We address the question of aggregating the preferences of voters in the context of participatory budgeting. We scrutinize the voting method currently used in practice, underline its drawbacks, and introduce a novel scheme tailored to this setting, which we call “Knapsack Voting”. We study its strategic properties - we show that it is strategy-proof under a natural model of utility (a dis-utility given by the $\ell_1$ distance between the outcome and the true preference of the voter), and “partially” strategy-proof under general additive utilities. We extend Knapsack Voting to more general settings with revenues, deficits or surpluses, and prove a similar strategy-proofness result. To further demonstrate the applicability of our scheme, we discuss its implementation on the digital voting platform that we have deployed in partnership with the local government bodies in many cities across the nation. From voting data thus collected, we present empirical evidence that Knapsack Voting works well in practice.

1. INTRODUCTION

Direct democracy has gained a lot of significance lately, with many novel attempts at engaging citizens very directly in policy-making [Pateman 2012; Smith 2009]. An exciting new development in this space is participatory budgeting [Cabannes 2004], in which a local government body asks residents to vote on project proposals to decide how they should allocate their budgetary spending. The proposals could be, in a particular city/ward for instance, resurfacing streets, adding street lights, building playgrounds for children or renovating recreational facilities like parks. Participatory budgeting has had a long history in South America [Schneider and Goldfrank 2002], where it was born out of a need to address inequalities through democratic reform. It is now gaining popularity in the US, with cities like San Francisco, Vallejo, Boston, Chicago and New York adopting this paradigm [PBP 2016]. This development necessitates a detailed look at the voting method used in current ballots, and motivates the following question: when there are projects with different costs, and a fixed budget, how can the varied preferences of voters be best aggregated? We address this question by proposing Knapsack Voting, and discuss its advantages over existing methods. We show that Knapsack Voting has desirable strategic properties, and provide empirical evidence that it works well in practice. We have been successful in implementing Knapsack Voting as the official ballot procedure in the Youth Lead the Change PB election, 2016, in Boston (https://youth.boston.gov/youth-lead-the-change/). Overall, this piece of work is a step in the direction of addressing the need for mechanisms to facilitate complex decision-making processes.

1.1. The Participatory Budgeting Problem

The participatory budgeting problem addresses the following scenario: the residents of a city, collectively the set of voters $V$, vote on a set $P$ of projects that they have
identified to be worthwhile, where project $j \in \mathcal{P}$ has a cost $c_j$ and there is a fixed total budget of $B$ Dollars. Denote the utility a voter $i \in \mathcal{V}$ gets from project $j$ as $v_{i,j}$.

Assuming additive utilities, one natural objective would be to find a feasible allocation among the various projects that maximizes the average benefit to the voters.

$$\arg \max_{W \subseteq \mathcal{P}} \sum_{j \in W} \frac{1}{|\mathcal{V}|} \sum_{i \in \mathcal{V}} v_{i,j}, \quad \text{subject to} \quad \sum_{j \in W} c_j \leq B.$$  

This decision problem is conceptually similar to the Knapsack problem with $\mathcal{P}$ as the set of items to be fit into a knapsack of size $B$. For a project $j$, its average utility ($\frac{1}{|\mathcal{V}|} \sum_{i \in \mathcal{V}} v_{i,j}$), and cost ($c_j$) correspond to its value, and size, respectively. Allowing fractional solutions, all we need to do is order the projects according to the average value-for-money ($\frac{1}{c_j} \sum_{i \in \mathcal{V}} v_{i,j}$) that each yields to the voters. The above general utility model is mainly for the purposes of exposition and analysis (we use it in Section 2.4), and we will not rely on voters’ knowing or reporting their values $v_{i,j}$. In fact, for the above problem, we will design a voting rule that, incorporating ideas from the Knapsack Problem, makes the voter choose projects under the budget constraint. In doing so, we align the constraints on the voters’ decisions, to those of the decision maker’s. We believe that this approach of aligning incentives is useful more broadly in designing mechanisms for complex participatory decision-making processes. Moreover, we want our schemes to be amenable to implementation as convenient digital interfaces to help individuals make informed budgeting decisions.

1.2. Participatory budgeting in practice

The first PB process in the United States was initiated in the 49th ward of Chicago in 2009. Since then PB has spread to many other cities like Vallejo, New York, Cambridge, to name a few. These elections traditionally used only paper ballots, and a voting method known as K-approval that we will describe shortly.

1.2.1. Our digital voting platform. To popularize digital voting, we built a digital voting platform (https://pbanet.org/) for the PB process of Chicago’s 49th ward in 2012. We have since customized this tool to be easily adaptable to other elections. Many of the cities/municipalities doing PB have taken to digital voting by adopting our platform, while some cities like Long Beach and Dieppe have used our platform for their very first attempt at PB. In all, our platform has been used in over 25 different elections in around a dozen cities/wards over the past few years. One thing to note is that digital voting here means voting in person at a polling booth but via the digital platform. Most of these elections are required to have a paper ballot for those voters that opt out of digital voting. Some elections, for example PB Cambridge, implemented internet voting, where voters can authenticate themselves and vote from anywhere, via a voting website designed using our platform. These elections had accompanying paper ballots also.

1.2.2. K-approval voting. The voting method currently used by most participatory budgeting elections is K-approval voting. Our interface for this method is shown in Figure 1. This is similar to the standard approval voting method, but with a cap on the number of alternatives a voter can choose.

**Definition 1.1 (K-approval voting).**

— Each voter chooses (“approves”) at most $K$ projects.
— The projects are ranked in descending order according to the total number of approving voters.
— The outcome is decided by picking projects in this order until the budget is exhausted.
While this method is simple and has wide acceptance, it doesn’t make the voters factor in the costs of the projects into their votes. Consider the following example:

**Example 1.2.** Think of one voter (or a homogeneous pool of voters) doing 1-approval. Let’s say the budget is $300 and Table I shows the projects on the ballot, with their costs and hypothetical utilities. It is natural to expect that the voter(s) will vote for **Project A**. But it has the least *value-for-money*, and should be given the least priority. Although this is a very simple example which makes a strong assumption on the voters’ response to the K-approval rule, it is an apt illustration of observed voting behavior (see Section 4.0.1).

A point to note here is that even truthful reporting under K-approval voting (choosing projects with highest utility) does not lead to good outcomes as seen in Example 1.2 above. It is conceivable that the outcome might be different if the voters vote strategically, taking aggregation with the budget constraint into account. However, we would like a natural way for the voters to express their preferences with respect to costs and benefits – one that works atleast for a homogeneous population under truthful reporting. This leads us to the following questions:

— What if the voter could choose projects up to $300?
— What if she is asked to compare/rank projects in terms of their *value-for-money*?

These ideas are derived from ways of solving the Knapsack Problem, and correspond to our Knapsack Voting and Value-for-money schemes, which will be treated in detail.
in later sections of this paper. We will give a short overview of our work in the next section.

1.3. Our contributions

Our overall goal in this paper will be to present voting mechanisms that are particularly suited to the Participatory Budgeting problem. These mechanisms, by getting voters to either optimize over the total budget, or compare projects based on their value-for-money, enable the voter to inherently consider costs and benefits, and moreover, are implementable using interactive digital tools.

1.3.1. Knapsack Voting. Our primary scheme is called Knapsack Voting. The main idea behind this scheme is that each voter has to adhere to the budget constraint, thus keeping costs in mind while giving her vote – thus internalizing the global constraints, while giving their preferences toward the outcome. Our interface for this method is shown in Figure 2.

Definition 1.3 (Knapsack Vote).

— Each voter $v \in V$ votes for a subset $S_v \subseteq P$, such that it satisfies the budget constraint $\sum_{p \in S_v} c_p \leq B$.
— The projects are arranged in decreasing order of number of their approval scores, which for any project $p$ is given by $\#\{v \in V : p \in S_v\}$.
— The projects are chosen in this order till the budget is exhausted.

As we can see, while the elicitation method is very different, the aggregation method here is the same as in K-approval. We assume that the last project can be fractionally implemented, and thus the budget is completely used up.

We analyze the strategic properties of Knapsack Voting assuming an $\ell_1$ utility model, in which, loosely speaking, the cost to a voter depends on how much the outcome differs from her preferred allocation as determined by the $\ell_1$ distance between the two (Definition 2.3). Under this model, we show that Knapsack Voting is strategy-proof and welfare-maximizing (Theorems 2.4, 2.5).
RESULT 1. Under the $\ell_1$ utility model, Knapsack Voting is strategy-proof and welfare-maximizing.

This result is analogous to the strategy-proofness results for Approval Voting under dichotomous preferences [Brams and Fishburn 1978].

We also extend Knapsack Voting to more general settings with revenues, surpluses or deficits (as opposed to one with a fixed budget) where approval voting has no known analog, and prove similar strategy-proofness and welfare-maximization results (Section 2.3, Theorem 2.8).

Also, under general additive utilities, we characterize a weaker yet interesting notion of strategy-proofness of a voter’s best response, based on the concept of sincerity [Niemi 1984]: it is in a voter’s best interest to vote for projects that she favors among those that are winning without her vote (Theorem 2.9).

RESULT 2. Under general additive utilities, a voter’s best response under Knapsack Voting is partially strategy-proof.

Together, these amount to substantial evidence that Knapsack Voting aligns the incentives of the voters with that of the decision maker. Note that these two results do not hold for K-approval voting.

An interesting, and standard, way of understanding voting rules is viewing them as Maximum Likelihood estimators (MLEs) [Conitzer and Sandholm 2012; Young 1988]. Votes are assumed to be drawn from a suitable noisy model parametrized by a “ground truth” outcome, such that the voting rule is the MLE of the “ground truth” given any realization of votes. For example, the Kemeny Young rule is the MLE of rankings drawn from the Mallows model. The Mallows model defines a probability distribution over all possible rankings, with the probability of each ranking depending on how much it differs from a given ground truth order. Since we are concerned with the problem of determining an outcome which is a subset of projects that satisfies the budget constraint, we will define a model that takes such a set as the ground truth, and defines a probability for all valid sets, depending on how much they differ from the ground truth. By interpreting Knapsack Voting (Section 2.5) as the MLE of this natural noise model, we reinforce the aggregation method of Knapsack Voting.

In our second scheme, we elicit the voters’ preferences based on their perceived value-for-money from projects. By value-for-money we mean the utility of a project normalized by its cost: if $v_{i,p}$ is the utility of voter $i$ from project $p$, then its value-for-money as perceived by $i$ is $\frac{v_{i,p}}{c_p}$. See Figures 3,7 for voting interfaces based on this idea, where voters compare pairs of projects, or rank their top projects, based on value-for-money.

In the previous section, we discussed how most elections use paper ballots for those voters that opt out of the digital platform. Using the idea of value-for-money, we designed a paper ballot that accompanies Knapsack Voting. We will discuss this and other details about value-for-money in Section 3.

1.3.2. Deployments and Data Analysis. Using our digital voting system that has gained widespread acceptance in many cities/municipalities across the nation, we tested our methods across various participatory budgeting elections. In most of these elections, K-approval was used as the official ballot, and in addition, we tested either the Knapsack or the value-for-money comparisons (Figure 3) or both. We do value-for-money to estimate the aggregate pairwise preferences of voters.

We describe our experimental procedure in greater detail in Section 4. Based on the data from our experiments to make the following observations:

(1) Knapsack Voting leads to a more economical consideration of the projects as compared to K-approval (4.0.1),
We present empirical evidence in support of these observations in Section 4.

Observation 1 just reinforces the fact that Knapsack Voting leads to a more economical consideration of projects by the voters. It is natural to expect voters to pay more attention to costs of projects under Knapsack Voting. And Observation 2 suggests that Knapsack Voting does not involve a much larger cognitive load on the voters than K-approval.

However, we stop short of claiming that Knapsack Voting leads to outcomes that are more beneficial to society as a whole. It could be worthwhile to think of ways to compare the outcomes in terms of their long-term societal value.

1.4. Related Work

A wide variety of voting procedures, both ranked and non-ranked, have been studied in social choice literature [Brams and Fishburn 2002]. Plurality and Approval Voting among non-ranked procedures, and Borda, Copeland and Kemeny-Young [Levin and Nalebuff 1995] among ranked, are perhaps the most well known. In approval voting, each voter can vote for, or “approve of”, any number of candidates on the ballot. Each candidate gets a vote from every voter that voted for her, and the candidate with the most votes wins. Approval voting has been widely studied as an alternative to plurality voting, with some compelling advantages [Brams 1993]. Various modifications to approval voting, such as added constraints [Brams 1990], and cumulative voting [Bhagat and Brickley 1984], that help improve the representation of minorities or under-represented groups in committees have been examined. In this paper, we devise the right adaptation of the approval paradigm to participatory budgeting elections.

Another parallel stream of literature in social choice theory has delved deeply into the question of manipulability of voting rules. It has been shown that the only rea-
sonable social choice functions with single/multiple winners that are not susceptible to strategic manipulation by voters are degenerate forms of dictatorships [Gibbard 1973; Satterthwaite 1975; Duggan and Schwartz 2000]. This result subsumes our setting and rules out any general non-manipulable voting rule in our case. However, strategy-proof schemes were shown to be possible in restricted domains like single-peaked preferences [Moulin 1980; Barberà et al. 1993; Nehring and Puppe 2002]. These results rule out non-dictatorial strategy-proof voting rules in our setting with three or more projects [Barberà et al. 1997]. We propose an aggregation rule that finds the geometric median of the votes on the simplex that satisfies the budget constraint and show that strategy-proofness holds under the $\ell_1$ utility model (Definition 2.3). It is also interesting to compare our setting with the axioms of Arrow’s Impossibility Theorem [Arrow 2012], especially the Independence of Irrelevant Alternatives (IIA). With a weaker form of IIA [Campbell and Kelly 2000], Knapsack Voting, akin to Approval Voting with dichotomous preferences, satisfies all three of Arrow’s axioms.

From the more recent literature in Computational Social Choice, our work is similar in spirit to some recent work on identifying the trade-offs between the utilities from various possible societal activities [Conitzer et al. 2015]. Our work also involves capturing tradeoffs, but only so far as to obtain the best allocation of the budget, and we propose voting methods that use ideas from the Knapsack Problem to do so. There has also been some work on selecting committees under weight or cost constraints given complete rankings from voters [Klamler et al. 2012]. However, our work is different in that we do not get complete rankings from voters, and design ways of eliciting preferences of voters.

There has also been some work on developing visualization tools for voters to make sense of different items in complex budget problems, e.g. the federal budget [Kim et al. 2016]. In our setting, the items on the ballot are simple and well defined, for instance building a park, or improving streets, and can be described concisely on our digital voting platform for voters to make an informed decision.

2. KNAPSACK VOTING: IMPOSING BUDGET CONSTRAINTS

As mentioned earlier, several impossibility results [Gibbard 1973; Satterthwaite 1975], and in particular [Duggan and Schwartz 2000], rule out the existence of strategy-proof mechanisms for our setting in all generality. The Gibbard-Satterthwaite Theorem inevitably led to the search for strategy-proof social choice functions on restricted domains, and several positive results in this direction. Inspired by Black’s Median Voter Theorem [Black 1948], [Moulin 1980] characterized Generalized Median Voter Schemes as the only strategy-proof rules under single-dimensional single-peaked preferences. This was generalized to multiple dimensions for preferences on the cartesian box [Border and Jordan 1983; Barberà et al. 1993], and general subsets of the cartesian box [Barberà et al. 1997]. The results of [Barberà et al. 1997] imply that, under single-peaked preferences, the only voting rules that could be be both strategy-proof and non-dictatorial for our problem are Generalized Median Voter Schemes (GMVS).

Unfortunately, GMVSs do not respect the budget constraint for three or more projects. Hence we look at an voting scheme (Knapsack Vote, Definition 2.2) that differs from GMVS in that it finds a geometric median of the votes on the simplex that respects the budget constraint. Adapting dichotomous preferences to our setting, we define a natural model of voter utilities (Definition 2.3) – a dis-utility given by the $\ell_1$ distance between the outcome and the true preference of the voter. We show that Knapsack Voting is strategy-proof (Theorem 2.4) and welfare-maximizing (2.5) under this model. In its simplest form, Theorem 2.4 is analogous to the strategy-proofness results on Approval voting [Brams and Fishburn 1978]. However, we show that Knapsack Voting can be extended to more complex settings with revenues, deficits or surpluses.
(Equation 1) for which there is no known analog of Approval voting. We prove that Knapsack Voting is strategy-proof and welfare-maximizing in this setting (Theorem 2.8).

Under approval voting, a voter gives a sincere vote if whenever she votes for a particular candidate, she also votes for more preferred candidates [Niemi 1984]. Based on this idea, and assuming the voter has full knowledge of all other votes, we characterize a weaker notion called “partial” strategy-proofness.

2.1. Per-dollar approach, and Fractional Knapsack Rule

There is a wide range of budget problems where the allocation per project is not a fixed cost, but can be variable, possibly up to a certain upper limit (see Figure 4 for an experimental interface). For example, the federal budget consists of a variable allocation of money to various sectors of the economy at large. To accommodate variable allocation, and to avoid combinatorial difficulty, we reinterpret Knapsack Voting using a “per-dollar” approach.

**Example 2.1.** Given a budget of 10, and three voters A, B, C, let’s say that the preferred allocations of the voters for projects $P_1, P_2, P_3$ of maximum costs $(5, 5, 10)$ are $(4, 5, 1)$, $(3, 5, 2)$, $(0, 0, 10)$ respectively (Figure 5). We divide each project into as many dollars as its maximum cost, and given that A allocates 4 to $P_1$, we give one vote to each of the first 4 dollars of $P_1$, and so on. Figure 5 shows how many votes each dollar of each project gets. If we select those dollars with the highest number of votes, we get the allocation $(3, 5, 2)$. Notice that here we didn’t have to break ties to get a feasible allocation.
More formally, we assume that each project is fractionally implementable, we split each project \( p \in \mathcal{P} \) into \( c_p \) different “per-dollar sub-projects” \( D^p_1, D^p_2, \ldots, D^p_{c_p} \), and collect all the sub-projects into a set \( \mathcal{P}' \). The problem then reduces to choosing \( B \) out of the \( C \) candidates in \( \mathcal{P}' \). Given a vote from a voter \( v \) using our Knapsack interface (Figure 2), it is guaranteed to be consistent after the per-dollar conversion, i.e., a set \( S_v \subset \mathcal{P}' \) such that for all \( p \in \mathcal{P}' \), if \( D^p_1 \in S_v \), then \( D^p_{c_p} \in S_v \) for all 1 \( \leq t' < t \). For any \( j \in \mathcal{P}' \), define its score as \( \text{score}(j) = \sum_{p \in S_j} \text{score}(p) \). We will use a consistent deterministic tie-breaking order: a strict ordering \( \prec \) on \( \mathcal{P}' \) such that for all \( p \in \mathcal{P}' \), 1 \( \leq t' < t \leq c_p \iff D^p_{t'} \prec D^p_{t} \) (if \( j < k \), then \( j \) gets priority over \( k \) when \( \text{score}(j) = \text{score}(k) \)).

**Definition 2.2 (Fractional Knapsack Vote).**

— Each voter \( i \in V \) submits a consistent subset \( S_i \subset \mathcal{P}' \), such that it satisfies a budget constraint \( |S_i| = B \).
— The winning set is given by \( \arg \max_{S:|S|=B} \sum_{j \in S} \text{score}(j) \), using a consistent deterministic tie-breaking order \( \prec \).

Note that given consistent votes, and a consistent tie-breaking order, the Fractional Knapsack Vote gives a “consistent” outcome. In this section, whenever we say “Knapsack Voting”, we mean “Fractional Knapsack Vote”. The results in this section holds even when each vote is not fractional, i.e., each voter \( v \) chooses a set of projects \( S_v \subset \mathcal{P} \) such that \( \sum_{p \in S_v} c_p \leq B \). In this case, we assume that the outcome, a set of projects \( S \) such that \( \sum_{p \in S} c_p \leq B \) together with an additional project \( p' \) that is funded only up to a cost of \( B - \sum_{p \in S} c_p \), chosen according to Definition 1.3. We adopt the fractional version as in Definition 2.2 above for the ease of exposition that it allows.

### 2.2. Strategy-proofness under the \( \ell_1 \) utility model

Given the set of projects \( \mathcal{P} \), let’s say that the true preference \( S_v \) of each voter \( v \) corresponds to allocating an (integral) amount \( w^*_p \) to project \( p \in \mathcal{P} \), such that \( \sum_{p \in \mathcal{P}} w^*_p = B \).

Any outcome that deviates from this allocation would result in some dis-utility for the voter. Let \( \{w^*_p\}_{p \in \mathcal{P}} \) denote the final outcome. Imagine that the dis-utility that each voter gets is equal to the \( \ell_1 \) distance between \( \{w^*_p\}_{p \in \mathcal{P}} \) and \( \{w^*_p\}_{p \in \mathcal{P}} \), i.e., \( \sum_{p \in \mathcal{P}} |w^*_p - w^*_p| \).

**Definition 2.3 (\( \ell_1 \) utility model).** Given an outcome \( \{w^*_p\}_{p \in \mathcal{P}} \), the utility of a voter \( v \) with a preferred allocation \( \{w^*_p\}_{p \in \mathcal{P}} \) is given by \( U(v) = -\sum_{p \in \mathcal{P}} |w^*_p - w^*_p| \).

In the above, the budget allocation to each project \( p \) is flexible, and can take any integral value in \([0, c_p]\), with the votes are restricted to the simplex determined by \( \sum_{p \in \mathcal{P}} w^*_p = B \). The outcome of Knapsack Voting can be defined as an \( \ell_1 \)-median restricted to the above-mentioned simplex (follows from Lemma 2.6 and Theorem 2.5), i.e., an allocation that minimizes the sum of the \( \ell_1 \) distances to the votes. ¹ Under the Overlap utility model, by defining the outcome as the geometric \( \ell_1 \) median, we get the following result:

**Theorem 2.4.** For Knapsack Voting, under the \( \ell_1 \) utility model, voting for her preferred allocation \( S_v \) is a weakly dominant strategy for voter \( v \).

**Proof.** Let \( S_{-v} \) and \( \text{score}_{-v}(\cdot) \) be the outcome determined by the votes of everyone except \( v \) using the Knapsack Voting rule (Definition 2.2). As mentioned in Section 2.1, for any \( i \in \mathcal{P}' \), \( \text{score}_{-v}(i) = |\{u \in V \setminus \{v\} : i \in S_u\}| \), where \( S_u \) denotes the vote of any

¹This works in continuous space too, with a suitably defined tie-breaking rule.
voter $u \in V \setminus \{v\}$ (since we are looking for a dominant strategy for voter $v$, $S_v$ here can be any valid vote, not necessarily the truthful report of voter $u$). Assume that $T_v \neq S_v$ is a best response of $i$. Let the outcome after incorporating $T_v$ be $\text{score}(\cdot)$ and $S$.

Let $j \in T_v \setminus S_v$ such that if $j = D'_t$ for some $p \in P$ (recall the definition of per-dollar sub-projects from Section 2.1), then $D'_t \notin T_v$ for all $t' > t$. Choose some $k \in S_v \setminus T_v$ such that if $k = D'_t$ for some $q \in P$, then $D'_t \notin T_v$ for all $t' < z$. Such a $k$ exists because $T_v$ and $S_v$ are consistent and of the same size $B$. Let $T'_v \triangleq T_v \cup \{k\} \setminus \{j\}$, and the outcome here be $\text{score}'(\cdot)$ and $S'$. We will show that $T'_v$ is also a best response for $v$. If $S = S'$, then we have nothing to prove. In what follows, we will assume $S \neq S'$.

We have that $\text{score}'(j) = \text{score}(j) - 1$, $\text{score}'(k) = \text{score}(k) + 1$, and for all $l \in P' \setminus \{j, k\}$, $\text{score}'(l) = \text{score}(l)$. Note that the only change here is that the score of $j$ decreases, and the score of $k$ increases. As a result, for any given consistent tie-breaking order (see Section 2.1 for definition), $S' \setminus S$ must be singleton. Further, we must have either $j \in S \setminus S'$ or $k \in S' \setminus S$, i.e., any change in outcome must involve either $j$ moving from within the winning set to without, or $k$ from without to within (or both).

If $j \in S \setminus S'$, it must be that $S \setminus \{j\} \subset S'$, since for any $l \in S \setminus \{j\}$, $\text{score}'(l) \geq \text{score}(l)$. And since $S' \setminus S$ is a singleton set, say it contains $m \in P'$ (possibly $m = k$), the change in utility of voter $v$ is $1(m \in S_v) - 1(j \in S_v) = 1(m \in S_v) \geq 0$ (here $1(\cdot)$ is the indicator function that takes the value 1 if the argument is true, and 0 if not). In other words, the change in utility from removing $j$ is 0, and that from adding any $m$ in its place cannot be negative.

Similarly, if $k \in S' \setminus S$, then $S \setminus S' = \{m'\}$ for some $m' \in P'$. And the change in utility is $1(k \in S_v) - 1(m' \in S_v) = 1 - 1(m' \in S_v) \geq 0$.

By repeating this process until we only have elements in $S_v$, we have not decreased the utility of voter $v$. Therefore, the utility obtained by voting for $S_v$ cannot be strictly dominated by that for any $T_v \neq S_v$. $\Box$

Under truthful voting, the Knapsack Voting outcome maximizes the social welfare.

**Theorem 2.5.** The truthful dominant strategy equilibrium for Knapsack Voting is welfare-maximizing.

Before proving the above theorem the above theorem, we will interpret the $\ell_1$ utility model in a slightly different way.

### 2.2.1. Overlap utility

Let $S_v$ and $S'$ be the allocations in the per-dollar sense corresponding to $\{w_p^v\}_{p \in P}$ and $\{w_p^*\}_{p \in P}$. Assume for a moment that the utility of the voter $v$ is given by $|S_v \cap S'|$, or equivalently, $\sum_{p \in P} \min\{w_p^v, w_p^*\}$. In this quantity, the term corresponding to $p$ is equal to the “overlap” between $w_p^v$ and $w_p^*$, and so we call this expression the Overlap Utility. This means, in other words, that the utility from $p$ for a voter $v$ is equal to $w_p$ when $w_p^v \geq w_p^*$, and $w_p$ otherwise. For example, if $w_p^v = 5$ and $w_p^* = 3$, then the utility derived by $v$ is 5. Adopting this model of utilities, we can extend the results of this section to the case where the votes are non-fractional and comprise just a set of projects that together cost at most $B$. In this case, we assume that the outcome includes at most one project that is fractionally implemented to exhaust the budget, and voters derive utility from this project proportional to the level to which it is funded.

In the budgeted case being treated in this section (cf. Section 2.3), the utility given by the $\ell_1$ model is equal to the Overlap utility. This equivalence is of independent interest, and is also useful in the proof of Theorem 2.4. We now prove this equivalence formally.

**Lemma 2.6.** $|S_v \cap S'| = B - \frac{1}{2} \sum_{p \in P} |w_p^v - w_p^*|.$
Proof. As discussed above, we know that \(|S_v \cap S^*| = \sum_{p \in P} \min \{w_v^p, w_p^*\}\). Partition \(P\) into two sets \(P_L\) and \(P_H\), where \(P_L \equiv \{p \in P : w_v^p > w_p^*\}\) is the set of projects for which the amount allocated in the outcome is less than that in the voter \(v\)'s preferred allocation, and \(P_H \equiv \{p \in P : w_v^p \leq w_p^*\}\) is the set of projects where the amount allocated in the outcome is at least as much as that in the voter \(v\)'s preferred allocation. Then we have \(|S_v \cap S^*| = \sum_{p \in P_L} w_v^p + \sum_{p \in P_H} w_p^*\). First we note that \(\sum_{p \in P_L} w_v^p + \sum_{p \in P_H} w_p^* = \sum_{p \in P} w_v^p - \sum_{p \in P_H} (w_v^p - w_p^*) = B - \sum_{p \in P_H} (w_v^p - w_p^*)\). Similarly, \(\sum_{p \in P_L} w_v^p + \sum_{p \in P_H} w_p^* = -\sum_{p \in P_L} (w_v^p - w_p^*) + \sum_{p \in P_H} w_p^* = -\sum_{p \in P_L} (w_v^p - w_p^*) + B\). Therefore \(|S_v \cap S^*| = \frac{1}{2}(2B - \sum_{p \in P_L} (w_v^p - w_p^*) - \sum_{p \in P_H} (w_v^p - w_p^*)) = B - \frac{1}{2} \sum_{p \in P} |w_v^p - w_p^*|\). \(\Box\)

Proof of Theorem 2.5. The Knapsack Voting rule outputs a set \(S^*\) that maximizes \(\sum_{j \in S} \text{score}(j)\) among all consistent sets \(S\) that satisfy the budget constraint.

If voters vote truthfully, then
\[
\sum_{j \in S^*} \text{score}(j) = \sum_{j \in S^*} \sum_{v \in V} 1(j \in S_v) = \sum_{v \in V} \sum_{j \in S^*} 1(j \in S_v) = \sum_{v \in V} |S_v \cap S^*|,
\]
and by Lemma 2.6, the proof follows. \(\Box\)

We would like take a moment here to discuss Knapsack Voting in terms of the conditions in Arrow's Theorem. Choosing a set of projects that fit the budget can be thought of as a partial ordering \(\prec\) between those within, and those without. In our case, if the preferences of voters between two projects, say \(a\) and \(b\) with respect to this partial ordering remains the same, then the final score of \(a\) and \(b\) remains the same. If the outcome is \(a \prec b\), then changing the other preferences cannot result in \(b\) being chosen over \(a\), i.e., change from \(a \prec b\) to \(b \prec a\). A similar property has been characterized as a weaker form of the Independence of Irrelevant Alternatives in the context of Approval Voting [Campbell and Kelly 2000].

We must also mention here that under the \(\ell_1\) utility model, Knapsack Voting is not group strategy-proof. For example, consider the following:

Example 2.7. There are 5 projects \(\{a, b, c, d, e\}\), 4 voters \(\{1, 2, 3, 4\}\), and a budget of $2. The true preferences are: Voters 1 and 2 allocate $2 to \(a\), and 3 allocates $1 to \(b\) and $1 to c, and 4 allocates $1 to d and $1 to e. If they voted truthfully, the outcome is $2 to \(a\), giving a utility of 0 to both 3 and 4. Assuming ties are broken in the order \(b, d, c, e, a\). Then if voters 3 and 4 both voted for $1 to \(b\), and $1 to \(d\), then the outcome changes to \(b\) and \(d\), giving both voters a utility of 1.

2.3. An extension to scenarios with revenues, deficits or surpluses
A similar result can be obtained in a case where there is no hard budget, and there are both expenditure terms, and revenue terms by extending the framework of approval voting under dichotomous preferences. A voter proposes both how to generate revenue from among various avenues in \(\mathcal{R}\), and how to spend it on various projects in \(\mathcal{P}\). We will now discuss a case where the budget is balanced. Extensions to cases where there are multiple revenue items, or where the budget is unbalanced, are easy to see.

As before, in the per-dollar sense, let \(\mathcal{P}'\) be the set of per-dollar sub-projects for expenditure, and \(\mathcal{R}'\) be the set of per-dollar sub-projects for revenue. We use a similar \(\ell_1\) utility metric for the project terms as for the budgeted case, with revenue included. Formally, we will assume that if \(\{x_v^r\}_{r \in \mathcal{R}}\) is the outcome revenue level, and the voter \(v\) prefers \(\{x_p^e\}_{p \in \mathcal{P}}\), then her dis-utility from the revenue term will be \(\sum_{r \in \mathcal{R}} |x_v^r - x_p^e|\), and her total dis-utility will be \(\sum_{p \in \mathcal{P}} |w_v^p - w_p^e| + \sum_{r \in \mathcal{R}} |x_v^r - x_p^e|\). Because we are considering balanced budgets, we will assume \(\sum_{r \in \mathcal{R}} x_v^r = \sum_{p \in \mathcal{P}} w_p^e\) for all voters \(v\).

ACM Journal Name, Vol. V, No. N, Article A, Publication date: January YYYY.
Let \( R_v \subseteq \mathcal{R} \) and \( S_v \subseteq \mathcal{P} \) denote the vote of voter \( v \) satisfying \(|R_v| = |S_v|\). For \( j \in \mathcal{R} \) we define \( \text{score}(j) \equiv -|\{v \in \mathcal{V} : j \notin R_v\}| \). The Knapsack Vote outcome \( R^* \subseteq \mathcal{R} \) and \( S^* \subseteq \mathcal{P} \) is defined as follows:

\[
(R^*, S^*) = \arg \max_{(R,S):|S|=|R|} \left( \sum_{i \in S} \text{score}(i) + \sum_{j \in R} \text{score}(j) \right) \tag{1}
\]

In this setting, revenue corresponds to voters paying the government in fees/taxes, and hence we make the score corresponding to revenue terms negative. We will need a consistent tie-breaking rule here as well. But since the size of the outcome is not fixed, we can just break ties in favor of the bigger (or smaller) sets. We state a similar result as in Theorem 2.4 for this case:

**Theorem 2.8.** With a balanced budget, Knapsack Voting is strategy-proof under the \( l_1 \) utility model.

We defer the proof to the appendix (A.1), as it is similar to the proof of Theorem, and not crucial to the exposition here.

We can also extend this to *unbalanced* settings by including the budget deficit (expenditure - revenue) as part of the voters' preferences. If \( \Delta_v = |S_v| - |R_v| \) is the preferred deficit of voter \( v \), and the deficit in the outcome is \( \Delta^* \), then the voter incurs an additional dis-utility of \(|\Delta^* - \Delta_v|\), i.e., how much the deficit in the outcome differs from her preferred level. As such, it is equivalent to adding an additional revenue term.

### 2.4. “Partial” strategy-proofness under general utilities

In the case with general utilities, we illustrate a weaker, yet interesting property which we call “partial” strategy-proofness. Consider a focal voter \( i \) responding to the votes of all others. Assume that she has full knowledge about the how the others voted in aggregate. If \( S_{-i} \) denotes the cumulative votes of all voters except \( i \), she knows \( W_{-i} \subseteq \mathcal{P} \), the winning set as determined by \( S_{-i} \). Let \( W(S_i, S_{-i}) \) denote the set of winners if her vote \( S_i \) is added. A best response for \( i \) is a consistent set

\[
S_i^* \in \arg \max_{S_i \subseteq \mathcal{P} : |S_i| = |S_{-i}|} \sum_{j \in W(S_i, S_{-i})} v'_{i,j},
\]

where \( v'_{i,j} \) is the utility of voter \( i \) for sub-project \( j \in \mathcal{P} \). We also assume that the utility from per-dollar sub-projects is *concave*, i.e., if \( x < y \) and \( a = D^p_x \) and \( b = D^p_y \) for some \( p \in \mathcal{P} \), then \( v'_{i,a} \geq v'_{i,b} \). With respect to voter \( i \), we say a candidate \( k \) dominates \( j \) if and only if

\[
k \in W_{-i} \text{ and } v'_{i,k} \geq v'_{i,j}.
\]

A consequence of the budget constraint is that it allows for partial strategy-proofness in the best response of a focal voter responding to all other votes.

**Theorem 2.9 (Partial Strategy-proofness).** There exists a best response \( S_i^* \) such that if \( k \) dominates \( j \), and \( j \in S_i^* \), then \( k \in S_i^* \).

The proof is not critical to the exposition here, and we defer it to the Appendix.

We can think of \( W_{-i} \) as representing the candidates that \( i \) thinks are popular. And projects with a higher “value-for-money” are preferred from her perspective. The theorem states that a simple way for a voter to act in her best interest is to vote for projects that to her are both popular and favorite. This notion is similar in spirit to the ideal of *sincerity* in approval voting [Brams and Fishburn 1978; Niemi 1984].
The reason partial strategy-proofness holds under Knapsack Voting is that the voters face the same constraints as the outcome they are collectively deciding. It is interesting to note that this property does not hold under K-approval voting, as there is a mismatch between the constraints on the voters (choosing a fixed number of projects $k$) and those on the outcome (a fixed budget $B$). We illustrate this with a small example.

Example 2.10. Consider 5 projects $a, b, c, d, e$ of costs $200, 100, 100, 100, 200$, and using the 2-approval rule. The budget is $400$ and they poll $100, 50, 50, 50, 20$ votes respectively without counting in $i$’s vote. The tie is broken in the order $e, d, c, b, a$, and in this case $d$ wins over $b, c$. Let’s say that $i$’s utilities for these are $500, 100, 150, 200, 500$ respectively. Based on the above, $i$’s best response is to not vote for $a$ but for $c, d$.

2.5. Maximum Likelihood Interpretation

One way of looking at voting rules is as follows: There exists a “ground truth” outcome, and each voter has a noisy perception of it. And the the voting rule is the Maximum likelihood estimator of the “ground truth” given any realization of votes. The Knapsack voting rule can be interpreted in this way since it belongs to a family of voting rules known as scoring rules [Conitzer and Sandholm 2012]. Selecting subsets of winners based on the MLE approach has received some attention lately [Procaccia et al. 2012] wherein voting rules to select subsets based on their performance on various metrics with respect to noisy comparisons or rankings drawn from a Mallows model.

In this section, we will use the per-dollar approach (see Section 2.1) and explicitly construct a natural noise model for the votes for which it is the Maximum Likelihood estimator. The noise model we construct is similar to the Mallows model, but defines a distribution over subsets directly as opposed to rankings.

Definition 2.11 (Noisy Knapsack Vote Model). There is “ground truth” set $S^* \in \mathcal{P}'$, which satisfies $|S^*| = B$. Each vote $S_i$ is drawn i.i.d. according to a distribution that is given by:

$$\Pr\{S_i | S^* \} \propto \exp(|S^* \cap S_i|), \quad \text{if } |S_i| \leq B$$

$$\Pr\{S_i | S^* \} = 0, \quad \text{otherwise}$$

The quantity $|S^* \cap S_i|$ is equal to the number of dollars in the allocation given by $S^*$ that agrees to that given by voter $i$ in her vote $S_i$, and by Lemma 2.6 is related to the Overlap/$\ell_1$ utility we discussed previously.

By taking the logarithm of the probabilities $\Pr(S_i | S^*)$, it is easy to see that the Maximum Likelihood estimate of the “ground truth” $S^*$ given all the votes, is the set $S \subseteq \mathcal{P}'$ satisfying $|S| = B$ that maximizes

$$\frac{1}{|V|} \sum_{i \in V} |S \cap S_i|. \quad (2)$$

Theorem 2.12. The Knapsack rule returns the maximum likelihood estimate under the Noisy Knapsack Vote Model.

Proof. Let’s rewrite the quantity in Equation 2 as

$$\sum_{i \in V} |S \cap S_i| = \sum_{i \in V} \sum_{j \in S} 1(j \in S_i) = \sum_{j \in S} \sum_{i \in V} 1(j \in S_i) = \sum_{j \in S} \text{score}(j),$$

where $1(.)$ is a $\{0,1\}$ variable that takes on a value 1 when the statement in the argument is true, and 0 otherwise. This quantity is maximized by picking $B$ candidates from $\mathcal{P}'$ that have the highest score (see Definition 2.2), which is essentially what the Knapsack rule chooses. □
3. VOTING BASED ON VALUE-FOR-MONEY

We have seen that Knapsack Voting has many advantages with respect to its strategic properties, and implementation in Participatory Budgeting elections. We now turn to another way of eliciting voters’ preferences in this setting - value-for-money comparisons. Given a single agent's Knapsack Problem, one way of computing the optimal solution is to order the items according to their value-to-size ratio and pick the higher ranked ones in order till the knapsack capacity is used up. This order can be ascertained by comparing pairs of projects according to their value-to-size ratio. Asking voters to compare/rank projects based on their value-for-money is a natural analog (see Figure 3) of this idea in a setting where multiple agents together have to decide an outcome.

Unlike Knapsack Voting, aggregation schemes based on value-for-money comparisons cannot be guaranteed to have good strategic properties. Further, these schemes are not as transparent as Knapsack Voting. Value-for-money schemes also do not extend naturally to settings with revenue, deficits and surpluses.

Despite these difficulties, they are useful in practice because they can be used to design paper ballots with Knapsack Voting, and also elicit the aggregate preferences of voters to make empirical observations from data. We will discuss these presently.

In addition, Value-for-money schemes have the following potential advantages:

— a smaller cognitive load on voters: especially with large ballots (see Section 4.0.2)
— aggregation in cases where the budget is not known or fixed a priori (perhaps using Kemeny-Young like ranking rules)

We will delve into these in more detail in the next section.

Fig. 7. Value-for-money ranking paper ballot for Knapsack Voting - PB Boston 2016

3.1. Value-for-money rankings and paper ballots

As mentioned before, some elections that employ digital voting also require a corresponding paper ballot. This was the case in the Youth Lead The Change 2016 PB election held in Boston, a set of projects was put to vote to youth between the ages 12
and 25. We implemented Knapsack Voting as the official ballot process in this election. Since it is tedious to do Knapsack Voting on paper, we designed a paper ballot that asked voters to rank their top 4 projects according to value-for-money (Figure 7). From a total of around 4000 voters, about 10% used the digital Knapsack interface, and the rest used the paper ballot with ranking.

Here we ask each voter to rank the projects on the ballot taking both costs and benefits into account. Voters take into account costs and benefits into account, and ranking reinforces this consideration. Interpreting these votes as Knapsack votes, we observe that the outcome from the rankings was exactly the same as that of Knapsack Voting. This gives us a way of using rankings to solve the budgeting problem in practice.

3.2. Value-for-money comparisons

We can elicit fine-grained information about the preferences of voters between pairs of projects by doing value-for-money comparisons between randomly chosen projects.

Definition 3.1 (Value-for-money comparison). For each pair of projects \{j, k\} from \(\mathcal{P}\) shown to her, voter \(i\) chooses a winner \(w_i(\{j, k\}) = \arg \max_{t \in \{j, k\}} v_{i,t}c_t\).

Doing these comparisons in practice, leads us to an interesting empirical observation of the structure in the aggregate preferences of voters. We observe that compiling this data from our experiments reveals a (nearly) transitive majority relation among the projects (Figure 8). By this we mean that there exists a Condorcet winner, and upon removal of that candidate, there again exists a Condorcet winner, and so on.

In figures 8 and 9, we show the aggregate strength of each comparison across the two elections – Cambridge 2015 and Vallejo 2015. More details about the two elections are in the next section. The number in the cell denotes the fraction of comparisons where the row project beats the column project. Green represents a fraction greater than 0.5, and red represents a fraction smaller than 0.5. The darker the green(red), the closer it is to 1(0).

Given this structure, there is a clear order among projects based on the aggregate strength of the comparisons. In fact, any Condorcet rule leads to the same outcomes on such data. If this structure does not hold, then the aggregation of these comparisons becomes more complicated.

\(^2\)There is an implicit assumption here that taking both costs and benefits into account is the same as considering value-for-money.
3.2.1. Comparing outcomes of Knapsack and K-approval. We will define a quantitative measure for any outcome that represents the level of its agreement with the pairwise comparisons. To do this, we generalize the Borda rule and define a Borda score for outcomes that are sets of projects.

In our experiments, we compare one project against another as shown in Figure 3, and to each voter we present randomly chosen pairs so as to ascertain the aggregate preferences of voters. Given the results of these comparisons, let \( n(j, k) \) denote the number of voters that chose \( j \) over \( k \).

We refer the reader to [Young 1974; 1988] for a standard definition of Borda’s rule, and its classical interpretation as an MLE. Generalizing the Borda score for a single winner, we define

**Definition 3.2 (Set-Borda score).** For any set of projects \( S \subseteq P \), the Set-Borda score of \( S \) is given by

\[
\frac{1}{C(M-C)} \sum_{j \in S} \sum_{k \in S} c_j c_k (n(j, k) - n(k, j)),
\]

where \( C = \sum_{p \in S} c_p \) and \( M = \sum_{p \in P} c_p \).

We can interpret a pairwise value-for-money comparison between two projects \( j \) and \( k \), as a pairwise relation between a dollar sub-project of \( j \) and a dollar sub-project of \( k \). The Set Borda score of a set of \( S \) corresponds to the average number of such dollar versus dollar comparisons that agree, minus those that disagree, with the partition induced by it. We can use this score, as a measure of social welfare that embodies the essence of the Borda rule, to empirically compare the outcome of Knapsack Voting with that of K-approval.

4. OUR DIGITAL PLATFORM, AND RESULTS FROM OUR EXPERIMENTS

Our work with Participatory Budgeting began with a partnership with Chicago’s 49th Ward to develop a digital voting system for their election. Since then we’ve have worked with almost a dozen cities/districts over the past couple of years (see http://pbstanford.org), but we will primarily look at data collected from the following elections:

- Boston 2016 - Youth Lead the Change
- Boston 2015 - Youth Lead the Change
- Cambridge 2015
- Cambridge 2014
- Vallejo 2015
- New York City District 5 2015 (NYC5)
- New York City District 8 2015 (NYC8)

The total number of voters in each of these elections were 4176, 2600, 3273, 2194, 1834, 704, 271 respectively.

In the Boston 2016 election, we implemented Knapsack Vote as the official ballot mechanism. In the other elections, besides implementing a interactive interface for the official K-approval vote (Fig. 1), our goal has been to experiment with Knapsack voting and Value-for-money comparisons and use the data so collected to complement a theoretical understanding of these methods. After the K-approval vote used for the formal election, we ask the voters to participate in our experiments on either Knapsack Voting or Value-for-money comparisons or both.

- In Cambridge 2015 and Boston 2015, we showed each voter either Knapsack and value-for-money experiments with a 50% chance.
- In NYC5 and NYC8, we did only the Knapsack vote.
- In Cambridge 2014 and Vallejo 2015 we did only pairwise comparisons.
The Knapsack interface (Figure 2 in Section 1.1) has a live budget bar that shows how much of the budget has used up with the current selection. For the value-for-money comparisons, we present each voter with a fixed number of pairs of projects chosen uniformly at random without replacement, and ask them the following question (in the spirit of Definition 3.1): “Which of these projects gives a higher benefit to the community per dollar spent?” (see Figure 3). We must mention that only a subset of the entire set of voters actually chose to take part in our experiments, and we report this percentage in Tables V and IV. This procedure was approved by Stanford University’s Institutional Review Board.

4.0.1. Cost consideration under Knapsack. Our data suggests that there is a bias towards projects of larger costs in the K-approval method, as compared to the Knapsack method. We empirically verify this effect in two ways. First, we present data from Cambridge and NYC District 8, where a budget of $600,000 and 6-approval, and $1,000,000 and 5-approval, were used respectively. In figures 10 and 11 we lay out the projects in descending order of cost, and plot the cumulative fraction of votes for projects above every cost threshold. We then compare K-approval and Knapsack against the uniform distribution. We see that this function for K-approval dominates that for Knapsack, which means that costlier projects are over-represented, thereby supporting our hypothesis.

Second, we look at the average cost of the winning projects under each method. Table II shows the average cost of the winning projects (normalized by the total budget) in each of those elections. On the average, across the three places, there is a reduction of about 30% in the average cost of the winning projects. The above two observations clearly suggest that Knapsack Voting leads to voters’ being more frugal while choosing which projects to vote for.

Table II. Average cost of winning projects, as a fraction of the budget

<table>
<thead>
<tr>
<th></th>
<th>K-approval</th>
<th>knapsack</th>
</tr>
</thead>
<tbody>
<tr>
<td>NYC District 5</td>
<td>0.18</td>
<td>0.14</td>
</tr>
<tr>
<td>NYC District 8</td>
<td>0.20</td>
<td>0.12</td>
</tr>
<tr>
<td>Boston 2015</td>
<td>0.27</td>
<td>0.18</td>
</tr>
<tr>
<td>Cambridge 2015</td>
<td>0.15</td>
<td>0.10</td>
</tr>
</tbody>
</table>

ACM Journal Name, Vol. V, No. N, Article A, Publication date: January YYYY.
4.0.2. Timing data. The data from Tables III and V suggests that the Knapsack interface is not much more time consuming than the K-approval interface. Of course, since the Knapsack interface follows the official K-approval interface, and so the voters were familiar with the projects when they attempted the Knapsack vote. Even if the time taken by Knapsack were the sum of times for both in our experiments, we can see that it is very reasonable. This is borne out in the Boston 2016 election (see Table V), where the number of projects was the same as in Boston 2015 and Knapsack was presented first as the official ballot.

Here are some more interesting observations on the ease of each voting method based on Tables III, IV, V:

— we presented to each voter either Knapsack or Value-for-money comparisons with a 50% chance in both Boston 2015 and Cambridge 2015, and the percentage of voters that completed the comparisons was at least a third greater than that of Knapsack;
— in Boston 2015, the median time taken by the voters was 61 seconds for K-approval vote (K=4 out of 10 projects on the ballot), 70 seconds for Knapsack and 14 seconds for value-for-money comparisons of 4 pairs. A similar trend is seen in the Cambridge experiment.

All this suggests that Knapsack Vote and K-approval have comparable times. value-for-money comparisons involve a smaller cognitive load, than the other voting methods, especially with a large number of projects as seen in Cambridge 2015.

<table>
<thead>
<tr>
<th>City</th>
<th>K</th>
<th>N</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston 2015</td>
<td>4</td>
<td>10</td>
<td>61s</td>
</tr>
<tr>
<td>Cambridge 2015</td>
<td>6</td>
<td>23</td>
<td>213s</td>
</tr>
</tbody>
</table>

Table IV. Timing data for value-for-money comparisons; n = number of comparisons, T = median completion time

<table>
<thead>
<tr>
<th>City</th>
<th>n</th>
<th>T</th>
<th>% of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston 2015</td>
<td>4</td>
<td>14s</td>
<td>10</td>
</tr>
<tr>
<td>Cambridge 2015</td>
<td>4</td>
<td>53s</td>
<td>40</td>
</tr>
</tbody>
</table>

Table V. Timing data for Knapsack Voting; N = number of projects, T = median completion time

<table>
<thead>
<tr>
<th>City</th>
<th>Budget</th>
<th>N</th>
<th>T</th>
<th>% of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston 2015</td>
<td>$1,000,000</td>
<td>10</td>
<td>70s</td>
<td>7</td>
</tr>
<tr>
<td>Cambridge 2015</td>
<td>$600,000</td>
<td>23</td>
<td>115s</td>
<td>30</td>
</tr>
<tr>
<td>Boston 2016</td>
<td>$1,000,000</td>
<td>10</td>
<td>86s</td>
<td>100</td>
</tr>
</tbody>
</table>

4.1. Comparison of Knapsack and K-approval against value-for-money comparisons

Using the data from random pairwise comparisons in the value-for-money experiment, and based on the Set Borda score (Definition 3.2) we calculate the average number of comparisons that agree and disagree with the winning sets as determined by the K-approval and Knapsack methods. If we picked a dollar allocated in the winning outcome, and a dollar not in it, both at random, the numbers in Table VI are a measure of the average fraction of votes that agree with the outcome minus those that disagree.

<table>
<thead>
<tr>
<th></th>
<th>K-approval</th>
<th>Knapsack</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Agreement</td>
<td>Standard error</td>
</tr>
<tr>
<td>Boston 2015</td>
<td>0.20</td>
<td>0.047</td>
</tr>
<tr>
<td>Cambridge 2015</td>
<td>0.30</td>
<td>0.022</td>
</tr>
</tbody>
</table>

ACM Journal Name, Vol. V, No. N, Article A, Publication date: January YYYY.
We see that on this score, Knapsack does better than the K-approval, thereby indicating a higher level of agreement with the voters' preferences. We also note the standard error in the sample mean calculated above, assuming the sample mean is normally distributed. We see that the difference between the sample means of Knapsack and K-approval is greater than the standard error.

5. CONCLUSIONS AND ONGOING WORK

Knapsack voting and Value-for-money comparisons are intuitive ways of eliciting voters' preferences for budgetary decisions. The Knapsack Vote admits interesting strategic properties: in particular, it is strategy-proof under a natural utility model that depends on the overlap (and equivalently the $\ell_1$ distance) between the outcome and the voters' true preferred allocations. It can also be extended to more complicated settings with revenues, deficits and surpluses. While Value-for-money comparisons have some drawbacks, they provide a way of eliciting the voters' preferences with a small cognitive load, especially in the case of large ballots. The fact that our schemes do better on many different measures finds support in the data we collected from participatory budgeting elections in various cities/municipalities. All our schemes are amenable to implementation using interactive digital tools, thereby enhancing the ability of voters to make more informed decisions in participatory budgeting. We have been able to make some initial progress along implementing Knapsack Voting as the official ballot, and we hope that this paper makes a strong case for its wider adoption in practice.

ACKNOWLEDGMENTS

This work is supported by the Army Research office (grant # 116388), the Office of Naval Research (grant # 11904718), and the Stanford Cyber Initiative.

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APPENDIX

A.1. Proof of Theorem 2.8

Let \( R_v, S_v, \) and \( \text{score}_v(\cdot) \) be the outcome determined by the votes of everyone except \( v \) using the Knapsack Voting rule (Equation 1). Assume that \( Q_v \subseteq R_v' \) and \( T_v \subseteq P' \) be a best response for voter \( v \) such that \( Q_v \neq R_v \) and \( T_v \neq S_v \). Let the outcome after incorporating \( Q_v, T_v \) be \( \text{score}_v(\cdot) \) and \( S \).

We will discuss the case when \( |Q_v| = |T_v| < |S_v| = |R_v| \). The other case of \( |Q_v| = |T_v| \geq |S_v| = |R_v| \) follows analogously (equality is similar to the proof of Theorem 2.4).

Choose some \( j \in S_v \setminus T_v \) such that if \( k = D_p^v \) for some \( p \in P \), then \( D_p^v \in T_v \) for all \( l' < t \).

Such a \( k \) exists because of consistency and the fact that \( |T_v| < |S_v| \). Let \( T'_v \equiv T_v \cup \{j\} \). Similarly, choose the \( k \in R_v \setminus Q_v \) such that if \( k = D_p^v \) for some \( q \in P' \), then \( D_p^v \in T_v \) for all \( l < z \) and define \( Q'_v \equiv Q_v \cup \{k\} \). Let the outcome here be \( \text{score}'(\cdot) \) and \( R', S' \). We will show that \( Q'_v, T'_v \) is also a best response for \( v \).

If \( R = R' \) and \( S = S' \), then the utility is unchanged and we have nothing to prove.

We have that \( \text{score}'(j) = \text{score}(j) + 1 \), \( \text{score}'(k) = \text{score}(k) + 1 \), and for all \( l \in P' \cup R' \setminus \{j, k\} \), \( \text{score}'(l) = \text{score}(l) \). Note that the only change from \( \text{score}(\cdot) \) to \( \text{score}'(\cdot) \) is that the score of \( j \) and \( k \) increases. So the outcomes must satisfy \( R' \supseteq R \) and \( S' \supseteq S \), and \( |R'| = |S'| \leq |S| + 2 = |R| + 2 \). For any given tie-breaking rule, if \( S' \neq S \), then we must have either \( j \in S' \setminus S \) or \( k \in R' \setminus R \), or both, i.e., any change in outcome must involve either \( j \) or \( k \) moving from outside the winning set to within.
If \( j \in S' \setminus S \), then there is a corresponding sub-project that is added to \( R \) to maintain the budget balance, say \( m \in R' \setminus R \). The change in utility of voter \( v \) is \( 1(j \in S_v) - 1(m \notin R_v) = 1 - 1(m \notin R_v) \geq 0 \). In other words, the change in utility from adding \( j \) to \( S \) is 0, and any \( m \) to \( R \) to maintain budget balance, cannot be negative.

Similarly, if \( k \in R' \setminus R \), then there is a corresponding \( m' \in S' \setminus S \). And the change in utility is \(-1(k \notin R_v) + 1(m' \in S_v) = 0 + 1(m' \in S_v) \geq 0 \).

By repeating this process, we reach a point at which we have a best response \( Q_v, T_v \) equal in size to \( R_v, S_v \) respectively. From here we do a procedure similar to the proof of Theorem 2.4 until we only have elements in \( S_v \). In this entire process, we do not decrease the utility.

To prove that the outcome is welfare-maximizing, note that it is given by \((R^*, S^*)\), where \( R^* \) and \( S^* \) are both consistent and \(|R^*| = |S^*|\), which maximizes the following:

\[
\sum_{i \in S} \text{score}(i) + \sum_{j \in R} \text{score}(j) = \sum_{i \in S} \sum_{v \in V} 1(i \in S_v) - \sum_{j \in S} \sum_{v \in V} 1(j \notin R_v)
\]

\[
= \sum_{v \in V} \sum_{i \in S} 1(i \in S_v) - \sum_{v \in V} \sum_{j \in R} 1(j \notin R_v) = \sum_{v \in V} |S_v \cap S| - |R \setminus R_v|,
\]

and the last quantity in the above is the welfare according to the Overlap Utility Model.

A.2. Proof of Theorem 2.9

We will use the per-dollar approach (see Section 2.1 above), i.e., each voter \( i \in V \) submits a vote \( S_i \subseteq \mathcal{P}' \) such that \( S_i \) is consistent and \(|S_i| = B\). Let us first reinterpret the notation from Section 2.4 per-dollar, and restate Theorem 2.9 in technical terms.

Consider a focal voter \( i \) responding to the votes of all others. Assume that she has full knowledge about the how the others voted in aggregate. If \( S_{-i} \) denotes the cumulative votes of all voters except \( i \), she knows \( \mathcal{W}_{-i} \), the set of winners as determined by \( S_{-i} \). Let \( \mathcal{W}(S_i, S_{-i}) \) denote the set of winners if her vote \( S_i \) is added.

A best response for \( i \) is then defined as a vote \( S_i^* \) that satisfies

\[
S_i^* = \arg \max_{S_i \in A} \sum_{j \in \mathcal{W}(S_i, S_{-i})} v'_{i,j},
\]

where \( A = \{S : S \text{ is consistent, } |S| = B\} \), and \( v'_{i,j} \) is the utility of voter \( i \) from sub-project \( j \in \mathcal{P}' \).

With respect to voter \( i \), we say a candidate \( q \in \mathcal{P}' \) dominates \( p \in \mathcal{P}' \) if and only if

\(- q \in \mathcal{W}_{-i}, \quad \text{ and} \quad q > v'_{i,q} \quad \text{ and} \quad v'_{i,q} > v_{i,p} \).

Denote the collection of candidates that dominate \( p \) with respect to voter \( i \) as \( \Lambda_{i,p} \), i.e.,

\[ \Lambda_{i,p} \triangleq \{j \in W_{-i} : v'_{i,j} > v_{i,p}\}. \]

Let’s say that \( p \in S_i^* \) and \( \Lambda_{i,p} \notin S_i^* \). We will first claim that \( \Lambda_{i,p} \subseteq \mathcal{W}(S_i^*, S_{-i}) \).

Let \( j \in \Lambda_{i,p} \). We have the following two possible cases:

(1) If \( j \in S_i^* \) since \( \Lambda_{i,p} \subseteq W_{-i} \), we have \( j \in S_i^* \cap W_{-i} \), and consequently \( j \in \mathcal{W}(S_i^*, S_{-i}) \).

(2) Else, if \( j \notin S_i^* \) Assume \( j \notin \mathcal{W}(S_i^*, S_{-i}) \). Then \( S_i^* \) cannot be a best response, because by switching her vote from \( S_i^* \) to \((S_i^* \setminus \{p\}) \cup \{j\} \), \( i \) can make \( j \) win instead of \( p \), and this strictly increases her total utility (since \( v_{i,j} > v_{i,p} \)).

In either case, \( j \in \mathcal{W}(S_i^*, S_{-i}) \).
Now, if $S_1 \triangleq S_i^* \setminus \Lambda_{i,p}$, then $|S_1| = B - |S_i^* \cap \Lambda_{i,p}|$. And since we have proved $\Lambda_{i,p} \subseteq W(S_i^*, S_{-i})$, it follows that $|S_1 \cap W(S_i^*, S_{-i})| \leq k - |\Lambda_{i,p}|$. These two facts together imply the following:

$$|S_1 \setminus \Lambda_{i,p}| = |S_1| - |S_1 \cap W(S_i^*, S_{-i})|$$

$$\geq (k - |S_i^* \cap \Lambda_{i,p}|) - (k - |\Lambda_{i,p}|)$$

$$\geq |\Lambda_{i,p}| - |S_i^* \cap \Lambda_{i,p}|$$

$$= |\Lambda_{i,p} \setminus S_i^*|.$$  

Hence, there exists $S_2 \subseteq S_1 \setminus W(S_i^*, S_{-i})$ such that $|S_2| = |\Lambda_{i,p} \setminus S_i^*|$. Let $S_i^{**} = (S_i^* \setminus S_2) \cup \Lambda_{i,p}$. Clearly, $|S_i^{**}| = k$ (since $S_2 \cap \Lambda_{i,p} = \emptyset$), and so it is a valid vote for voter $i$.

Also $W(S_i^{**}, S_{-i}) = W(S_i^*, S_{-i})$, since, we have replaced $S_2 \subseteq W(S_i^*, S_{-i})$ with $\Lambda_{i,p} \setminus S_i^* \subseteq W(S_i^*, S_{-i})$. Because $S_i^*$ is a best response, so is $S_i^{**}$.

Note that

$$S_2 = S_i^* \setminus S_i^{**} \subseteq W(S_i^*, S_{-i})$$  \hspace{1cm} (4)

Now if there is a $p' \in S_i^{**}$ such that $\Lambda_{i,p'} \not\subseteq S_i^{**}$, we can do a similar replacement procedure to define another best response $S_i^{***} = (S_i^{**} \setminus S_2') \cup \Lambda_{i,p'}$, such that $S_2' \subseteq W(S_i^{**}, S_{-i})$ (from equation 4). Since $\Lambda_{i,p} \subseteq W(S_i^{**}, S_{-i})$, this implies that $S_2' \cap \Lambda_{i,p} = \emptyset$ and so $S_i^{***}$ includes both $\Lambda_{i,p}$ and $\Lambda_{i,p'}$. By a series of replacements, we have the best response as required by the theorem. \qed