Mathematical Programming and Combinatorial Optimization
MS&E 212, Spring 2005-2006, Stanford University
Instructor: Ashish Goel
Handout 14, practice problems for the final

1. A company is considering n possible new products for next year's product line. A decision now needs to be made regarding which products to manufacture and in what quantities. Product i has a fixed cost of f_i and a marginal cost of m_i per unit. Each unit of product i will fetch a revenue of r_i. No more than d_i units of product i can be sold. The total production budget is B. The goal of the company is to maximize its revenue without exceeding the production budget. Give a dynamic programming solution for deciding the amount of each product that should be produced to meet this goal.

   Assume that B, d_i, f_i, r_i, m_i are all integers.

2. The following theorem is a classical result due to König. Prove it using the max-flow min-cut theorem. *Note: A vertex cover of a graph is a set of vertices S such that each edge in the graph has at least one endpoint in S. Also, a graph is bipartite if the vertices can be partitioned into two sets P and Q such that all edges are between a vertex in P and a vertex in Q.*

   König's Theorem: The size of the largest matching in an undirected bipartite graph is the same as the size of the smallest vertex cover

3. You are given an undirected bipartite graph. Present a polynomial time algorithm to find the smallest vertex cover of this graph.

4. Present an algorithm to find the maximum weight spanning tree of a graph. Give its running time and explain why it is correct.

5. You are given a set of eight men and eight women. Man i and woman j are compatible if i is prime and j is composite, or if i is composite and j is prime. Draw the corresponding bipartite graph and find a maximum matching. Prove that your matching is the largest possible. For this problem, the number "1" is assumed to be prime.

6. Suppose you are given a min-cost flow problem on a graph with n vertices and m edges. Prove that there exists an optimal solution which can be decomposed into at most $m + n$ paths.

7. Assume P = NP. Prove that there is a polynomial time algorithm for factorizing a number n.

8. You are given a set of n elements U and a collection of k subsets of U, denoted $S_1, S_2, \ldots S_k$. Each element in U is covered by at most 6 sets. You have to determine the smallest number of sets which cover U. Formulate the decision version of this problem and prove that it is NP-complete.