Harmonic Oscillator in Magnetic Field

Consider the 2D harmonic oscillator

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega_0^2(x^2 + y^2).$$

As usual we can define the annihilation operators $a_x$ and $a_y$, and associated Hermitian conjugates. Furthermore, define

$$a_\pm \equiv \frac{a_x \pm i a_y}{\sqrt{2}}.$$

(a) Show that $[a_\pm, a_\pm^\dagger] = 1$.

(b) Show that

$$H = \hbar \omega_0 \left( a_+^\dagger a_+ + a_-^\dagger a_- + 1 \right).$$

(c) Show that

$$L_z = xp_y - yp_x = \hbar \left( a_+^\dagger a_- - a_-^\dagger a_+ \right).$$

We can now use this trick to use the harmonic oscillator creation/annihilation operators to find the spectra of Hamiltonians where the harmonic oscillator has an angular momentum term. Here’s a simple example of this.

(d) Verify that the vector potential

$$\mathbf{A} = \frac{B\mathbf{z} \times \mathbf{r}}{2}$$

corresponds to a constant magnetic field of magnitude $B$.

(e) Write down the Hamiltonian for the harmonic oscillator of charge $q$, placed in a magnetic field of constant strength $B$.

(f) By exploiting the tricks developed in parts (a)-(c), find the eigenvalues of this Hamiltonian. Express your answer in terms of $\hbar, \omega_0$, and

$$\omega_\text{c} \equiv \frac{qB}{2m}.$$