Discussion Section – Monday, Dec. 3, 2007

Duality

- for every LP (primal), there is another, related LP (dual)
- very deep theory – beyond LP + scope of class

symmetric forms

① $\max\ c^T x$ st $Ax \le b$, $x \ge 0$ ⟷ $\min\ b^T y$ st $A^T y \ge c$, $y \ge 0$

② $\min\ c^T x$ st $Ax \ge b$, $x \ge 0$ ⟷ $\max\ b^T y$ st $A^T y \le c$, $y \ge 0$

primal — dual

sizes

Note: A is $m \times n$
x is n b is m
y is m – one for each constraint

Forming Duals

- convert LP to one of the symmetric forms using "tricks"
- use above table, then simplify (if you can)

ex: Find dual of:

convert

$$\begin{aligned} \max \quad & c^T x \\ \text{st} \quad & Ax \geq b \\ & x \text{ free} \end{aligned} \Rightarrow \begin{aligned} \min \quad & -c^T(x^+ - x^-) \\ \text{st} \quad & A(x^+ - x^-) \geq b \\ & \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \begin{bmatrix} -c \\ c \end{bmatrix}^T \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \\ \text{st} \quad & \begin{bmatrix} A & \vdots & -A \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq b \\ & \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \geq 0 \end{aligned} \xRightarrow{\text{take dual}} \begin{aligned} \max \quad & b^T y \\ \text{st} \quad & \begin{bmatrix} A^T \\ \cdots \\ -A^T \end{bmatrix} y \leq \begin{bmatrix} -c \\ c \end{bmatrix} \\ & y \geq 0 \end{aligned}$$

simplify

$$\begin{aligned} \max \quad & b^T y \\ \text{st} \quad & A^T y \leq -c \\ & -A^T y \leq c \\ & y \geq 0 \end{aligned} \Rightarrow \begin{aligned} \max \quad & b^T y \\ \text{st} \quad & A^T y \leq -c \\ & A^T y \geq -c \\ & y \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} \max \quad & b^T y \\ \text{st} \quad & A^T y = -c \\ & y \geq 0 \end{aligned}}$$

- there is short cut! (tables in VRM notes)
- but, not responsible for knowing these

Interpretations of Dual

1. shadow prices
2. sensitivities

} idea:- one dual var. for each primal constraint
- gives amount opt. changes for small change in constraint rhs

3. application specific interpretations
   - (a) min cut / max flow
   - (b) knapsack
   - (c) shortest path / Ford's algorithm

Properties of Dual

(Not responsible for proofs)

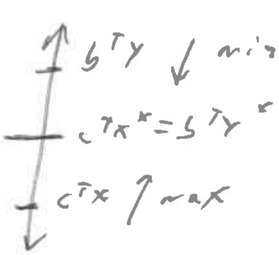
1. strong duality: given x^* (primal opt. soln.), y^* (dual opt. soln.)

$$\Rightarrow \quad c^T x^* = b^T y^*$$

2. weak duality: given x (primal feasible soln.), y (dual feasible soln.)

$$c^T x \leq b^T y$$

(assuming first symmetric form)



(3) complimentary slackness : given x^* (primal opt. soln.)
y^* (dual opt. sol.)

$$\Longleftrightarrow \quad (b - Ax^*)^T y^* = 0 \quad \underline{\text{and}}$$

$$(A^T y^* - c)^T x^* = 0$$

- look back at first symmetric form

- makes sense in terms of sensitivity :
  if ~~to~~ there is slack, changing rhs
  yields nothing

minor properties

(4) dual of dual $\Rightarrow$ primal (by construction)

(5) primal unbounded $\Rightarrow$ dual infeasible
dual unbounded $\Rightarrow$ primal infeasible

(from weak duality)

primal infeasible $\Rightarrow$ dual infeasible ? NO

(see HW5)