**Introduction to Optimization**
MS&E 111/ENGR 62, Autumn 2007-2008, Stanford University
Instructor: Ashish Goel
Homework 2. Given 10/8/07. Due 10/15/07 in class.

Collaboration policy: You can solve Problems 4 and 5 with a partner. If you choose to do so, both of you should turn in a copy of your Answer Reports and clearly indicate who you worked with. Additionally, on any problem you can discuss general strategies with other students in this class but cannot collaborate on the actual final answer. You cannot discuss the HW with anyone not in the class.

**Problem 1** List all basic feasible solutions of the following LP:

$$\begin{array}{lrcl} \text{maximize} & x_1 + 3x_2 & & \\ \text{subject to} & x_1 + 0.1x_2 & \leq & 2 \\ & 0.4x_1 + 2x_2 & \leq & 3 \\ & x_1 + 1.1x_2 & \leq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

**Problem 2** Recall that the problem of finding an arbitrage opportunity in a market with $N$ assets whose prices are given as $\rho \in \mathcal{R}^N$ and whose payoffs are given as $P \in \mathcal{R}^{M \times N}$ can be posed as the LP:

$$\begin{array}{ll} \text{minimize} & \rho^T x \\ \text{subject to} & Px \geq 0 \\ & \rho^T x = -1 \end{array}$$

Suppose now that there are transaction costs. In particular, for each $j \in \{1, \ldots, N\}$ we must pay a transaction cost of $q_j > 0$ per unit of the $j^{\text{th}}$ contingent claim bought or sold short. Provide a linear program that finds an arbitrage opportunity (if it exists) which minimizes the transaction cost incurred for every unit of current profit. Make sure that the optimization problem you provide is written as a linear program; that is the objective function and constraints are linear.

**Problem 3** A policy maker in Sacramento county needs to allocate some water to a group of $n$ farmers in California from $m$ reservoirs. Because the media will highlight any farmer who is very unhappy, the policy maker wants to maximize the minimum amount that any of the farmers receives. Reservoir $i$ has capacity $u_i$. If farmer $j$ is served from reservoir $i$, then a fraction $f_{ij}$ of the water gets lost in evaporation while being channeled to the farmer.

(a) Formulate this as a linear program.

(b) Suppose farmer 2 cannot be served by reservoir 3 because the farmer is at a higher altitude than the the reservoir. How can you take this into account in your model without increasing the number of constraints or variables?

**Problem 4 (VRM 3.13)** MSE Airlines (pronounced messy) needs to hire customer service agents. Research on customer demands has led to the following requirements on the minimum number of customer service agents that need to be on duty at various times in any given day:

| Time Period | Staff Required |
|---|---|
| 6am to 8am | 68 |
| 8am to 10am | 90 |
| 10am to noon | 56 |
| Noon to 2pm | 107 |
| 2pm to 4pm | 80 |
| 4pm to 6pm | 93 |
| 6pm to 8pm | 62 |
| 8pm to 10pm | 56 |
| 10pm to midnight | 40 |
| Midnight to 6am | 15 |

The head of personnel would like to determine the least expensive way to meet these staffing requirements. Each agent works an 8 hour shift, but not all shifts are available. The following table gives the available shifts and daily wages for agents working various shifts:

| Shift | Daily Wages |
|---|---|
| 6am-2pm | \$180 |
| 8am-4pm | \$170 |
| 10am-6pm | \$160 |
| Noon-8pm | \$190 |
| 2pm-10pm | \$200 |
| 4pm-Midnight | \$210 |
| 10pm-6am | \$225 |
| Midnight-8am | \$210 |

(a) Write a linear program that determines the least expensive way to meet the staffing requirements.

(b) Solve the linear program using Excel.

**Problem 5** Consider a stock that can take on prices in {\$1, . . . , \$100}. Assume that the current price of the stock is \$50, there is a zero-coupon bond available at a price of \$0.9, and there is a European call option expiring in one year with strike price of \$50 available at a price of \$10. There is also a European put option expiring in one year with strike price of \$50 available but we don't know its price.

A butterfly spread is a contingent claim that pays its holders $\max(K - |L - S|, 0)$ if the price of the stock is $S$ at some future time $T$, where $K$, $L$, and $T$ are parameters of the contract. Consider a butterfly spread that pays its holder one year from now, with $K = 10$ and $L = 50$.

(a) What is a reasonable, arbitrage-free price for the put option described above? (*hint:* see VRM 2.4.2)

(b) What are the payoff vectors for the available securities?

(c) What is the payoff vector for the given butterfly spread?

(d) Formulate a linear program to find the cheapest super-replicating portfolio, and solve in Excel.