Jaynes-Cummings Hamiltonian

One of the chief problems of quantum computing is finding an effective way to measure the state of a quantum system without disturbing it too much. One proposal to do this is to use so-called cavity quantum electrodynamics. In this problem, we’ll consider a simple model for such a system, which consists of a single atom trapped in a container, which can interact with photons at frequency \( \omega \). We may write the Hilbert space of the system to be given by \( |n,s\rangle \) where \( n = 0, 1, 2, \ldots \) counts the number of photons at frequency \( \omega \), and \( s = \uparrow, \downarrow \) determines whether the atom is in spin up or spin down. The Hamiltonian for this system is called the Jaynes-Cummings Hamiltonian:

\[
H = \frac{\hbar \omega}{2} \sigma_z + \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) + g \left( a^\dagger \sigma_- + a \sigma_+ \right),
\]

where \( a^\dagger \) and \( a \) are the creation/annihilation operators for the photon, and the Pauli matrices act on the atomic state. Note that

\[
\sigma_\pm = \frac{\sigma_x \pm i \sigma_y}{2}.
\]

(a) Define

\[
N = a^\dagger a + \frac{\sigma_z}{2}.
\]

Show that \( [H, N] = 0 \).

(b) Find the eigenvectors of \( H \), and show that the eigenvalues are

\[
E_{N,\pm} = \hbar \omega \left( N + \frac{1}{2} \right) \pm \sqrt{N + 1} g.
\]

For the remainder of the problem, you may assume that \( g \gg \hbar \omega \), and ignore the \( \omega \) terms in \( H \).

Our goal is now to understand how this interaction can help us build a quantum computer. An ideal quantum computer would be able to perform the unitary transform

\[
U = \begin{pmatrix}
\cos(\alpha \theta) & -i \sin(\alpha \theta) \\
i \sin(\alpha \theta) & \cos(\alpha \theta)
\end{pmatrix}
\]

where \( \alpha \) and \( \theta \) are real constants. Our “cavity QED” quantum computer performs the unitary transform

\[
\hat{U} = e^{-i H t / \hbar} \equiv e^{-i \theta H / g}
\]

with \( \theta \equiv gt / \hbar \). We’d like for \( U \) and \( \hat{U} \) to roughly correspond to each other.

To understand in what limits this will be true, we first use the fact that most quantum optics systems rely on coherent states for the photons, as opposed to eigenstates of \( H \). Let \( |\alpha\rangle \) be a coherent state for the photons, with \( a|\alpha\rangle = \alpha |\alpha\rangle \). Assume that \( \alpha \) is real, without loss of generality.

(c) Define the operator (on atomic states only) \( \hat{U}(n) = \langle n|\hat{U}|\alpha\rangle \), and show that in the \( |\uparrow\rangle, |\downarrow\rangle \) basis:

\[
\hat{U}(n) = e^{-\alpha^2 / 2} \frac{\alpha^n}{\sqrt{n!}} \begin{pmatrix}
\cos(\theta \sqrt{n}) & -i \frac{\sqrt{n}}{\alpha} \sin(\theta \sqrt{n}) \\
i \frac{\alpha}{\sqrt{n+1}} \sin(\theta \sqrt{n+1}) & \cos(\theta \sqrt{n+1})
\end{pmatrix}.
\]
(d) As you may remember, the probability distribution for finding \( n \) photons in a coherent state is a Poisson distribution, which is sharply peaked for large \( \alpha \). Let \( n = \alpha^2 + L\alpha \), and show that

\[
e^{-\alpha^2/2} \frac{\alpha^n}{\sqrt{n!}} \approx \frac{e^{-L^2/4}}{(2\pi)^{1/4}}.
\]

(e) For large \( \alpha \), we can approximate \( L \) as a continuous parameter. Verify

\[
\int_{-\infty}^{\infty} dL \, \hat{U}(L)^\dagger \hat{U}(L) = 1.
\]

(f) Define the fidelity of this measuring device as

\[
f = \min_{|\psi\rangle, \theta} \int_{-\infty}^{\infty} dL \, \left| \langle \psi | \hat{U}(L) |\psi\rangle \right|^2
\]

where \(|\psi\rangle\) refers to an atomic state. Taylor expand this quantity about \( \alpha = \infty \), in powers of \( 1/\alpha \), to the lowest non-trivial order.

(g) Comment on the results of part (f) briefly; how would you want a “cavity QED” quantum computer to operate for optimal results? Can you make it as close to a perfect measuring device as you’d like?