The Orr-Sommerfeld equation describes the perturbations around the pressure-driven (Poiseuille) viscous flow of fluid in a channel in two dimensions. Consider a fluid with viscosity \( \eta \), flowing with velocity \( u_x(y) \) down a channel which extends infinitely in the \( x \)-direction, and between \( |y| \leq h/2 \). This flow is driven by a pressure gradient, which we take to be a constant: \( dP/dx \equiv -\alpha \) (take \( \alpha > 0 \)). The flow is independent of \( x \) and \( t \).

(a) What is the solution to the background problem? You probably know this already, and can just write it down.

(b) Now, suppose that we perturb this Couette flow with a perturbation of the form

\[
\delta u_x = \delta u_x(y)e^{ikx-i\omega t},
\]

\[
\delta u_y = \delta u_y(y)e^{ikx-i\omega t},
\]

\[
\delta P = \delta P(y)e^{ikx-i\omega t}.
\]

Show that, after appropriate non-dimensionalizations, the incompressible Navier-Stokes equations reduce to the following single differential equation, where \( \mathcal{R} \) is the Reynolds number of the flow (written in terms of the usual stream function \( \psi \), whose Laplacian is the vorticity):

\[
i(k(1-y^2)-\omega)(\partial_y^2 \psi - \psi) = \mathcal{R}^{-1}(\partial_y^2 - 1)^2 \psi - 2k \psi
\]

Show that the boundary conditions on the stream function \( \psi \) are that \( \psi \) and \( \partial_y \psi \) vanish at the boundaries. This equation is called the **Orr-Sommerfeld equation**.

(c) In 1972, it was shown that spectral methods are a very accurate and easy way of computing the eigenfrequencies \( \omega \). We’re looking for the onset of an instability (to turbulence). Use the fourth-order spectral method based on Chebyshev polynomials in the interval \([-1, 1]\) with the clamped boundary conditions relevant for this problem. Write some code to calculate the values of \( \omega \). Then, explore the eigenfrequencies for various discretizations (with \( N \) points). You should find that the critical value of \( \mathcal{R} \approx 5772 \), and the critical value of \( k \approx 1.02 \), at which an instability occurs.