MS&E 339
Fisher Market — precursor to Understanding Decentralized Exchanges

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Centralized Allocation of Resources: Canonical Problem

Given N agents, and one unit of M divisible goods each

Need to divide these goods among the agents

x_{ij} = amount of good j allocated to agent i

Constraints: $x \geq 0$; For all goods j, $\sum_i x_{ij} \leq 1$.

Objective function: Two parts:

(A) What does each agent want?
— $U_i(x)$ = the utility of agent i given allocation x

— Simplifying assumption for today: $U_i(x) = \sum_j w_{ij}x_{ij}$ where the w_{ij} are given and known (no strategic behavior)

(B) Remaining issue: how to **fairly** trade-off between different agents?

Basic Desiderata and Objective Functions

Two Desiderata:

— Pareto-Optimal solution: Not possible to strictly improve utility for one or more agents without decreasing utility for any agent

— Envy-Freeness: No agent prefers another agent's allocation over its own

— Proportionality: No agent would prefer getting a 1/N share of each good over its own allocation

Two simple objectives:

— Utilitarian Objective: Maximize the total utility of all players, i.e. $\sum_i U_i(x)$

— Egalitarian Objective: Maximize Min $\{U_i(x)\}$

Both are easy to obtain using Linear Programming

The Nash Welfare

Maximize $\sum_i \log(U_i(x))$

— Pareto-optimal

— Scale-invariant for each agent (i.e. the allocation does not change if any agent's weights are all multiplied by some positive constant factor)

— Appears to be a nice tradeoff between egalitarian and utilitarian welfare

— A small utility increase for an agent who is receiving very little equality is equivalent to a large utility increase for an agent who is receiving a lot of utility

Fisher Market

Same setting as before

— Given N agents, and one unit of M divisible goods each

— Need to divide these goods among the agents

— x_{ij} = amount of good j allocated to agent i

But not allocated by a planner. Rather by a market

— Each agent given a unit of currency (monopoly money, with no external value) outside this market

— A per-unit price $p_j \geq 0$ is declared for each good

— Each agent can buy goods that maximize its happiness, without worrying about the amount available of each good (and can buy fractional amounts)

Fisher Market: Market clearing

A price vector p and an allocation x such that

— The amount allocated to each agent is optimal for that agent, given the prices

— Agent i's maximum value: Maximize $\sum_j w_{ij}z_j$

subject to: $z \geq 0$, $\sum_j p_j z_j \leq 1$

(i.e. ignore amount of every good)

— $x \geq 0$; For all goods j, $\sum_i x_{ij} = 1$ (i.e. the goods get fully allocated)

Known as Competitive Equilibrium with Equal Endowments

Incredibly: always exists, and is achieved by maximizing the Nash Welfare (proof on whiteboard)

Nash Welfare: Properties

Note: remember that these are for the specific assumptions that we made (e.g. linear utilities)

— Proportionality. Discuss why.

— Envy-Freeness. Discuss why.

— Objective function is concave. Discuss why, and consequences.

— The optimization problem is also known as the Gale-Eisenberg convex program.

— Convex program: Maximize $c^T x$ s.t. x lies in a convex set

Arrow-Debreu markets (defined later)

Market clearing prices exist in very general settings, more relevant for decentralized exchanges

An explicit convex program and/or efficient convex program exists in some settings

This class will not discuss the convex program for Arrow-Debreu Markets or algorithms. Rather, we will draw upon intuition from Fisher markets to gain understanding of Arrow-Debreu markets