MS&E 214
Optimization via Case Studies

Week 6: Finance

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(based on slides by Professor Benjamin Van Roy)

Roadmap

- Finished two case studies
  - Social Choice and Welfare
    - Matchings
    - Participatory Budgeting
    - Scoring Rules
    - Fair Division
    - Walrasian Equilibrium
  - Machine Learning
    - Regression
    - Pattern Classification
- Techniques
  - LPs (Everywhere)
  - Quadratic Programming (Regression, SVMs)
  - Basic Feasible Solutions (Matchings)
  - Duality (Fair Division)
  - Max-min and Min-max objective functions (Fair Division, Regression, Pattern Classification, Participatory Budgeting)
  - Still to come: The logarithmic trick, Convex Optimization in general, Network Flows, Revisit Duality, Revisit Basic Feasible Solutions
- Next: Finance
  - Options (as a linear payoff function)
  - Designing portfolios to cover a specific liability under diverse market scenarios (LP)
  - Arbitrage (LP); Currency arbitrage (network flows with the logarithmic trick) to come later
  - Portfolio optimization (Quadratic Programming)

Two different approaches to finance in this class

- Assume that you can enumerate all future market scenarios (useful for financial derivatives whose values depend on a small number of well defined market variables), and enumerate your future payoffs and liabilities in all possible scenarios

vs

- Assume that the market is complex, and different products interact in ways that are impossible (or too hard) to fully characterize. Hence, we use past statistical measures (such as average returns and variance of returns of different stocks, the correlations between different stocks) to optimize a risk-adjusted measure of expected future returns

Two different approaches to finance in this class

Today

- Assume that you can enumerate all future market scenarios and enumerate your future payoffs and liabilities in all possible scenarios. Useful for financial derivatives whose values depend on a small number of well defined market variables.

Next lecture

vs

- Assume that the market is complex, and different products interact in ways that are impossible (or too hard) to fully characterize. Hence, we use past statistical measures (such as average returns and variance of returns of different stocks, the correlations between different stocks) to optimize a risk-adjusted measure of expected future returns

Contingent Claims

- A context and application area for Linear Programs

- A contingent claim is a contract
  - Receive a **payoff** in the future depending on (or, contingent upon) an uncertain outcome, e.g. the price of a stock in the future
  - May pay a **price** to purchase the contract
  - Current money is not the same as future money

- Examples
  - Insurance
  - Negotiated contract with contingencies
  - Stocks, bonds, options, and other derivatives

- Mathematical representation
  - Enumerate possible outcomes 1, 2, ..., M
  - Specify outcome-contingent payoffs $\mathbf{a} \in \Re^M$

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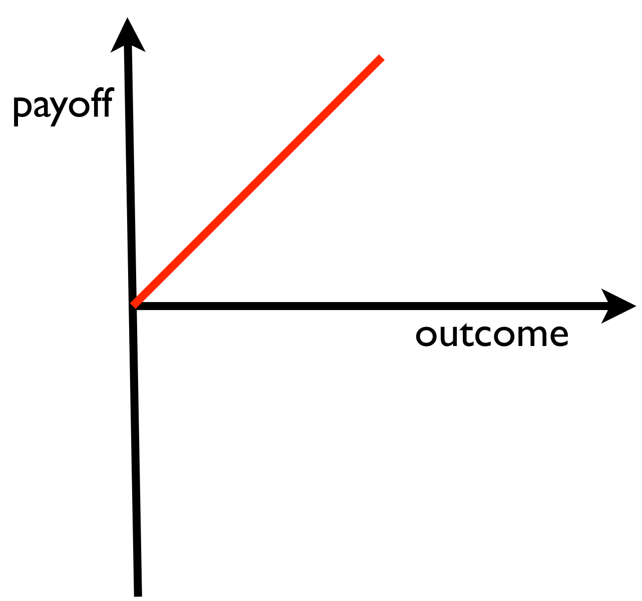
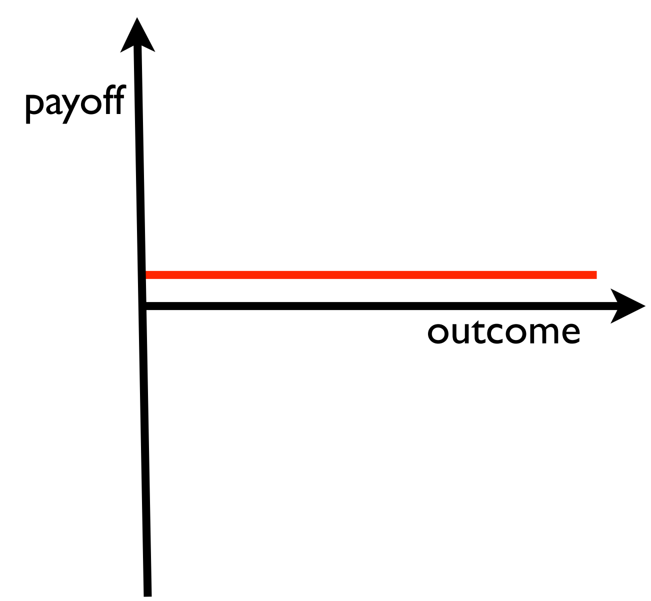
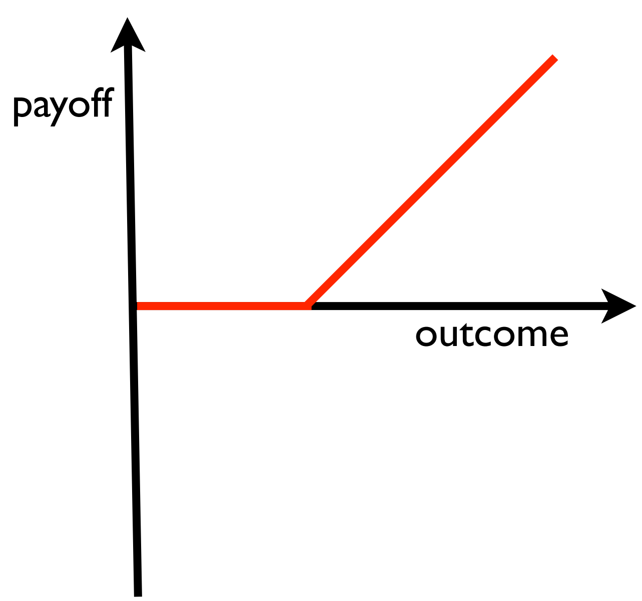
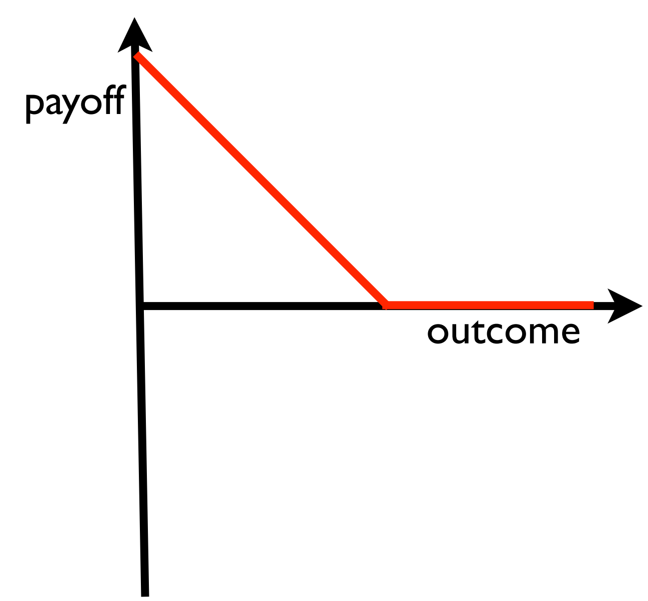
Stocks and Bonds

- Assume one year holding period

	stock	zero-coupon bond
(Current) price	p_1	p_2
outcomes	future stock price = 1, …, M	future stock price = 1, …, M
(Future) payoff vector	$\mathbf{a}^1 \in \Re^M$	$\mathbf{a}^2 \in \Re^M$
illustration		

European Calls and Puts

	European call option	European put option
price	p_3	p_4
expiration date	1 year	1 year
strike price	$40	$60
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
payoff vector	$\mathbf{a}^3 \in \Re^M$	$\mathbf{a}^4 \in \Re^M$
illustration		

Call and Put Options

t = Maturity date

s = Strike Price

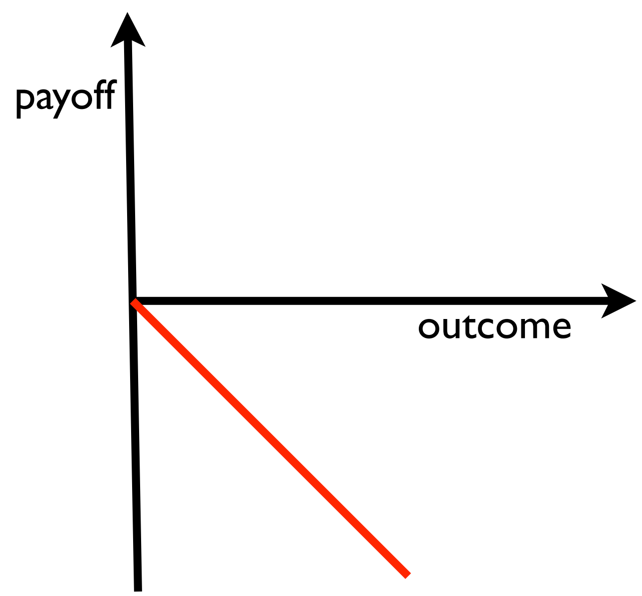
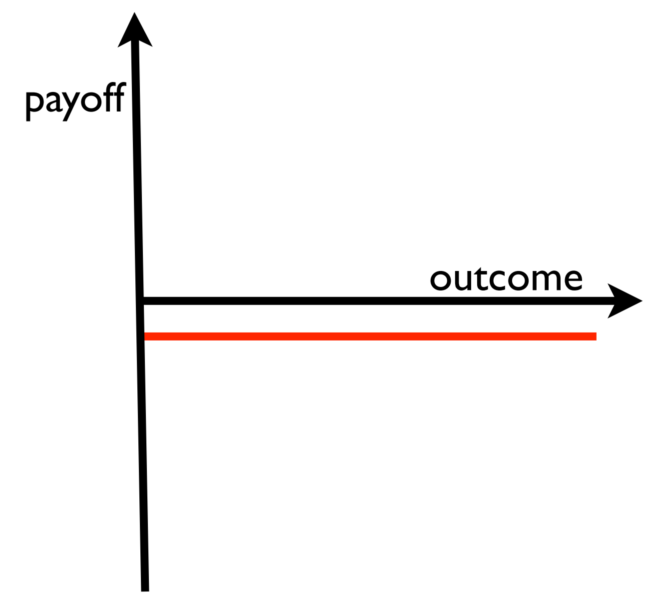
z = future price of stock on maturity date

- Call option: Payoff = $\max\{0, z-s\}$

- Put option: Payoff = $\max\{0, s-z\}$

Short Selling

- Broker borrows/sells contingent claim you don't have

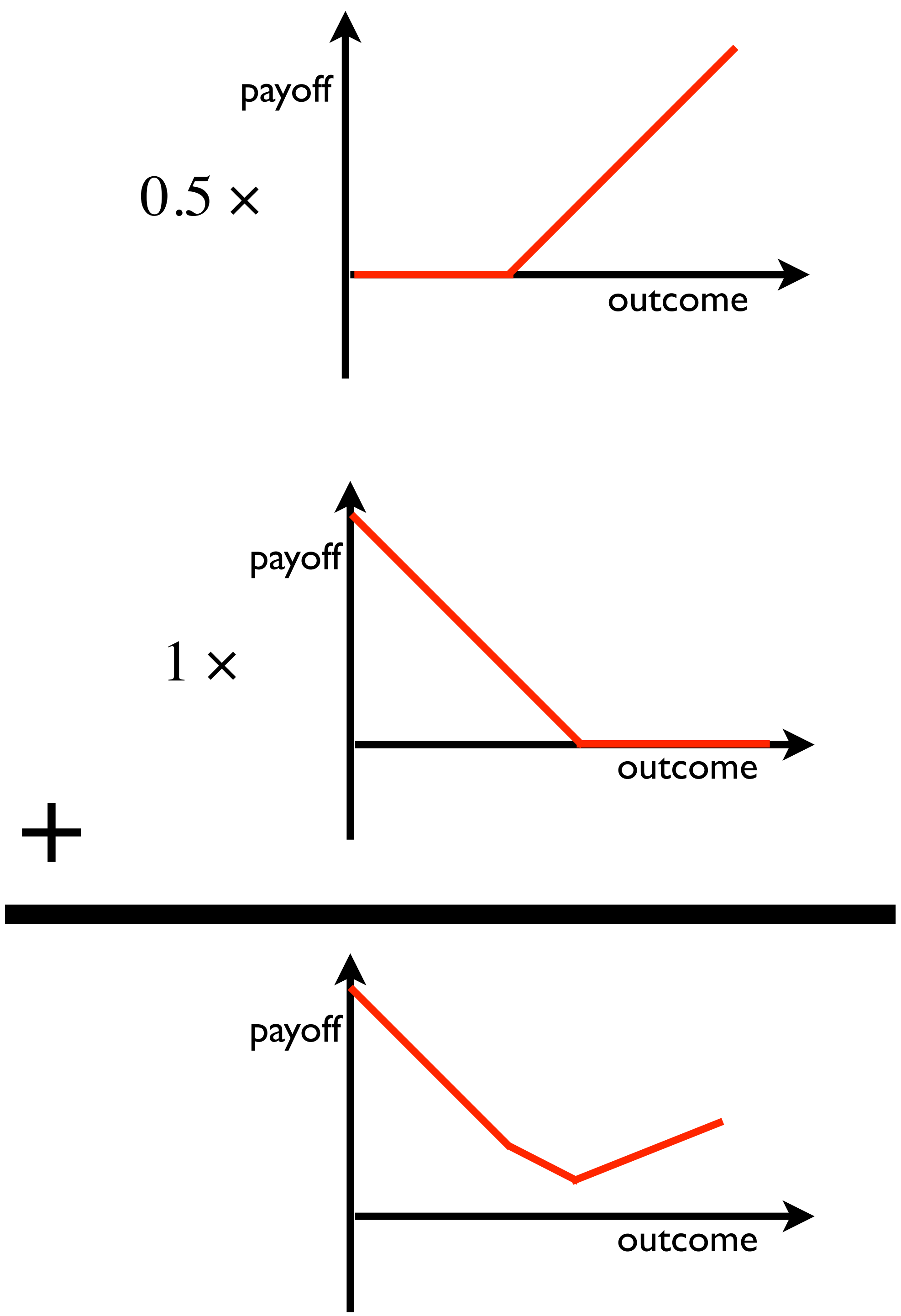
	short sell stock	short sell zero-coupon bond
price	$-p_1$	$-p_2$
outcomes	future stock price = 1, …, M	future stock price = 1, …, M
payoff vector	$-\boldsymbol{a}^1$	$-\boldsymbol{a}^2$
illustration		

- Modeling simplifications: no transaction costs or margin requirements

Markets and Portfolios

- N contingent claims, M payoff-relevant outcomes
- Payoff matrix (future) $\mathbf{P} \in \Re^{M \times N}$
- Market prices (current) $\rho \in \Re^{N}$ (row vector)
- Portfolio vector $\mathbf{x} \in \Re^{N}$
- Portfolio payoff $\mathbf{Px} \in \Re^{M}$
- Portfolio price $\rho\mathbf{x}$

Example



$$P = \begin{bmatrix} 0 & 8 \\ 0 & 7 \\ 0 & 6 \\ 0 & 5 \\ 0 & 4 \\ 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & 0 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 3 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$Px = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2.5 \\ 2 \\ 1.5 \\ 2 \\ 2.5 \\ 3 \\ 3.5 \\ 4 \end{bmatrix}$$

$$\rho x = 3.5$$

Replication

- Liabilities $\mathbf{b} \in \Re^M$
- Replicating Portfolio $\mathbf{Px} = \mathbf{b}$
- Price of Replication $\rho \mathbf{x}$

Example of Replication

A banker has sold a call option on a stock with strike price \$50, maturing in one year. The banker has access to a zero-coupon bond with payoff \$1 cost 0.95 maturing in one year. The banker also can buy the stock at the current price \$40 and keep it. And the banker can also buy a put option on the same stock with a strike price of \$50 at cost \$1. Design a portfolio that replicates the payoff from the call option that the banker has sold.

(Of minimum current cost)

Solve using excel.

Super-Replication

- What if there is no replicating portfolio?
  - Incomplete market

- Super-Replication $\mathbf{Px} \geq \mathbf{b}$

- Minimize price

$$\begin{array}{rlcl} \text{minimize} & \rho\mathbf{x} & & \\ \text{subject to:} & \mathbf{Px} & \geq & \mathbf{b} \end{array}$$

Example of Super-Replication

A banker has sold a call option on a stock with strike price $50, maturing in one year. The banker has access to a zero-coupon bond with cost 0.95 maturing in one year. The banker also can buy the stock at the current price $40 and keep it. And the banker can also buy a put option on the same stock with a strike price of $45 at cost $1. Design a portfolio that super-replicates the payoff from the call option that the banker has sold. ($Px >= b$ as opposed to $Px = b$)

Solve using excel.

Example of Super-Replication

A banker has sold a call option on a stock with strike price \$50, maturing in one year. The banker has access to a zero-coupon bond with cost 0.95 maturing in one year. The banker also can buy the stock at the current price \$40 and keep it. And the banker can also buy a put option on the same stock with a strike price of \$45 at cost \$1. Design a portfolio that super-replicates the payoff from the call option that the banker has sold. (Px >= b as opposed to Px = b)

Solve using excel.

Which scenarios to use: In this case, since the payoffs are all piecewise linear functions, it is enough to ensure that every "change point" is represented and at least one scenario smaller than all the change points and at least one scenario larger than all the change points

Arbitrage

- **Def.** arbitrage opportunity

$$\mathbf{x} \in \Re^N \quad \text{s.t.} \quad \rho\mathbf{x} < 0 \quad \textit{and} \quad \mathbf{P}\mathbf{x} \geq 0$$

- Most lucrative arbitrage opportunity

$$\begin{aligned} \min \quad & \rho\mathbf{x} \\ \text{s.t.} \quad & \mathbf{P}\mathbf{x} \geq 0 \end{aligned}$$

- Is there a problem with this?

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- Is there a problem with this?

It is always infeasible

It always has an unbounded optimum

Both of the above

It does not identify an arbitrage opportunity

Arbitrage

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- Most lucrative arbitrage opportunity

$$\begin{array}{lll} \min & \rho\mathbf{x} & \\ \text{s.t.} & \mathbf{P}\mathbf{x} \geq & 0 \end{array} \qquad \mathbf{x} \to \infty$$

- Arbitrage opportunity that makes $1

$$\begin{array}{llll} \min & \ldots & & \\ \text{s.t.} & \rho\mathbf{x} & = & -1 \\ & \mathbf{P}\mathbf{x} & \geq & 0 \end{array}$$

Minimizing Shares Traded

$$\begin{array}{lrcl} \min & \sum_{i=1}^{N} |x_i| & & \\ \text{s.t.} & \rho\mathbf{x} & = & -1 \\ & \mathbf{P}\mathbf{x} & > & 0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{lrcl} \min & \sum_{i=1}^{N} (x_i^+ + x_i^-) & & \\ \text{s.t.} & \rho(\mathbf{x}^+ - \mathbf{x}^-) & = & -1 \\ & \mathbf{P}(\mathbf{x}^+ - \mathbf{x}^-) & \geq & 0 \\ & \mathbf{x}^+, \mathbf{x}^- & \geq & 0 \end{array}$$