Mathematical Programming and Combinatorial Optimization
MS&E 212, Spring 2005-2006, Stanford University
Instructor: Ashish Goel
Handout 11: Homework 4. Given 5/30/06. Due 6/7/06 by midnight.

Collaboration policy: Limited collaboration is allowed – you can discuss general strategies with other students in this class but can not collaborate on the actual final answer. Please do not look at someone else's solution and do not share your solution with anyone else.

1. **[20 pts]** Reduce the following problems to 0-1 knapsack, or solve them from scratch using dynamic programming:

   (a) Given N positive integers $x_1, x_2, \ldots x_N$, find whether there exists a subset S which has sum exactly Y.

   (b) Here, $\mathcal{Z}^+$ refers to the set of positive integers. Assume that a_i, b_i, and c are also posiitive integers which are known to you, and that x_i's are the decision variables.

$$\begin{array}{rl} \textbf{maximize} & \prod_i x_i^{a_i} \\ \text{subject to:} & \\ (\forall i): & x_i \in \mathcal{Z}^+, \\ \text{and} & \sum_i b_i x_i \leq c \end{array}$$

   For each problem analyze the running time of your solution and the state space of the underlying dynamic program. Can you think of any real-life examples where either of the two problems might arise? Implement either of these problems using your favorite programming language and send us the code and some test inputs.

2. **[10 pts]** A total of of $n-1$ stepping stones were placed at one meter intervals across a river that is n meters wide. Over time, some of these stones got washed away. For $1 \leq i \leq n-1, A[i] = 1$ if the i-th stepping stone is still there, and $A[i] = 0$ if this stone got washed away. A baby kangaroo wants to cross this river. The kangaroo can jump at most k meters at a time. Also, a jump gives a kangaroo some momentum; so two successive jumps can only differ by one meter. For example, if the first jump is 10 meters, the next jump can be 9, 10, or 11 meters. Assume the kangaroo never jumps backwards. You are given n, A, k. You need to find out whether the kangaroo can cross the river without falling into the water, and if yes, the minimum number of leaps it needs to make. Assume that the kangaroo must start jumping from the bank i.e. from the point $i = 0$ and is allowed to land anywhere on the opposite side i.e. at any $i \geq n$. The kangaroo can take a running start, so the first jump can be anything between 1 and k. *Note: For a dynamic programming problem where you are just asked to give a solution, you must clearly state the recursive formula, define the terms in the formula, analyze the atate space size and the running time, explain why the recursion does not lead to cycles, and explain what decision variables you would store.*

3. **[10 pts]** Suppose there are n cities along the Amazon river, at positions $x_1, x_2, , x_n$. We have to choose p of these cities as docks such that the sum, over all cities, of the distance of the city

to the nearest *downstream* dock is minimized. Present a dynamic programming algorithm for this problem.

4. **[15 pts]** A marketing executive has an advertising budget of B dollars, which needs to be allocated to n products. The expected return from investing x_i dollars in advertising product i is $f_i(x_i)$ where the function f_i is given either as a table or a formula. The executive wants to maximize total expected return.

   (a) Assume that each f_i is concave. Reduce this problem to a min-cost flow problem.

   (b) In reality, f_i would often be non-concave for this problem. Explain why (this requires a common-sense answer, not a technical one).

   (c) Explain how you could solve this problem using a dynamic program, assuming that the f_i's are completely general.