CS261 Winter 2018 - 2019
Lecture 7: Basic Feasible Solution; Min-cost Flow

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1 Formulation of Min-cost Flow

Suppose that are solving a max-flow problem, and as such are routing flow across a graph, but routing flow along an edge incurs some cost dependent on the edge. We would like to find a max-flow that minimizes total cost. What algorithm can we use to solve this problem?

In fact, we will generalize this problem slightly to permit analysis of any flow pattern. Consider a graph $G = (V, E)$. With every vertex in V, we associate some quantity d_v, which we will call its "demand" for flow. If $d_v > 0$, we say that the vertex "consumes" flow, and if $d_v < 0$ it produces flow. In the max flow problem, if D is the value of a max flow from s to t, then we would set $d_s = -D$ and $d_t = D$.

To finish formalizing the problem, similarly to the max-flow problem, we will say that each edge $e \in E$ has a flow capacity u_e. In addition, each $e \in E$ has a cost c_e, which is the cost of sending one unit of flow through e.

Ultimately, the goal of the min-cost flow problem is to send a flow f through G such that the total cost is minimized while capacity constraints and conservation constraints are satisfied for every edge and node respectively. We can solve min-cost flow by solving the following linear programming problem:

$$\begin{aligned}
\text{minimize } & \sum_{e \in E} c_e f_e \\
\text{subject to } & 0 \leq f_e \leq u_e, \text{ for all } e \in E \\
& \sum_{(u,v) \in E} f_{(u,v)} = d_v + \sum_{(v,u) \in E} f_{(v,u)}, \text{ for all } v \in V
\end{aligned}$$

2 Integrality of BFS to Min-cost Flow

Assume that d_v and u_e are all integers. Then given a min-cost flow problem, every basic feasible solution (BFS) to the LP above must be integral (the flow f given by each BFS must have integer flow on each edge of the graph).

To prove this result, consider a feasible flow f which is not integral — there exists at least one edge $e = (u_1, v_1) \in E$ such that f_e is fractional. Since for v_1 the equality $d_{v_1} + \sum_{(u,v_1)\in E} f_{(u,v_1)} = \sum_{(v_1,u)\in E} f_{(v_1,u)}$ holds and d_{v_1} is an integer, we know that there is at least another edge (u_2, v_1) or (v_1, u_2) with $u_2 \neq u_1$ carrying fractional flow. Applying the same argument on u_2, we can find another edge carrying fractional flow.

We can repeat this process until we hit a cycle (which must happen eventually, as the number of edges is finite). Let the cycle be $u_1 - v_1, ... - u_k - v_k - u_1$. Each edge in this cycle carries a fractional flow, and hence we can choose a small number δ such that $\delta > 0$ and we guarantee that for each edge e in this cycle:

$$f_e + \delta \leq u_e$$
$$f_e - \delta \geq 0$$

Next, construct a new flow f' by sending an additional flow with flow amount δ along the cycle in the direction of $u_1 \to v_1 \to ... \to u_k \to v_k \to u_1$ on top of f. It's easily to verify that all edges in the cycle satisfy capacity constraints and all vertices in the cycle satisfy conservation constraints; the remaining edges and vertices remain unchanged and hence still satisfy the constraints. Hence f' is a new feasible solution to the min-cost flow problem.

Similarly, we can construct another flow f'' by sending flow in the opposite direction along the cycle. We can also verify that f'' is a feasible solution to the min-cost flow problem.

Consider the flow g induced by routing δ units of flow around the above cycle. If the cost of g is strictly positive, then the cost of its reverse must be strictly negative. If one of g or its reverse is strictly negative in cost, then either f' or f'' has a lower cost than f, which is a contradiction. Hence, the costs of f, f' and f'' are all equal (since for any two flows g_1 and g_2, $cost(g_1) + cost(g_2) = cost(g_1 + g_2)$).

What is the relationship between f, f' and f''? For any edge e not in the cycle, $(f' + f'')_e = 2f_e$, and for any edge e in the cycle, $(f' + f'')_e = (f_e + \delta) + (f_e - \delta) = 2f_e$. Hence $f = (f' + f'')/2$, and thus that f is a convex combination of f' and f''. Therefore f is by definition not a BFS (from which it follows that any BFS solution is integral).

In practice, many LP solvers will provide BFS solutions to the given LP due to the usage of the simplex method. Hence, the fact that BFS's of min-cost flow problems are integral means that if we send a min-cost problem as the input to a LP solver, we are able to get an integer solution back, which is exactly what we require in many practical cases.