{Sym}^2(H^0(C, \Omega^1_{C})) \to H^0(C, (\Omega^1_{C})^\otimes 2)$$

$$x \otimes y \mapsto x \cdot y$$

**Proof.** First, we need some deformation theory. Recall that the tangent space to a curve $[C]$ in $M_g$ is the same as maps of $k[\epsilon]/(\epsilon^2) \to M_g$ sending the special fiber to $[C]$. In other words, it is a family of genus $g$ curves over $k[\epsilon]/(\epsilon^2)$ with special fiber $C$, also known as a abstract deformation of $C$. Deformations of $C$ are classified up to isomorphism by $H^1(C, T_C)$.

Additionally, the tangent space to $A_g$ (remember we keep track of the principal polarization, so it is not just $H^1(\text{Jac } C, T_{\text{Jac } C})$ at $(\text{Jac } C, W_{g-1}(C))$ can be identified with $\text{Sym}^2(H^1(\text{Jac } C, O_{\text{Jac } C}))$. This can be seen from the complex analytic perspective that $A_g$ corresponds to forms $Q$ satisfying the Riemann bilinear relations, which means that the matrix corresponding to the form $Q$ must be symmetric and have positive
definite imaginary part. In an infinitesimal neighborhood, the latter condition will automatically be satisfied, and so the point is that the tangent space can be identified with how symmetric matrices vary, which is precisely $\text{Sym}^2(V)$, where $V = H^1(\text{Jac } C, \mathcal{O}_{\text{Jac } C})$.

Summing up the above, the map on tangent spaces is given by $H^1(C, T_C) \to \text{Sym}^2(H^0(\text{Jac } C, \mathcal{O}_{\text{Jac } C}))$. By Serre duality, since $H^1(C, T_C) \cong H^0(C, \Omega^1_C \otimes T_C^\vee) \cong H^0(C, (\Omega^1_C)^\otimes 2)$, and $H^1(\text{Jac } C, \mathcal{O}_{\text{Jac } C}) \cong H^1(C, \mathcal{O}_C) \cong H^0(C, \Omega^1_C)$, this is dual to the natural multiplication map $\text{Sym}^2(H^0(C, \Omega^1_C)) \to H^0(C, (\Omega^1_C)^\otimes 2)$ $x \otimes y \mapsto x \cdot y$.

**Exercise 5.2** (Highly strenuous exercise, see [OS79, Theorem 2.6]). Verify that the multiplication map is indeed dual to the map gotten from the tangent space of the Torelli map. Of course, this is the “only” natural map floating around, but it takes serious effort to follow these identifications.

Hence, we have reduced to verifying that the natural map $\text{Sym}^2(H^0(C, \Omega^1_C)) \to H^0(C, (\Omega^1_C)^\otimes 2)$ is surjective for curves if and only if the curve is not hyperelliptic or has genus at most 2. Indeed, this is precisely the content of Noether’s famous theorem:

**Theorem 5.3** (Noether). For $C$ a curve of genus at least 3, the natural map $\text{Sym}^2(H^0(C, \Omega^1_C)) \to H^0(C, (\Omega^1_C)^\otimes 2)$ is surjective if and only if $C$ is hyperelliptic or has genus $\leq 2$.

□

**Remark 5.4.** All hope is not lost for hyperelliptic curves! A similar proof shows that if $\mathcal{H} \subset \mathcal{M}_g$ denotes the locus of hyperelliptic curves, then the Torelli map restricted to the hyperelliptic locus $\tau|_{\mathcal{H}} : \mathcal{H} \to \mathcal{A}_g$ is in fact injective on differentials (and hence an immersion). The proof is quite similar to that above, and boils down to the statement that since the hyperelliptic locus has dimension $2g - 1$, we want the image of the map $\text{Sym}^2(H^0(C, \Omega^1_C)) \to H^0(C, (\Omega^1_C)^\otimes 2)$ to have $2g - 1$, which indeed it does. This can be checked explicitly by writing down a basis of differentials for a hyperelliptic curve.
We can now further state a slightly souped-up version of the infinitesimal Torelli, which even lets us recover the curve from the infinitesimal variation.

**Theorem 5.5.** Suppose $C$ is not trigonal, not hyperelliptic, and not a plane quintic. Then, map on tangent spaces $\tau : M_g \to A_g$ at $[C]$ determines $C$ up to isomorphism.

**Remark 5.6.** An alternate way to phrase Theorem 5.5 is to say that when $C$ is not hyperelliptic, trigonal, or a plane quintic, $C$ can be recovered from the infinitesimal variation of its polarized Hodge structure, though we won’t make this precise.

**Proof.** By Theorem 5.1, we know that the map on tangent spaces is dual to the multiplication map

$$m : \text{Sym}^2(H^0(C, \Omega_C^1)) \to H^0(C, (\Omega_C^1)^\otimes 2).$$

It suffices to show we can recover $C$ from this multiplication map. The key input here is Petri’s theorem:

**Theorem 5.7 (Petri).** If $C$ is not trigonal, not hyperelliptic, and not a plane quintic, then the map

$$\bigoplus_n \text{Sym}^n(H^0(C, \Omega_C^1)) \to \bigoplus_n H^0(C, (\Omega_C^1)^\otimes n)$$

is generated in degree 1 with relations in degree 2.

By Petri’s theorem, we obtain that the canonical ring $\bigoplus_n H^0(C, (\Omega_C^1)^\otimes n)$ is generated in degree 1 with relations in degree 2. In other words, we can write $\bigoplus_n H^0(C, (\Omega_C^1)^\otimes n) \cong \text{Sym}^* H^0(C, \Omega_C^1)/ \ker m$. But then, since $C \cong \text{Proj} H^0(C, \Omega_C^\infty)$, (since the canonical map is an embedding for non-hyperelliptic curves) we can reconstruct $C$ from this ring. $\square$

**References**


