The Child-Langmuir Law

A vacuum diode consists of electrons of charge $-e$ and mass $m$ flowing between two metal plates, separated by a distance $d$. Electrons flow from the anode at potential $\varphi = 0$ to the cathode at potential $\varphi = \varphi_0$. Assume a steady-state (time-independent) potential $\varphi(x)$, dependent only on $x$, the distance from the anode.

(a) Why is the current density $J$ a constant function of $x$?

(b) Relate the electron speed $v(x)$, the potential $\varphi(x)$ and the charge density $\rho(x)$ to obtain an ODE for $\varphi(x)$.

(c) Solve the ODE from part (b) under the proper boundary conditions to show that

$$J = k\varphi_0^{3/2},$$

and find the constant of proportionality $k$. This equation is called the Child-Langmuir law.