Introduction to Optimization
MS&E 111/ENGR 62, Autumn 2007-2008, Stanford University
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Handout 15: Practice problems for the final

The following are a set of practice problems for the final. They are not intended to comprise a practice final. The actual final will not be this long nor will it necessarily have the same type of problem mix as this problem set.

Problem 1 For each of the following statements, circle TRUE on your answer sheet if the statement is *always* true; otherwise circle FALSE. No explanations/proofs/counterexamples are necessary.

1. If the objectives of the primal and dual symmetric programs (pg. 90) are reversed (i.e., the primal is written as "min $c^T x$" and the dual is written as "max $b^T y$") then the direction of the weak duality inequality in Theorem 4.2.1 (pg. 90) is also reversed.
2. Consider a min-cost-flow problem with all integer demands and capacities. If this problem has multiple optimal solutions, then it has at least one optimal solution containing a non-integer component (i.e, the flow on at least one edge is not an integer).
3. If a linear program has an optimal solution, then the set of optimal solutions is either a point or line (or subset thereof - i.e., a segment, ray, etc.).
4. A polytope defined by M linear inequality constraints in 2-dimensional space can have no more than M basic feasible solutions.
5. A polytope defined by M linear inequality constraints in 3-dimensional space can have no more than $M + 1$ basic feasible solutions.
6. If a contingent claims market has an arbitrage opportunity (pg. 40), then there are two portfolios with the same payoff but different prices. In other words, there exist some x and y in $\mathcal{R}^N$ such that $Px = Py$ but $\rho^T x \neq \rho^T y$.
7. If the edge costs in a *feasible* min-cost-flow problem are all nonnegative, then there must exist at least one optimal solution.

Problem 2 Consider the inventory management problem from HW5 (problem 1). Recall that for each day i, there was a cost to produce goods of c_i per unit and a demand for goods d_i. Recall also that there was a holding cost of h per unit and a backorder cost (of deferring demand) of b per unit. In practice, the holding cost is usually concave in the number of the items (e.g. $h(x) = \sqrt(x)$ is the holding cost for x units in a period). Also, the cost of backlog is convex (e.g. $b(x) = x^2$). In such circumstances, our min cost flow formulation will not work.

Create a dynamic program that will solve this problem under such circumstances. Also, let's ignore the revenue side of the equation, and just worry about cost minimization (since the revenue is fixed anyway). Use the subproblem $C(i, I)$ to represent the minimum cost from day i onwards beginning with inventory of I, where negative I represents a backlog. Specify this subproblem as well as the range of values of i and I you will need to solve the problem for. Finally, remember that all demand must be satisfied eventually.

Problem 3 A total of of $n-1$ stepping stones were placed at one meter intervals across a river that is n meters wide. Over time, some of these stones got washed away. For $1 \le i \le n-1, A[i] = 1$ if the i-th stepping stone is still there, and $A[i] = 0$ if this stone got washed away. A baby kangaroo wants to cross this river. The kangaroo can jump at most k meters at a time. Also, a jump gives a kangaroo some momentum; so two successive jumps can only differ by one meter. For example, if the first jump is 10 meters, the next jump can be 9, 10, or 11 meters. Assume the kangaroo never jumps backwards. You are given n, A, k. You need to find out whether the kangaroo can cross the river without falling into the water, and if yes, the minimum number of leaps it needs to make. Assume that the kangaroo must start jumping from the bank i.e. from the point $i = 0$ and is allowed to land anywhere on the opposite side i.e. at any $i \ge n$. The kangaroo can take a running start, so the first jump can be anything between 1 and k. Solve this with a dynamic program. *Hint: Think of solving a series of sub-problems wherein you are finding the minimum number of jumps it would take to reach the other side. It is up to you to determine what other information will be given in your subproblem and how to formulate it.*

Problem 4 Suppose you are in charge of coordinating an aid campaign consisting of bringing medicaments from different cities of America and Europe to certain cities in Africa. The logistics group of the project determined that the best way to do it is in two phases. First bring the medicaments from the American and European origin cities to some selected big cities in Africa where the aid would be organized to be sent to the destination cities. Assume that there is only one type of medicament.

The American or European city i $(i = 1, ..., I)$ has an availability of medicaments equal to S_i. Our warehouse at transfer city j $(j = 1, ..., J)$ has a capacity T_j; that is, no more than T_j units of medicaments can go into city j. Everything that goes into the transfer city j must go out to some destination city. The African destination city k $(k = 1, ..., K)$ has a demand D_k of medicaments, which must be satisfied. The unit transportation cost from city i to the transfer city j is c_{ij}, for $i = 1, ..., I$ and $j = 1, ..., J$. The unit transportation cost from the transfer city j to the destination city k is d_{jk}, for $j = 1, ..., J$ and $k = 1, ..., K$.

Formulate a linear program that minimizes the total transportation costs while satisfying the demand requirements of each city, without violating the availability and capacity constraints.

Problem 5 Suppose you are given a min-cost flow problem where the capacities are on the edges, but the cost is on vertices, i.e., it costs c_v to send one unit of flow through v. No cost is incurred if the flow originates or terminates at v. How would you reduce this problem to a standard min-cost flow problem where both costs and capacities are on the edges? *Hint: Observe that we can view flow as only originating at nodes with negative demand and terminating at nodes with positive demand.*

Problem 6 Consider the traditional knapsack problem where you have N items each with value v_i and weight w_i. The capacity of the knapsack is W, and you are to maximize the value of the items put in your knapsack assuming items can be placed fractionally.

a) Set up this LP.

b) Set up the dual to this LP.

c) Write the complementary slackness conditions.

d) Interpret the complementary slackness conditions for this problem. It may be useful to recall our interpretation of the dual variables as shadow prices of the resources.