Basic Feasible Solutions: A Quick Introduction

CS 261

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TEN STEPS TOWARDS UNDERSTANDING VERTEX OPTIMALITY AND BASIC FEASIBLE SOLUTIONS

THESE SLIDES: MOSTLY INTUITION; PROOFS OMITTED

Step 0: Notation

- Assume an LP in the following form

  Maximize $\mathbf{c}^T\mathbf{x}$

  Subject to:

  $$\mathbf{Ax} \leq \mathbf{b}$$

  $$\mathbf{x} \geq 0$$

- N Variables, M constraints
- U = Set of all feasible solutions

Step 1: Convex Sets and Convex Combinations

- Convex set: If two points belong to the set, then any point on the line segment joining them also belongs to the set

- Convex combination: weighted average of two or more points, such that the sum of weights is 1 and all weights are non-negative
  - Simple example: average

Convex Combination

- Given points x^1, x^2, …, x^K, in N dimensions
- For any scalars a_1, a_2, …, a_K such that
  - Each a_i is non-negative
  - $a_1 + a_2 + \ldots + a_K = 1$

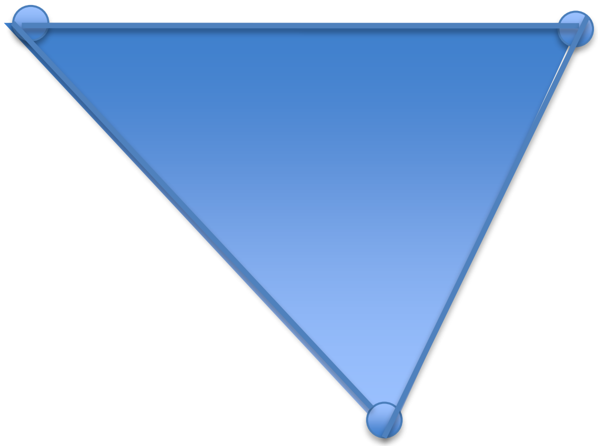
$a_1x^1 + a_2x^2 + \ldots + a_Kx^K$ is a convex combination of x^1, x^2, …, x^K

Example: Convex combination of two points

Exercise

- If **w** is a convex combination of **x** and **y**, and **z** is a convex combination of **x** and **w**, then must **z** be a convex combination of **x** and **y**?

Example: Convex combination of three points



Exercise

Is the set of all convex combinations of N points always convex?

Exercise

Can any convex set be represented as a convex combination of N points, for finite N?

Step 2: Half spaces are convex

- $\mathbf{a}^T\mathbf{x} \leq b$, $\mathbf{a}^T\mathbf{y} \leq b$

- Choose $\mathbf{z} = (\mathbf{x}+\mathbf{y})/2$
- Then, we must have $\mathbf{a}^T\mathbf{z} \leq b$

Step 3: Intersection of convex sets is convex

- Two convex sets S, T
- Suppose **x**, **y** both belong to S ∩T
- **z** = (**x**+**y**)/2

    (1) **z** belongs to S since S is convex

    (2) **z** belongs to T since T is convex
- Hence, **z** belongs to S ∩ T
- Implication: S∩T is convex

Step 4: Feasible solutions to LP

- The set U of feasible solutions to a Linear Program is convex

- Proof: U is the intersection of Half Spaces

Exercise

- Must every convex set be bounded?

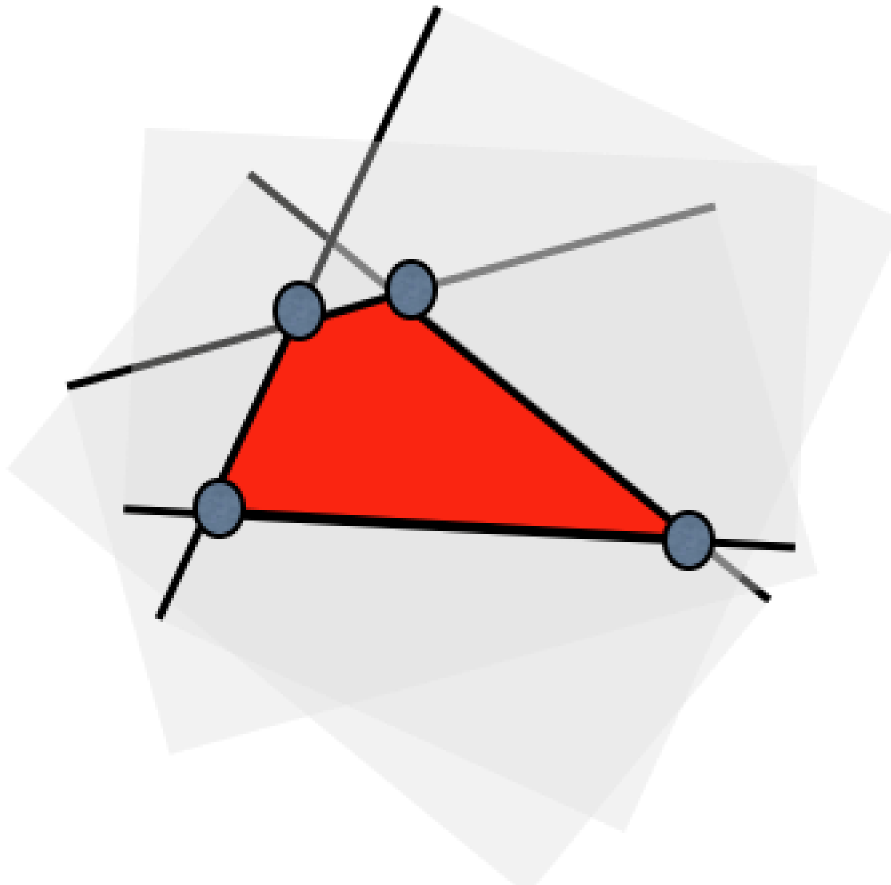
Step 5: Basic Feasible Solution

- **x** is a basic feasible solution to a LP, if

  (1) **x** is a feasible solution

  (2) There do not exist two other feasible solutions **y**, **z** such that **x** = (**y**+**z**)/2

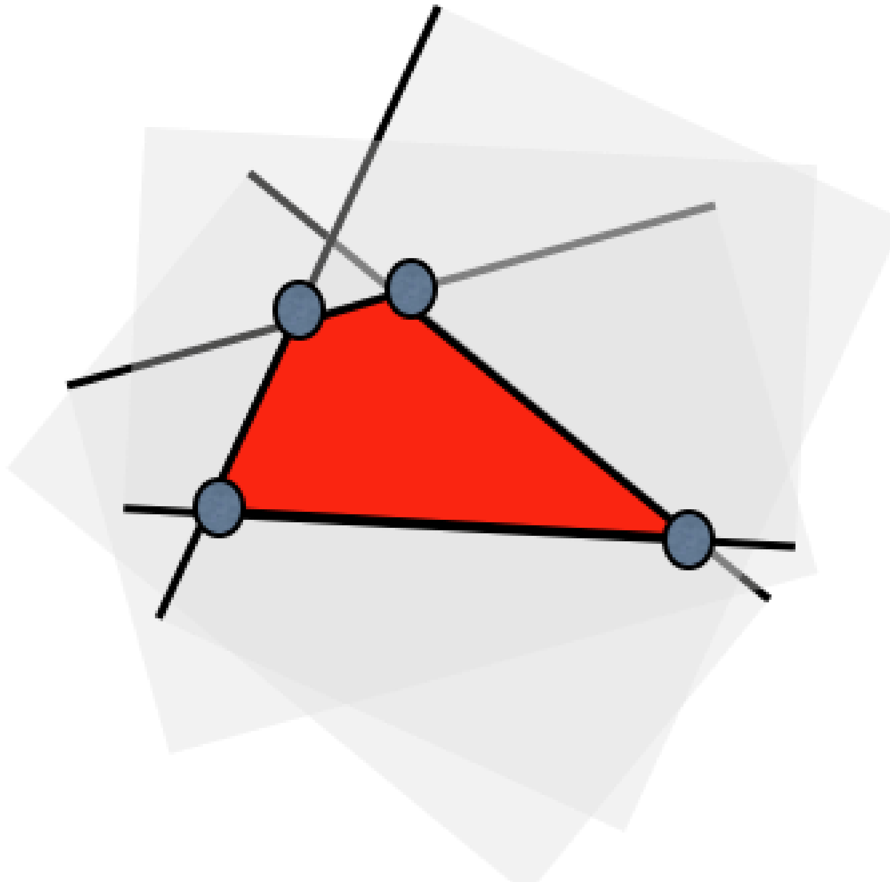
ALSO known as vertex solution, extreme point solution, corner-point solution



Step 6: BFS and Bounded Polytopes

- If U is bounded, then any point in U is the convex combination of its basic feasible solutions

Use $\mathbf{b}_1$, $\mathbf{b}_2$, …, $\mathbf{b}_K$ to denote the K basic feasible solutions



Step 7: BFS and Optimality for Bounded Polytopes

- If U is bounded and non-empty, then there exists an optimal solution that is also basic feasible

Step 8: BFS and Optimality for General LPs

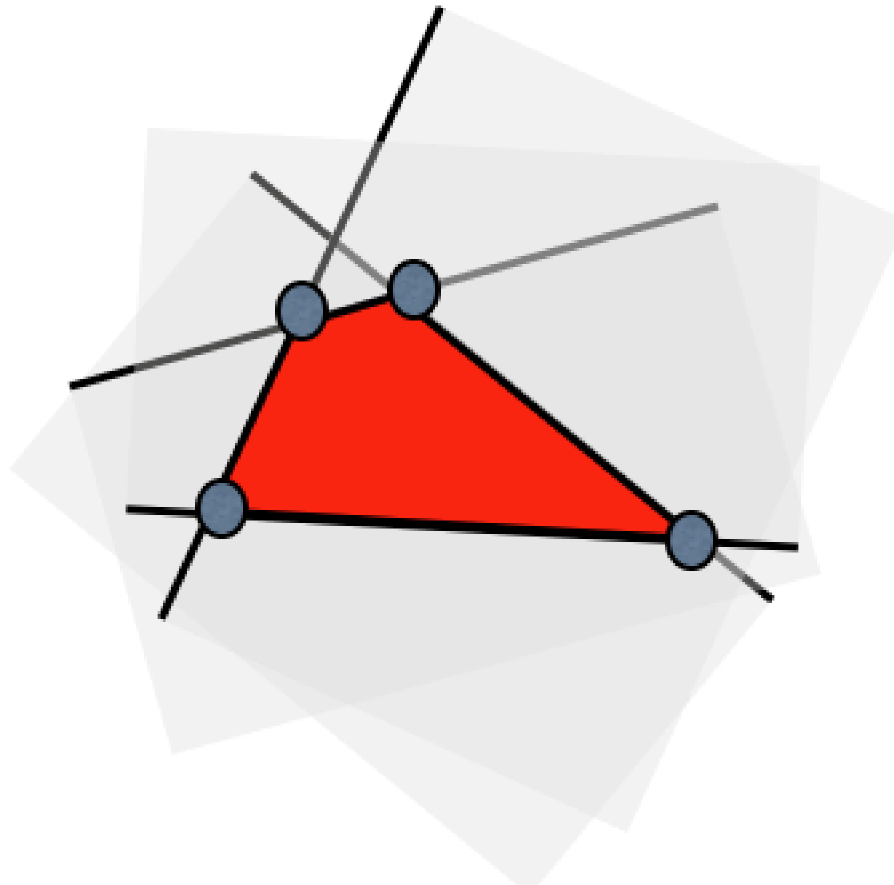
- If an LP has a basic feasible solution and an optimum solution, then there exists an optimal solution that is also basic feasible

- Will call these solutions "Vertex Optimal" or "Optimum Basic Feasible"

Step 9: Two Implications

(1) SOLUTION OF LPs: The Simplex method which walks from bfs to bfs is correct for bounded polytopes

(2) STRUCTURAL CONSEQUENCES: The Vertex Optimal Solutions often have very interesting properties. Example: For the matching LP in the next video, every vertex optimal solution is integral



In Practice

- Most LP Solvers return an optimum basic feasible solution, when one exists.
  - Either, they use Simplex
  - Or, they transform the solution that they do find to a basic feasible solution
- Hence, when we solve a problem using Excel we get an optimum basic feasible solution, when one exists.

Step 10: A useful property

- At every basic feasible solution, at least N constraints are tight
- More specifically: Exactly N Linearly Independent Constraints are tight
- Basic solution: Not necessarily feasible, but exactly N Linearly Independent Constraints are tight

Exercise

Is it possible for a LP to have an optimum solution but no basic feasible solution?

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Exercise

You are given a LP with three decision variables where some of the constraints are

$$0 \le x \le 1$$

$$0 \le y \le 1$$

$$0 \le z \le 1$$

Is it possible to add other constraints and an objective function such that the LP has an optimum solution but not an optimum basic feasible solution?

LP in standard form

Maximize $\mathbf{c}^T\mathbf{x}$

Subject to:

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geq 0$$

- N Variables, M constraints, assume that the M constraints are Linearly Independent
- At any BFS, at most M variables are strictly positive

Recap

- A feasible solution is basic feasible if it is not the average of two other feasible solutions
- If the feasibility region U for a LP is bounded and non-empty, then there exists an optimal solution that is also basic feasible
- If an LP has a basic feasible solution and an optimum solution, then there exists an optimal solution that is also basic feasible
- Commercial LP solvers return an optimum basic feasible solution by default, where one exists
- If there are N decision variables, then a basic feasible solution has N linearly independent constraints tight