Polarization Degrees of Freedom

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Table I
MULTIPLICATIVE GAIN IN DEGREES OF FREEDOM FROM POLARIZATION FOR DIFFERENT ARRAY GEOMETRIES AND SCATTERING CONDITIONS.

Scattering Condition	Array Geometries			
	Point	Linear	Planar	Spherical
Fully scattered	6	6	4	2
Azimuth only	6	4	2	2

***Abstract*—This paper unifies the different conclusions on the polarization degrees of freedom in multiple-antenna channels. It shows that the multiplicative gain in the degrees of freedom from polarization depends on the array geometry and the scattering condition.**

I. MAIN RESULTS

The number of degrees of freedom supported by a communication channel is a simple but fundamental performance measure. In multiple-antenna channels, there are different conclusions on the multiplicative gain in the degrees of freedom from polarization. Andrews et al. [1] demonstrated that there are 6 degrees of freedom from co-located antennas – 3 orthogonal current loops plus 3 orthogonal electric dipoles. Marzetta [2] stated that there are 4-fold increase in the degrees of freedom from using polarimetric antenna elements. In our earlier paper [3], we claimed that there are only 2-fold increase and therefore, scatterers would not increase the degrees of freedom from using polarimetric antenna elements. This paper unifies the different perspectives on the polarization degrees of freedom. While all the reported results hold, they are true only under certain geometry of the antenna arrays and scattering condition. With reference to the number of degrees of freedom from single polarization arrays derived in [3], Table I summarizes the multiplicative gain from polarization for different array geometries and scattering conditions. The 6-fold multiplicative gain only holds for a point source (with co-located orthogonal antennas) and a linear array of these point sources in a fully scattered environment. Using higher dimensional antenna arrays reduces the multiplicative gain.

II. VECTOR POINT SOURCES

We will first understand the 6 degrees of freedom from a pair of transmit and receive polarimetric antennas. Consider the radiation pattern from one of the six components made up the polarimetric antennas – the electric dipole oriented along the z-axis. In the far field, the radiated field $\mathbf{E}$ is perpendicular to the direction of propagation $\hat{\mathbf{k}} = (\theta, \phi)$. Along $\hat{\mathbf{k}}$, the radiated field lies on the plane consisting of the dipole and $\hat{\mathbf{k}}$ and therefore, the radiated field consists of only the $\hat{\theta}$ component. Its magnitude is proportional to the projection of the z-axis (antenna orientation) to $\hat{\theta}$ (field direction). Its radiation pattern can then be written as[1]

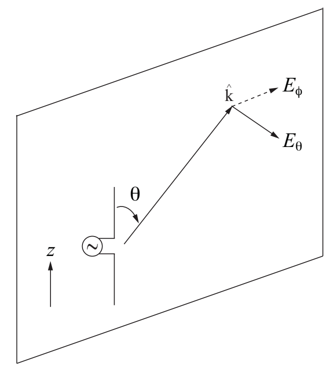

Figure 1. Coordinates for an electric dipole oriented along the z-axis.

$$\mathbf{a}_0(\theta, \phi) = \sqrt{\tfrac{3}{8\pi}}\,\hat{\theta}\sin\theta$$

The factor of $\sqrt{\frac{3}{8\pi}}$ is to normalize $\|\mathbf{a}_0(\theta,\phi)\|$ to unity. The radiation patterns for the electric field oriented along the x-axis and the y-axis are rotated version of $\mathbf{a}_0(\theta,\phi)$, and are given by

$$\mathbf{a}_1(\theta, \phi) = \sqrt{\tfrac{3}{8\pi}}\sqrt{\tfrac{2}{3}}\left(-\hat{\theta}\cos\theta\cos\phi + \hat{\phi}\sin\phi\right)$$
$$\mathbf{a}_2(\theta, \phi) = \sqrt{\tfrac{3}{8\pi}}\sqrt{\tfrac{2}{3}}\left(-\hat{\theta}\cos\theta\sin\phi - \hat{\phi}\cos\phi\right)$$

The radiation patterns for the 3 orthogonal current loops are the dual of the electric dipoles and are given by

$$\mathbf{a}_3(\theta, \phi) = -\sqrt{\tfrac{3}{8\pi}}\,\hat{\phi}\sin\theta$$
$$\mathbf{a}_4(\theta, \phi) = \sqrt{\tfrac{3}{8\pi}}\sqrt{\tfrac{2}{3}}\left(-\hat{\theta}\cos\phi + \hat{\phi}\cos\theta\sin\phi\right)$$
$$\mathbf{a}_5(\theta, \phi) = \sqrt{\tfrac{3}{8\pi}}\sqrt{\tfrac{2}{3}}\left(\hat{\theta}\sin\phi + \hat{\phi}\cos\theta\cos\phi\right)$$

Now the signal that is radiated by the mth component of the transmit polarimetric antennas and intercepted by the nth

[1] For ease of exposition, we do not include the factor $\frac{e^{ik_0 r}}{kr}$ in the expressions.

component at the receiver, has a channel gain of

$$H_{nm} = \iint \mathbf{a}_n^\dagger(\vartheta,\varphi)\,\mathbf{H}(\vartheta,\varphi,\theta,\phi)\,\mathbf{a}_m(\theta,\phi)\,d\Omega_t\,d\Omega_r \quad (1)$$

for $n,m = 0,\cdots,5$, where $\hat{\boldsymbol{\kappa}} = (\vartheta,\varphi)$ is the direction of the incident field at the receiver, $d\Omega_t = \sin\theta\,d\theta\,d\phi$, and $d\Omega_r = \sin\vartheta\,d\vartheta\,d\varphi$. The channel response $\mathbf{H}(\hat{\boldsymbol{\kappa}},\hat{\mathbf{k}})$ is a 2×2 complex integral kernel. It models the channel gain, and the change in polarization due to scatterers. Now we consider two scattering conditions: (1) multipaths spread over the entire (θ,ϕ) propagation space and (2) multipaths only spread over the azimuth directions, ϕ. In the first case, the scattering response satisfies

$$\begin{aligned} &\mathrm{E}\big[H_{ij}(\hat{\boldsymbol{\kappa}},\hat{\mathbf{k}})H^*_{i'j'}(\hat{\boldsymbol{\kappa}}',\hat{\mathbf{k}}')\big] \\ &\quad = \delta_{ii',jj'}\delta(\hat{\boldsymbol{\kappa}}-\hat{\boldsymbol{\kappa}}',\hat{\mathbf{k}}-\hat{\mathbf{k}}') \qquad i,i',j,j'=0,1 \end{aligned} \quad (2)$$

In the second case, we assume that scatterers are lying on the xy-plane and therefore the scattering response is non-zero only at $\theta = \pi/2$. The response satisfies

$$\begin{aligned} &\mathrm{E}\big[H_{ij}(\tfrac{\pi}{2},\varphi,\tfrac{\pi}{2},\phi)H^*_{i'j'}(\tfrac{\pi}{2},\varphi',\tfrac{\pi}{2},\phi')\big] \\ &\quad = \delta_{ii',jj'}\delta(\varphi-\varphi',\phi-\phi') \qquad i,i',j,j'=0,1 \end{aligned} \quad (3)$$

In both cases, we model $H_{ij}(\hat{\boldsymbol{\kappa}},\hat{\mathbf{k}})$'s as zero-mean white Gaussian random processes.

A. Fully Scattered in θ and ϕ

Now we look into the correlation among entries of the channel matrix $\bar{\mathbf{H}}$ in (1) with property given in (2). As $\mathbf{a}_n(\theta,\phi)$'s are orthonormal over (θ,ϕ), H_{nm}'s are *i.i.d.* complex Gaussian random variables. Thus, the channel is equivalent to the *i.i.d.* Rayleigh fading MIMO channel. Furthermore, $\bar{\mathbf{H}}$ is a 6×6 random matrix, the number of degrees of freedom is 6.

B. Scattered in ϕ Only

The radiation patterns $\{\mathbf{a}_n(\frac{\pi}{2},\phi), n = 0,\cdots,5\}$ are orthogonal over ϕ. Denoting the norm of $\mathbf{a}_n(\frac{\pi}{2}r,\phi)$ by σ_n, they are given by

$$\sigma_0^2 = \sigma_3^2 = \frac{3}{4} \qquad \sigma_1^2 = \sigma_2^2 = \sigma_4^2 = \sigma_5^2 = \frac{3}{8}$$

Therefore, elements of the channel matrix $\bar{\mathbf{H}}$ are independent with variances given by

$$\mathrm{var}(H_{nm}) = \sigma_n^2\sigma_m^2$$

The ratio of the largest to the smallest variances is 4. The degrees of freedom should be close to that of *i.i.d.* Rayleigh fading channel. Thus, the number of degrees of freedom is 6.

III. Linear Arrays

Now we consider linear arrays of vector point sources.

A. Fully Scattered in θ and ϕ

Suppose the linear arrays are oriented along the z-axis such that the array responses involve θ only. This choice of coordinates is for convenience. Suppose the element spacing on the transmit array is Δ_t, and the number of vector point sources is n_t, and therefore, the length of the transmit array is $L_t = n_t\Delta_t$. All length quantities are normalized to a wavelength. Similar definitions apply to Δ_r, n_r, and L_r. The channel matrix is given by

$$\begin{aligned} \bar{\mathbf{H}} = \iint & \mathbf{e}_r(\cos\vartheta)\otimes\bar{\mathbf{B}}^\dagger(\vartheta,\varphi)\,\mathbf{H}(\vartheta,\varphi,\theta,\phi) \\ & \mathbf{e}_t^\dagger(\cos\theta)\otimes\bar{\mathbf{B}}(\theta,\phi)\,d\Omega_t\,d\Omega_r \end{aligned} \quad (4)$$

and is of dimension $6n_r\times 6n_t$, where the transmit and the receive array responses are

$$\mathbf{e}_t(\alpha) = \frac{1}{\sqrt{n_t}}\begin{bmatrix}1\\ e^{i2\pi\Delta_t\alpha}\\ e^{i2\pi\cdot 2\Delta_t\alpha}\\ \vdots\end{bmatrix} \quad \mathbf{e}_r(\alpha) = \frac{1}{\sqrt{n_r}}\begin{bmatrix}1\\ e^{i2\pi\Delta_r\alpha}\\ e^{i2\pi\cdot 2\Delta_r\alpha}\\ \vdots\end{bmatrix}$$

respectively, the element response is

$$\bar{\mathbf{B}}(\theta,\phi) = \big[\mathbf{a}_0(\theta,\phi) \quad \cdots \quad \mathbf{a}_5(\theta,\phi)\big]\,\mathbf{T}_{\theta\phi}$$

and the transformation matrix is

$$\bar{\mathbf{T}}_{\theta\phi} = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2} & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & \sqrt{2} & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$$

The narrowest beam from the transmit array is given by

$$f_t(\alpha) := \mathbf{e}_t^\dagger(0)\,\mathbf{e}_t(\alpha) = e^{-i\pi(n_t-1)\Delta_t\alpha}\frac{\sin(\pi L_t\alpha)}{n_t\sin(\pi\Delta_t\alpha)} \quad (5)$$

It attains its main lobe at $\alpha = 0$, has zeros at multiples of $1/L_t$, and is periodic with period $1/\Delta_t$. As α spans over $[-1,1)$, the number of antenna elements is optimized when the period equals to 2, that is, $\Delta_t = 1/2$ and $n_t = 2L_t$. Same conclusion can be drawn by looking at the array resolution over $[-1,1)$ which is $1/L_t$. Hence, the number of antenna elements is $2/(1/L_t) = 2L_t$. Under the optimal sampling, the transmit array response can be approximated by

$$\mathbf{e}_t(\alpha) \approx \sum_{p=-n_t/2}^{n_t/2-1} f_t\big(\alpha-\frac{p}{L_t}\big)\mathbf{e}_t\big(\frac{p}{L_t}\big) \quad \text{as } L_t \gg 1 \quad (6)$$

The $f_t(\cos\theta-\frac{p}{L_t})$'s are the sampling functions over θ. Similarly, we define $f_r(\alpha)$. As $\mathbf{e}_t(p/L_t), p = -\frac{n_t}{2},\cdots,\frac{n_t}{2}-1$ and $\mathbf{e}_r(q/L_r), q = -\frac{n_r}{2},\cdots,\frac{n_r}{2}-1$ are individually orthonormal, the statistical properties of $\bar{\mathbf{H}}$ can be approximated by $\hat{\bar{\mathbf{H}}}$

where its elements are given by

$$\hat{H}_{6(q+n_r/2)+n,6(p+n_t/2)+m}$$
$$= \iint f_r\big(\cos\vartheta - \frac{q}{L_r}\big)\mathbf{b}_n^\dagger\big(\cos^{-1}\frac{q}{L_r},\varphi\big)\,\bar{\mathbf{H}}(\vartheta,\varphi,\theta,\phi)$$
$$\mathbf{b}_m\big(\cos^{-1}\frac{p}{L_t},\phi\big)f_t^*\big(\cos\theta - \frac{p}{L_t}\big)\,d\Omega_t\,d\Omega_r \quad (7)$$

and $\mathbf{b}_n(\theta,\phi)$ is the nth column of $\bar{\mathbf{B}}(\theta,\phi)$. The use of a new set of fundamental modes given by the transformation by $\bar{\mathbf{T}}_{\theta\phi}$ makes sure that $\mathbf{b}_m(\cos^{-1}\frac{p}{L_t},\phi)f_t^*(\cos\theta - \frac{p}{L_t}), \forall m, p$ and $\mathbf{b}_n(\cos^{-1}\frac{q}{L_r},\phi)f_r^*(\cos\theta - \frac{q}{L_r}), \forall n, q$ are individually orthogonal over (θ,ϕ). Consequently, elements of $\hat{\bar{\mathbf{H}}}$ are independent with variances given by

$$\mathrm{var}(\hat{H}_{6(q+n_r/2)+n,6(p+n_t/2)+m}) = \varsigma_{q,n}^2\sigma_{p,m}^2$$

where

$$\sigma_{p,0}^2 = \sigma_{p,3}^2 = \frac{2}{\pi}(1-\frac{p}{L_t})(1+\frac{p}{L_t})$$
$$\sigma_{p,1}^2 = \sigma_{p,5}^2 = \frac{2}{3\pi}\big(1+\frac{p}{L_t}\big)^2 \qquad \sigma_{p,2}^2 = \sigma_{p,3}^2 = \frac{2}{3\pi}\big(1-\frac{p}{L_t}\big)^2$$

Similar definitions apply to the $\varsigma_{q,n}^2$'s. The variances are close to 0 at the corners of $\hat{\bar{\mathbf{H}}}$. When $L_t, L_r \gg 1$, the number of degrees of freedom of $\hat{\bar{\mathbf{H}}}$ should be close to that of *i.i.d.* Rayleigh fading channel, and it is also close to that of $\bar{\mathbf{H}}$. Therefore, the asymptotic number of degrees of freedom is $\min\{6n_t, 6n_r\}$ which is 6 times of the scalar channel.

B. Scattered in ϕ Only

We consider linear arrays oriented along the x-axis. The array responses are $\mathbf{e}_t(\sin\theta\cos\phi)$ and $\mathbf{e}_r(\sin\theta\cos\phi)$. At $\theta = \pi/2$, they become $\mathbf{e}_t(\cos\phi)$ and $\mathbf{e}_r(\cos\phi)$ which are the same as those oriented along the z-axis except that ϕ spans between 0 and 2π while θ spans between 0 and π. At $\cos\phi = \frac{p}{L_t}$, it is seemingly that the 2×6 element response $\bar{\mathbf{B}}(\theta,\phi)$ is a rank 2 constant matrix. However, $\sin\phi$ in the response takes different values over $0 \le \phi < \pi$ and $\pi \le \phi < 2\pi$. Therefore, the element response is not a constant matrix but is a function of ϕ. To derive the dimension of the functional space spanned by the columns of $\bar{\mathbf{B}}(\theta,\phi)$, we use a different transformation matrix:

$$\bar{\mathbf{T}}_\phi = \mathbf{I}$$

Now the element response is

$$\bar{\mathbf{B}}(\tfrac{\pi}{2},\phi) = \sqrt{\tfrac{3}{8\pi}}\begin{bmatrix}1 & 0 & 0 & 0 & -\cos\phi & \sin\phi\\ 0 & \sin\phi & -\cos\phi & -1 & 0 & 0\end{bmatrix}$$

At $\cos\phi = \frac{p}{L_t}$, the 1st and the 5th columns span the same space. Similarly, the 3rd and the 4th columns span the same space but differ from that of the 1st column. The 2nd, 3rd, 5th, and 6th columns are orthogonal. Consequently, the asymptotic number of degrees of freedom is $\min\{4n_t, 4n_r\}$ which is 4 times of the scalar channel.

Linear arrays can only resolve either θ or ϕ between 0 and π. In each of the resolved directions, we could obtain 2 degrees of freedom from polarization. When the angular spread of scatterers is over both θ and ϕ, the extra 3-fold increase in the degrees of freedom is due to the use of vector point sources to resolve the remaining angular direction, not resolved by the arrays. When the angular spread is over ϕ only, there is no need to resolve the θ by the vector point sources. Instead, the vector sources are used to distinguish between $0 \le \phi < \pi$ and $\pi \le \phi < 2\pi$, which yields the extra 2-fold increase in the degrees of freedom.

IV. Planar Arrays

We consider disk-like planar arrays lying on the xy-plane. Suppose (x_p, y_p) is the coordinates of the pth vector source normalized to a wavelength. Then its response in the far field is

$$e^{i2\pi(x_p\sin\theta\cos\phi + y_p\sin\theta\sin\phi)}\,\bar{\mathbf{B}}(\theta,\phi)$$

It is more convenient to use the polar coordinates where the response becomes

$$e^{i2\pi\rho_p\sin\theta\cos(\phi-\phi_p)}\,\bar{\mathbf{B}}(\theta,\phi)$$

and (ρ_p,ϕ_p) is the corresponding coordinates of the pth vector source. We arrange the vector sources on the array in concentric circles:

$$\rho_p = n\Delta\rho_t, \qquad n = 1,2,\cdots,n_{t,\rho} \quad (8a)$$
$$\phi_p = \frac{m}{n}\Delta\phi_t, \quad m = 0,1,\cdots,2\pi n - 1 \quad (8b)$$

The transmit array response is

$$e_{t,\pi n(n-1)+m}(\sin\theta,\phi) = \frac{1}{\sqrt{n_t}}e^{i2\pi n\Delta\rho_t\sin\theta\cos(\phi - \frac{m}{n}\Delta\phi_t)}$$

A. Fully Scattered in θ and ϕ

As the array response involves both θ and ϕ, the sampling functions are derived in two steps. The resolution over θ is derived from

$$\begin{aligned} f_{t,\theta}(\sin\theta) &:= \mathbf{e}_t^\dagger(0,0)\,\mathbf{e}_t(\sin\theta,\phi)\\ &= \frac{1}{n_t}\sum_{n=1}^{n_{t,\rho}}\sum_{m=0}^{n-1} e^{i2\pi n\Delta\rho_t\sin\theta\cos(\phi-\frac{m}{n}\Delta\phi_t)}\\ &\to \frac{1}{\pi R_t^2}\int_0^{R_t}\int_0^{2\pi} e^{i2\pi\rho'\sin\theta\cos(\phi-\phi')}\rho'\,d\phi' d\rho'\\ &= \frac{2}{R_t^2}\int_0^{R_t} J_0(2\pi\rho'\sin\theta)\rho'\,d\rho'\\ &= \frac{2J_1(2\pi R_t\sin\theta)}{2\pi R_t\sin\theta}\\ &= \mathrm{jinc}(2R_t\sin\theta) \end{aligned} \quad (9)$$

as $\Delta\rho_t, \Delta\phi_t \to 0$, where R_t is the radius of the transmit array, $J_n(\cdot)$ is the nth order Bessel function of the first kind, and $\mathrm{jinc}(\cdot)$ is the jinc function. The jinc function is similar to the sinc function: $\mathrm{jinc}(x)$ is maximized at $x = 0$ and $\mathrm{jinc}(0) = 1$; and asymptotically, $\mathrm{jinc}(x)$ has zeros at multiples of 1. Thus, the resolution over $\sin\theta$ is approximately equal to $\frac{1}{2R_t}$.

At $\sin\theta = \frac{p_\theta}{2R_t}$, the narrowest beam over ϕ is given by

$$f_{t,\phi}(p_\theta,\phi) := \mathbf{e}_t^\dagger(\frac{p_\theta}{2R_t},0)\,\mathbf{e}_t(\frac{p_\theta}{2R_t},\phi) \rightarrow \mathrm{jinc}(2p_\theta\sin\frac{\phi}{2}) \quad (10)$$

The resolution over ϕ is approximately equal to $2\sin^{-1}\frac{1}{2p_\theta}$. Putting together, the sampling functions over (θ,ϕ) are

$$f_{t,\theta}(\sin\theta - \frac{p_\theta}{2R_t})f_{t,\phi}(p_\theta,\phi - 2p_\phi\sin^{-1}\frac{1}{2p_\theta})$$

for $p_\theta = 1,\cdots,2R_t$ and $p_\phi = 0,\cdots,\frac{2\pi}{2\sin^{-1}\frac{1}{2p_\theta}} - 1$. When $p_\theta \gg 1$, the resolution can be further approximated by $\frac{1}{p_\theta}$ and $p_\phi = 0,\cdots,2\pi p_\theta - 1$. Comparing the ranges of (p_θ,p_ϕ) to those of (n,m) in (8), we obtain $\Delta\rho_t = 1/2$ and $\Delta\phi_t = 1$ at optimal sampling.

At $\sin\theta = \frac{p_\theta}{2R_t}$ and $\phi = 2p_\phi\sin^{-1}\frac{1}{2p_\theta}$, ϕ is uniquely defined, but $\cos\theta$ in $\bar{\mathbf{B}}(\theta,\phi)$ takes different values over $0 \le \theta < \pi/2$ and $\pi/2 \le \theta < \pi$. Therefore, the element response $\bar{\mathbf{B}}(\theta,\phi)$ is a function of θ, and the dimension of the functional space spanned by the columns of $\bar{\mathbf{B}}(\theta,\phi)$ is 4. The asymptotic number of degrees of freedom is $\min\{4n_t,4n_r\}$ which is 4 times of the scalar channel.

B. Scattered in ϕ Only

At $\theta = \pi/2$ and $\phi = 2p_\phi\sin^{-1}\frac{1}{2p_\theta}$, $\bar{\mathbf{B}}(\theta,\phi)$ is a 2×6 constant matrix which is of rank 2 only. Therefore, the asymptotic number of degrees of freedom is $\min\{2n_t,2n_r\}$ which is 2 times of the scalar channel.

Unlike linear arrays, planar arrays can resolve both θ and ϕ. In each resolved directions, we obtain 2 degrees of freedom from polarization. When the angular spread of scatterers is over both θ and ϕ, vector point sources are used to distinguish between $0 \le \theta < \pi/2$ and $\pi/2 \le \theta < \pi$, which yields the extra 2-fold increase in the degrees of freedom. However, when the angular spread is over ϕ only, there is no extra increase in the degrees of freedom.

V. Spherical Arrays

We can continue using the sampling representation and derive the degrees of freedom for spherical arrays. However, it will involve more additional approximations. We therefore switch gear to a physical approach that can apply to all field regions (the near and the far fields) and all sizes of antenna arrays.

Suppose the radius of the transmit array is R_t. Huygens' principle states that knowledge of the wave field on a surface enclosing the source currents is sufficient to determine the wave field outside the surface. We therefore observe the wave field on a sphere enclosing the transmit array and of radius R_s. The mapping from current sources on the source sphere to wave field on the observation sphere is obtained from the free-space dyadic Green's function:

$$\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \Big(\mathbf{I} + \frac{\nabla\nabla}{k_0^2}\Big)\frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (11)$$

This yields the transmit array response:

$$\bar{\mathbf{A}}_t(\hat{\mathbf{k}},\mathbf{p}) = \bar{\mathbf{G}}(R_s\hat{\mathbf{k}},R_t\hat{\mathbf{p}}), \quad (\hat{\mathbf{k}},\hat{\mathbf{p}}) \in \mathcal{S}\times\mathcal{S} \quad (12)$$

Now we are looking for the decomposition similar to (6) which decomposes the array response into a sequence of dyads. That is, we are looking for two sets of orthonormal functions, $\{\mathbf{u}_i(\hat{\mathbf{k}})\}$ and $\{\mathbf{v}_i^t(\hat{\mathbf{p}})\}$ such that

$$\bar{\mathbf{G}}(R_s\hat{\mathbf{k}},R_t\hat{\mathbf{p}}) = ik_0\sum_n \sigma_n\mathbf{u}_n(\hat{\mathbf{k}})\mathbf{v}_n^{t\dagger}(\hat{\mathbf{p}})$$

The set of singular values $\{\sigma_i\}$ in descending order yields the spectrum of the array response which determines the number of degrees of freedom.

A. Spectrum of the Array Responses

We first make use of the multipole expansion of the Green's function in spherical coordinates [4, Ch. 7]:

$$\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}') = ik_0\sum_{n,m}\frac{1}{n(n+1)}\big[\mathbf{M}_{nm}(\mathbf{r})\hat{\mathbf{M}}_{nm}^\dagger(\mathbf{r}') + \mathbf{N}_{nm}(\mathbf{r})\hat{\mathbf{N}}_{nm}^\dagger(\mathbf{r}')\big]$$

where $\mathbf{M}_{nm}(\cdot)$ and $\mathbf{N}_{nm}(\cdot)$ are the magnetic and the electric multipole fields respectively, and $\hat{\mathbf{M}}_{nm}(\cdot)$ and $\hat{\mathbf{N}}_{nm}(\cdot)$ are the corresponding source functions. As stated in [5], $\big\{\mathbf{M}_{nm}(R\hat{\mathbf{r}}),\mathbf{N}_{nm}(R\hat{\mathbf{r}})\big\}$ and $\big\{\hat{\mathbf{M}}_{nm}(R\hat{\mathbf{r}}),\hat{\mathbf{N}}_{nm}(R\hat{\mathbf{r}})\big\}$ are individually orthogonal over the unit sphere $\mathcal{S}$.

Next, we find the norm of these functions. From definition,

$$\mathbf{M}_{nm}(\mathbf{r}) = \nabla\times\big[\mathbf{r}h_n^{(1)}(k_0r)Y_{nm}(\hat{\mathbf{r}})\big] = -i\sqrt{n(n+1)}h_n^{(1)}(k_0r)\mathbf{X}_{nm}(\hat{\mathbf{r}})$$

where $h_n^{(1)}(\cdot)$ is the spherical Hankel function of the first kind, $Y_{nm}(\cdot)$ is the spherical harmonic function, and $\mathbf{X}_{nm}(\hat{\mathbf{r}})$ is defined as the vector spherical harmonic function in [6, Sec. 9.7]. Using (9.120) in [6], we obtain

$$\int\mathbf{M}_{nm}^\dagger(R\hat{\mathbf{r}})\mathbf{M}_{n'm'}(R\hat{\mathbf{r}})\,d\Omega = n(n{+}1)\big|h_n^{(1)}(k_0R)\big|^2\delta_{nn',mm'}$$

To find the norm of $\mathbf{N}_{nm}(R\hat{\mathbf{r}})$'s, it requires some work that is not in standard electromagnetic textbooks. From definition,

$$\mathbf{N}_{nm}(\mathbf{r}) = \frac{1}{k_0}\nabla\times\nabla\times\big[\mathbf{r}h_n^{(1)}(k_0r)Y_{nm}(\hat{\mathbf{r}})\big] = -\frac{1}{k_0}\nabla\times\big[h_n^{(1)}(k_0r)\mathbf{r}\times\nabla Y_{nm}(\hat{\mathbf{r}})\big]$$

Using (3.12c) in [7], we obtain

$$\begin{aligned}\mathbf{N}_{nm}(\mathbf{r}) &= n(n+1)\frac{h_n^{(1)}(k_0r)}{k_0r}\hat{\mathbf{r}}Y_{nm}(\hat{\mathbf{r}}) + \frac{1}{k_0r}\frac{d}{dr}\big[rh_n^{(1)}(k_0r)\big]r\nabla Y_{nm}(\hat{\mathbf{r}}) \\ &= n(n+1)\frac{h_n^{(1)}(k_0r)}{k_0r}\hat{\mathbf{r}}Y_{nm}(\hat{\mathbf{r}}) + \Big[h_{n-1}^{(1)}(k_0r) - n\frac{h_n^{(1)}(k_0r)}{k_0r}\Big]r\nabla Y_{nm}(\hat{\mathbf{r}})\end{aligned}$$

Furthermore, we have

$$\int \hat{\mathbf{r}} Y_{nm}(\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} Y^*_{n'm'}(\hat{\mathbf{r}})\, d\Omega = \delta_{nn'}\delta_{mm'}$$

$$\int \hat{\mathbf{r}} Y_{nm}(\hat{\mathbf{r}}) \cdot r\nabla Y_{n'm'}(\hat{\mathbf{r}})\, d\Omega = 0$$

Thus, the norm of $\mathbf{N}_{nm}(R\hat{\mathbf{r}})$ is given by

$$\int \mathbf{N}^\dagger_{nm}(R\hat{\mathbf{r}})\mathbf{N}_{n'm'}(R\hat{\mathbf{r}})\, d\Omega = \Big[\frac{n(n+1)^2}{2n+1}\big|h^{(1)}_{n-1}(k_0R)\big|^2 + \frac{n^2(n+1)}{2n+1}\big|h^{(1)}_{n+1}(k_0R)\big|^2\Big]\delta_{nn'}\delta_{mm'} \tag{13}$$

Replacing $h^{(1)}_n(k_0 r)$ with $j_n(k_0 r)$ yields the norm of $\hat{\mathbf{M}}_{nm}(R\hat{\mathbf{r}})$ and $\hat{\mathbf{N}}_{nm}(R\hat{\mathbf{r}})\}$ where $j_n(\cdot)$ is the spherical Bessel function. As a whole, the singular value decomposition of $\bar{\mathbf{G}}(R_s\hat{\mathbf{k}}, R_t\hat{\mathbf{p}})$ is

$$\bar{\mathbf{G}}(R_s\hat{\mathbf{k}}, R_t\hat{\mathbf{p}}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\big[\sigma_{nm1}\mathbf{u}_{nm1}(\hat{\mathbf{k}})\mathbf{v}^{t\dagger}_{nm1}(\hat{\mathbf{p}}) + \sigma_{nm2}\mathbf{u}_{nm2}(\hat{\mathbf{k}})\mathbf{v}^{t\dagger(\hat{\mathbf{p}})}_{nm2}\big] \tag{14}$$

where the singular values are given by

$$\sigma^2_{nm1} = \big|h^{(1)}_n(k_0R_s)\big|^2 j^2_n(k_0R_t) \tag{15a}$$

$$\sigma^2_{nm2} = \Big[\frac{n+1}{2n+1}\big|h^{(1)}_{n-1}(k_0R_s)\big|^2 + \frac{n}{2n+1}\big|h^{(1)}_{n+1}(k_0R_s)\big|^2\Big] \cdot \Big[\frac{n+1}{2n+1}j^2_{n-1}(k_0R_t) + \frac{n}{2n+1}j^2_{n+1}(k_0R_t)\Big] \tag{15b}$$

and the corresponding orthonormal functions are

$$\mathbf{u}_{nm1}(\hat{\mathbf{k}}) = \frac{\mathbf{M}_{nm}(R_s\hat{\mathbf{k}})}{\|\mathbf{M}_{nm}(R_s\hat{\mathbf{k}})\|} \quad \mathbf{u}_{nm2}(\hat{\mathbf{k}}) = \frac{\mathbf{N}_{nm}(R_s\hat{\mathbf{k}})}{\|\mathbf{N}_{nm}(R_s\hat{\mathbf{k}})\|}$$

$$\mathbf{v}_{nm1}(\hat{\mathbf{p}}) = \frac{\hat{\mathbf{M}}_{nm}(R_t\hat{\mathbf{p}})}{\|\hat{\mathbf{M}}_{nm}(R_t\hat{\mathbf{p}})\|} \quad \mathbf{v}_{nm2}(\hat{\mathbf{p}}) = \frac{\hat{\mathbf{N}}_{nm}(R_t\hat{\mathbf{p}})}{\|\hat{\mathbf{M}}_{nm}(R_t\hat{\mathbf{p}})\|}$$

Similar definitions apply to ς_{nml} as singular values and $\mathbf{v}^r_{nml}(\hat{\mathbf{q}})$ as source orthonormal functions of the receive array.

B. Fully Scattered in θ and ϕ

Following (1), a discrete channel model is obtained:

$$\begin{aligned} &H_{n_r^2+n_r+m_r+l_r-2,n_t^2+n_t+m_t+l_t-2} \\ &= \iint \mathbf{v}^{r\dagger}_{n_rm_rl_r}(\mathbf{q})\bar{\mathbf{C}}(\mathbf{q},\mathbf{p})\mathbf{v}^t_{n_tm_tl_t}(\mathbf{p})\, d\mathbf{p}d\mathbf{q} \\ &= \varsigma_{n_rm_rl_r}\sigma_{n_tm_tl_t}\iint \mathbf{u}^\dagger_{n_rm_rl_r}(\hat{\boldsymbol{\kappa}})\bar{\mathbf{H}}(\hat{\boldsymbol{\kappa}},\hat{\mathbf{k}})\mathbf{u}_{n_tm_tl_t}(\hat{\mathbf{k}})\, d\hat{\mathbf{k}}d\hat{\boldsymbol{\kappa}} \end{aligned}$$

As $\mathbf{u}_{nml}(\hat{\mathbf{k}})$'s are orthonormal, entries of the channel matrix $\bar{\mathbf{H}}$ are independent with variances given by

$$\mathrm{var}(H_{n_r^2+n_r+m_r+l_r-2,n_t^2+n_t+m_t+l_t-2}) = \varsigma^2_{n_rm_rl_r}\sigma^2_{n_tm_tl_t}$$

The number of degrees of freedom is determined by the minimum of the number of significant ς_{nml}'s and that of σ_{nml}'s. By studying the distributions summarized in (15), we can obtain this number in any field region with arbitrary array sizes.

Now let us study the distribution in the far field with large arrays. In the far field, $k_0R_s \gg 1$. The spherical Hankel function can be approximated by

$$h^{(1)}_n(k_0R_s) = (-1)^{n+1}\frac{e^{ik_0R_s}}{k_0R_s}\Big[1 + \mathcal{O}\big(\frac{1}{k_0R_s}\big)\Big]$$

and the magnitude of the asymptotic term is independent of n and equals to $\frac{1}{k_0R_s}$. The singular values become

$$\sigma^2_{nm1} = \frac{j^2_n(kR_t)}{(kR_s)^2} \tag{16a}$$

$$\sigma^2_{nm2} = \frac{1}{(kR_s)^2}\Big[\frac{n+1}{2n+1}j^2_{n-1}(k_0R_t) + \frac{n}{2n+1}j^2_{n+1}(k_0R_t)\Big] \tag{16b}$$

When $k_0R_t \gg 1$, $j_n(kR_t) \approx 0$ for $n > k_0R_t$. Thus, both σ_{nm1} and σ_{nm2} vanish when $n > k_0R_t$. The numbers of degrees of freedom contributed by the magnetic multipoles and the electric multipoles are asymptotically the same, and equal to $2(k_0R_t)^2 = 8\pi\mathcal{A}_t$ where $\mathcal{A}_t = \pi(\frac{k_0R_t}{2\pi})^2$ is defined as the effective aperture of the transmit array in our earlier paper [3]. Consequently, the asymptotic number of degrees of freedom is $\min\{8\pi\mathcal{A}_t, 8\pi\mathcal{A}_r\}$ which is 2 times of the scalar channel derived in [3].

C. Scattered in ϕ Only

As scatterers do not increase the degrees of freedom in the fully scattered case, they will not increase them neither in the current case. At $\theta = \pi/2$, $\mathbf{u}_{nml}(\hat{\mathbf{k}})$'s are orthogonal for different m and l only. That is, for the same m, l, the dimension of the functional space spanned by $\mathbf{u}_{nml}(\hat{\mathbf{k}})$'s is 1. If the magnetic (electric) multipoles vanish at $n = N_1$ ($n = N_2$), the maximum m is $2N_1 + 1$ ($2N_2 + 1$). In the far field, the asymptotic number of degrees of freedom is $\min\{4k_0R_t, 4k_0R_r\} = \min\{8\sqrt{\pi\mathcal{A}_t}, 8\sqrt{\pi\mathcal{A}_r}\}$.

Finally, the vector point sources $\mathbf{a}_n(\hat{\mathbf{k}})$ in Section II are the lowest order modes $\mathbf{u}_{1ml}(\hat{\mathbf{k}})$ in the far field. Thus in this section, we consider large array in the far field while the small-array counterpart has been considered in Section II.

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