

Introduction to Optimization

MS&E 111/ENGR 62, Autumn 2008-2009, Stanford University

Instructor: Ashish Goel

Practice problems for the final

The following are a set of practice problems for the final. They are not intended to comprise a practice final. The actual final will not be this long nor will it necessarily have the same type of problem mix as this problem set.

Problem 1 For each of the following statements, circle TRUE on your answer sheet if the statement is *always* true; otherwise circle FALSE. No explanations/proofs/counterexamples are necessary.

1. A polytope defined by M linear inequality constraints in 2-dimensional space can have no more than M basic feasible solutions.
2. A polytope defined by M linear inequality constraints in 3-dimensional space can have no more than $M + 1$ basic feasible solutions.
3. If a contingent claims market has an arbitrage opportunity (pg. 40), then there are two portfolios with the same payoff but different prices. In other words, there exist some x and y in \mathcal{R}^N such that $Px = Py$ but $\rho^T x \neq \rho^T y$.
4. If the edge costs in a *feasible* min-cost-flow problem are all nonnegative, then there must exist at least one optimal solution.
5. If the polytope corresponding to the feasible region of a linear program is bounded and non-empty, the linear program must always have an optimum solution that is basic feasible.
6. If the polytope corresponding to the feasible region of a linear program is unbounded but the linear program has an optimum solution, the linear program must always have an optimum solution that is basic feasible.

Problem 2 You are given a collection of N plots of land of areas a_1, a_2, \dots, a_N respectively. Because of zoning restrictions, you can not further subdivide these plots. You have to present a dynamic programming formulation to determine if the plots can be allocated to two farmers such that each farmer gets exactly the same total area. Define subproblems $D(i, Z)$ where $D(i, Z) = 1$ if the first i plots can be allocated such that the difference between the area allocated to the two farmers is exactly Z , and $D(i, Z) = 0$ otherwise.

1. Show (i.e. write down a formula) how these subproblems can be solved using the solutions to “smaller” subproblems.
2. Specify the ranges of i, Z .
3. Specify which of these values contains the final answer.

Problem 3 Suppose you are in charge of coordinating an aid campaign consisting of bringing medicaments from different cities of America and Europe to certain cities in Africa. The logistics group of the project determined that the best way to do it is in two phases. First bring the medicaments from the American and European origin cities to some selected big cities in Africa where the aid would be organized to be sent to the destination cities. Assume that there is only one type of medicament.

The American or European city i ($i = 1, \dots, I$) has an availability of medicaments equal to S_i . Our warehouse at transfer city j ($j = 1, \dots, J$) has a capacity T_j ; that is, no more than T_j units of medicaments

can go into city j . Everything that goes into the transfer city j must go out to some destination city. The African destination city k ($k = 1, \dots, K$) has a demand D_k of medicaments, which must be satisfied. The unit transportation cost from city i to the transfer city j is c_{ij} , for $i = 1, \dots, I$ and $j = 1, \dots, J$. The unit transportation cost from the transfer city j to the destination city k is d_{jk} , for $j = 1, \dots, J$ and $k = 1, \dots, K$.

Formulate a linear program that minimizes the total transportation costs while satisfying the demand requirements of each city, without violating the availability and capacity constraints.

Problem 4 Suppose you are given a min-cost flow problem where the capacities are on the edges, but the cost is on vertices, i.e., it costs c_v to send one unit of flow through v . No cost is incurred if the flow originates or terminates at v . How would you reduce this problem to a standard min-cost flow problem where both costs and capacities are on the edges? *Hint: Observe that we can view flow as only originating at nodes with negative demand and terminating at nodes with positive demand.*

Problem 5 Consider the traditional knapsack problem where you have N items each with value v_i and weight w_i . The capacity of the knapsack is W , and you are to maximize the value of the items put in your knapsack assuming items can be placed fractionally.

- a) Set up this LP.
- b) Set up the dual to this LP.
- c) Write the complementary slackness conditions.

d) Interpret the complementary slackness conditions for this problem. It may be useful to recall our interpretation of the dual variables as shadow prices of the resources.

Problem 6 State and give a (brief) proof of the complementary slackness conditions, assuming the primal is in the standard form used in class (i.e. maximize $c^T x$ s.t. $Ax \leq b, x \geq 0$)